Diffusion-Reaction in Branched Structures: Theory and Application to the Lung Acinus

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An exact “branch by branch” calculation of the diffusional flux is proposed for partially absorbed random walks on arbitrary tree structures. In the particular case of symmetric trees, an explicit analytical expression is found which is valid whatever the size of the tree. Its application to the respiratory phenomena in pulmonary acini gives an analytical description of the crossover regime governing the human lung efficiency.

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The PDE problem for oxygen concentration is expressed as

$$\frac{\partial c}{\partial n} = \frac{1}{\Lambda} c,$$

(1)

where $n$ is the normal to the surface. A fixed concentration $c_0$ is set at the source of diffusion. The parameter $\Lambda$ is the ratio $D/W$ of the oxygen diffusivity in air $D$ and membrane permeability $W$ [7]. In the healthy human lungs, the value of $\Lambda$ is around 30 cm. It is of practical importance to know how the transport properties of the human acini depend on this physical (and physiological) parameter as, for example, pulmonary edema degrades the membrane permeability leading to a significant increase of $\Lambda$. The oxygen flux through the membrane of total surface $S$ is given by $\Phi = W \int c \, dS$. The system behavior as a gas exchanger is well described by a quantity $\eta$ called efficiency and defined as

$$\eta = \frac{\Phi}{W c_0 S}. \tag{2}$$

Therefore, $\eta$ is a number between zero and one representing the fraction of the surface which is active. It only depends on the physical parameter $\Lambda$ and the morphology of the branched structure [7].

Let us introduce the discrete representation of this PDE problem. First, we note that the stationary diffusion with partial absorption at the boundary can be modeled by $d$-dimensional partially absorbed random walks on a lattice of parameter $a$ [Fig. 1(a)]. In this frame, the mixed boundary condition (1) means that a particle hitting the boundary can be absorbed with probability $\sigma$, or reflected to its preceding position with probability $(1 - \sigma)$. The absorption probability $\sigma$ is related to the parameter $\Lambda$ of the equivalent PDE problem by the following relation: $\sigma = (1 + \Lambda/a)^{-1}$ [8]. Dealing with thin channels of square profiles, there are $2d - 2$ directions to the boundary. Then, the total probability to be absorbed at one step is $\sigma(2d - 2)/(2d)$. So, $d$-dimensional random walks in a thin channel of “diameter” $a$ can be considered as one-dimensional longitudinal walks with the following dynamic “rules”: being on an intermediate site $k$, the random particle can jump to the left (site $k - 1$) with probability...
The entering flux $\phi_{\text{ent}}$ and the exiting flux $\phi_{\text{exit}}$ can be defined as 
\[ \phi_{\text{ent}} = DS_0 \frac{c_0 - c_1}{a}, \quad \phi_{\text{exit}} = DS_0 \frac{c_\ell - c_{\ell+1}}{a}, \]
where the constant $D$ corresponds to the diffusion coefficient, and $S_0$ stands for cross section area ($S_0 = a^{d-1}$ for a square profile). In conclusion, the concentration $c_0$ and the entering flux $\phi_{\text{ent}}$ depend on $c_\ell$ and $\phi_{\text{exit}}$ through linear functions 
\[ c_0 = \frac{u_{\sigma,\ell}}{v_{\sigma,\ell}} c_{\text{exit}} + \frac{a}{DS_0} \frac{1}{v_{\sigma,\ell}} \phi_{\text{exit}}, \]
\[ \phi_{\text{ent}} = \frac{u_{\sigma,\ell}}{v_{\sigma,\ell}} \phi_{\text{exit}} + \frac{DS_0}{a} \frac{1}{v_{\sigma,\ell}} (u_{\sigma,\ell} - v_{\sigma,\ell}) c_{\text{exit}}. \]

The coefficients are complicated but explicit functions of the parameters $\sigma$ and $\ell$.

Now, in order to compute the flux within a complex tree, we divide it into branches. The thin channel description holds for each single branch. It is then possible to solve the problem for the entire system by using an iterative branch by branch procedure from the last generation up to the root, once some suitable condition has been defined for the branching points.

Let us consider the last branches. The mixed boundary condition (1) is applied on the terminal site $\ell + 1$

\[ \phi_{\text{exit}} = DS_0 (\frac{\partial c}{\partial n})_{\text{exit}} = DS_0 \frac{1}{\Lambda} c_{\text{exit}}. \]

From the last equations, one obtains a direct relation between $c_{\text{exit}}$ and $\phi_{\text{exit}}$:

\[ \phi_{\text{exit}} = DS_0 \frac{f_{\sigma,\ell}(\Lambda)}{c_{\text{exit}}}. \]

Here $f_{\sigma,\ell}(\Lambda)$ is the new function

\[ f_{\sigma,\ell}(\Lambda) = \frac{a}{(\Lambda/a) (u_{\sigma,\ell}^2 - v_{\sigma,\ell}^2) + u_{\sigma,\ell}}, \]

This means that the relation (6) remains valid at the entrance but with a modified parameter $\Lambda' = f_{\sigma,\ell}(\Lambda)$. In the particular case of a single deep pore of length $\ell \gg 1$ (one branch alone), there is a region of values $\Lambda/a$ (order of $\ell^2$) where the function $f_{\sigma,\ell}(\Lambda)$ behaves like $(\Lambda/a)^{1/2}$. One thus retrieves the flux $\phi_{\text{exit}}$ following a power law with the classical exponent $-1/2$.

The next step is to consider the branching point where the parent branch divides into $M$ daughter branches of lengths $\ell_1, \ldots, \ell_M$ [see Fig. 1(c)]. For each of these daughter branches, one can apply the relation (7) between the concentration $c_{\text{exit}}$ and the entering flux $\phi_{\text{ent}}$. We will use...
the superscript to distinguish different daughter branches,
\[ \phi_{\text{ent}}^{(m)} = \frac{DS_0}{\int_{\sigma, \epsilon_n}(\Lambda)} c_{\text{ent}}^{(m)} \quad (m = 1, \ldots, M) \]  

(9)

At the branching point, one has the following conditions:
\[ c_{\text{ent}}^{(1)} = c_{\text{ent}}^{(2)} = \ldots = c_{\text{ent}}^{(M)} = c_{\text{par}} \]

\[ \phi_{\text{ent}}^{(1)} + \phi_{\text{ent}}^{(2)} + \ldots + \phi_{\text{ent}}^{(M)} = \phi_{\text{par}}. \]

(10)

The first condition holds since the branching point connects the parent branch exit with the daughter branch entrances. The second condition provides the conservation of flux at the branching point: the exiting flux \( \phi_{\text{par}} \) of the parent branch is distributed into \( M \) daughter branches. If the branching point can also absorb the particles, the corresponding flux ought to be taken into account on the left hand side. Using these two conditions and relations (9), one has

\[ \phi_{\text{par}} = \frac{DS_0}{\Lambda \epsilon} c_{\text{par}} \]

\[ \frac{1}{\Lambda} = \sum_{m=1}^{M} \frac{1}{\int_{\sigma, \epsilon_n}(\Lambda)}. \]

(11)

Now, one can forget the branching point and the daughter branches and use the relation between the concentration and flux at the end of the parent branch. Performing iteratively this branch by branch procedure up to the tree entrance, one obtains a similar expression for the total flux \( \Phi \),

\[ \Phi = \frac{DS_0}{\Lambda \epsilon} c_0. \]

(12)

where \( c_0 \) is the concentration at the main entrance. The function \( \Lambda_{\text{eff}} \) (depending on \( \sigma \) and, consequently, on \( \Lambda \)) provides the response of the branched tree, and the efficiency \( \eta \) can be written as

\[ \eta(\Lambda) = \frac{\Phi}{W c_0 S} = \frac{S}{S} \frac{\Lambda}{\Lambda_{\text{eff}}}. \]

(13)

A particular simplification appears in the case of a symmetric tree of constant branching number \( M \) when all branches have the same length \( \ell \). Let us consider such a tree of depth \( n \) (containing \( n \) branching levels). In this case, the relation (11) becomes

\[ \Lambda_1 = \kappa f_{\sigma, \epsilon}(\Lambda) = \tilde{f}_{\sigma, \epsilon}(\Lambda), \]

with \( \kappa = 1/M \). On the second level, one applies again this relation: \( \Lambda_2 = \tilde{f}_{\sigma, \epsilon}(\Lambda_1) = \tilde{f}_{\sigma, \epsilon}(\tilde{f}_{\sigma, \epsilon}(\Lambda)). \) Repeating this procedure, one finds for the main entrance:

\[ \Lambda_{\text{eff}} = \Lambda_n = \tilde{f}_{\sigma, \epsilon}(\tilde{f}_{\sigma, \epsilon}(\cdots \tilde{f}_{\sigma, \epsilon}(\Lambda))\cdots), \]

(14)

Since \( \tilde{f}_{\sigma, \epsilon}(\Lambda) \) is the linear fractional transformation of \( \Lambda \), its successive application of itself gives again a linear fractional transformation. One can calculate its coefficients explicitly

\[ \Lambda_{\text{eff}} = \kappa a \frac{\Lambda(u_{\sigma, \epsilon} - v_{\sigma, \epsilon}^2)}{\Lambda(u_{\sigma, \epsilon} - v_{\sigma, \epsilon}^2) - \Lambda u_{\sigma, \epsilon}} + a, \]

(15)

with

\[ \Lambda_{1,2} = \frac{(1 + \kappa)u_{\sigma, \epsilon} + \sqrt{(1 + \kappa)^2 u_{\sigma, \epsilon}^2 - 4\kappa v_{\sigma, \epsilon}^2}}{2}. \]

(16)

Eqs. (13) and (15) give then an analytical expression for the efficiency \( \eta \) of a symmetric tree as a function of \( \Lambda \) and of the tree characteristics \( \kappa, \ell, \) and \( n \).

Figure 2 shows the dependency of the efficiency \( \eta \) as a function of \( \Lambda/L_p \), where the length scale \( L_p = S/a \) qualitatively corresponds to the perimeter of a cut of the acinus from the model of Kitaoka [12]. The observed behavior may be quite different from that of large trees for which one expects a power law with an exponent equal to 1 [13]. This last behavior can be derived from the above equations to the tree depth \( n \) sufficiently large. It can be shown that in this case the length \( \Lambda_{\text{eff}} \) becomes very close to the fixed point \( \Lambda_{\text{eff}}^{(\infty)} = a(\ell + 1)/(M - 1) \) of the linear fractional transformation \( \tilde{f}_{\sigma, \epsilon}(\Lambda) \) (for small \( \sigma \)). In this regime, the efficiency \( \eta \) given by (14) varies linearly with \( \Lambda \). This behavior is observed in Fig. 2 for the larger tree corresponding to \( M = 4 \) and \( \Lambda/L_p \) not too large. Consequently, a particular property of a large tree is that the flux given by Eq. (12) is constant and independent of \( \Lambda \); this is representative of the human acinus, one observes a nontrivial behavior. This means that, due to finite size effects, the system always works in the crossover regime for which we now have an exact theory. In other words, the whole function \( \eta(\Lambda) \) (instead of a single power

![FIG. 2. Efficiency of finite symmetric trees with different branching numbers \( M = 2, 3, 4 \), fixed branch length \( \ell = 2 \), and five levels of branching (depth \( n = 5 \)).](image)
The particular application of this approach to the respiratory processes in human lungs provides an explicit analytical approximation for the efficiency of pulmonary acini for which a symmetrized acinus approximation is shown to be valuable. This last result is important for future studies of the convection-diffusion transition in the human lung. This theoretical approach could be useful for studying other phenomena on branched structures such as, e.g., spectral dimension of fractal trees [14,15].