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Geometrical Structure of Laplacian Eigenfunctions*

D. S. Grebenkov†
B.-T. Nguyen‡

Dedicated to Professor Bernard Sapoval on the occasion of his 75th birthday

Abstract. We summarize the properties of eigenvalues and eigenfunctions of the Laplace operator in bounded Euclidean domains with Dirichlet, Neumann, or Robin boundary condition. We keep the presentation at a level accessible to scientists from various disciplines ranging from mathematics to physics and computer sciences. The main focus is placed onto multiple intricate relations between the shape of a domain and the geometrical structure of eigenfunctions.

Key words. Laplace operator, eigenfunctions, eigenvalues, localization

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†Corresponding author. Laboratoire de Physique de la Matière Condensée, CNRS – Ecole Polytechnique, 91128 Palaiseau, France; Laboratoire Poncelet, CNRS – Independent University of Moscow, Bolshoy Vlasyevskiy Pereulok 11, 119002 Moscow, Russia; and Chebyshev Laboratory, Saint Petersburg State University, Saint Petersburg, Russia (denis.grebenkov@polytechnique.edu).
‡Laboratoire de Physique de la Matière Condensée, CNRS – Ecole Polytechnique, 91128 Palaiseau, France.

1. Introduction. This review focuses on the classical eigenvalue problem for the Laplace operator $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ in an open bounded connected domain $\Omega \subset \mathbb{R}^d$ (where $d = 2, 3, \ldots$ is the space dimension),

$$-\Delta u_m(x) = \lambda_m u_m(x) \quad (x \in \Omega), \tag{1.1}$$

with Dirichlet, Neumann, or Robin boundary condition on a piecewise smooth boundary $\partial\Omega$:

$$\begin{aligned} u_m(x) &= 0 \quad (x \in \partial\Omega) \quad \text{(Dirichlet)}, \\ \frac{\partial}{\partial n} u_m(x) &= 0 \quad (x \in \partial\Omega) \quad \text{(Neumann)}, \\ \frac{\partial}{\partial n} u_m(x) + h u_m(x) &= 0 \quad (x \in \partial\Omega) \quad \text{(Robin)}, \end{aligned} \tag{1.2}$$

where $\partial/\partial n$ is the normal derivative pointed outwards from the domain, and h is a positive constant. The spectrum of the Laplace operator is known to be discrete; the eigenvalues λ_m are nonnegative and ordered in an ascending order by the index

$m = 1, 2, 3, \ldots,$

$$(0 \le)\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \nearrow \infty \tag{1.3}$$

(with possible multiplicities), while the eigenfunctions $\{u_m(x)\}$ form a complete basis in the functional space $L_2(\Omega)$ of measurable and square-integrable functions on Ω [138, 422]. By definition, the function 0 satisfying (1.1)–(1.2) is excluded from the set of eigenfunctions. Since the eigenfunctions are defined up to a multiplicative factor, it is sometimes convenient to normalize them to get the unit L_2-norm:

$$\|u_m\|_2 \equiv \|u_m\|_{L_2(\Omega)} \equiv \left(\int_\Omega dx\ |u_m(x)|^2\right)^{1/2} = 1 \tag{1.4}$$

(note that there is still ambiguity up to multiplication by $e^{i\alpha}$, with $\alpha \in \mathbb{R}$).

Laplacian eigenfunctions appear as vibration modes in acoustics, as electron wave functions in quantum waveguides, as a natural basis for constructing heat kernels in the theory of diffusion, etc. For instance, vibration modes of a thin membrane (a drum) with a fixed boundary are given by Dirichlet Laplacian eigenfunctions u_m, with the drum frequencies proportional to $\sqrt{\lambda_m}$ [419]. A particular eigenmode can be excited at the corresponding frequency [444, 445, 446]. In the theory of diffusion, an interpretation of eigenfunctions is less explicit. The first eigenfunction represents the long-time asymptotic spatial distribution of particles diffusing in a bounded domain (see below). A conjectural probabilistic representation of higher-order eigenfunctions through a Fleming–Viot-type model was developed by Burdzy et al. [102, 103].

The eigenvalue problem (1.1)–(1.2) is archetypical in the theory of elliptic operators, while the properties of the underlying eigenfunctions have been thoroughly investigated in various mathematical and physical disciplines, including spectral theory, probability and stochastic processes, dynamical systems and quantum billiards, condensed matter physics and quantum mechanics, the theory of acoustical, optical, and quantum waveguides, and the computer sciences. Many books and reviews have been dedicated to different aspects of Laplacian eigenvalues and eigenfunctions and their applications (see, e.g., [11, 22, 28, 36, 58, 125, 149, 165, 226, 243, 255, 300, 389, 406]). The diversity of notions and methods developed by mathematicians, physicists, and computer scientists often means that the progress in one discipline is almost unknown or barely accessible to scientists from other disciplines. One of the goals of this review is to bring together various facts about Laplacian eigenvalues and eigenfunctions and to present them at a level accessible to scientists from various disciplines. To this end, many technical details and generalities are omitted in favor of simple illustrations. While the presentation is focused on the Laplace operator in bounded Euclidean domains with piecewise smooth boundaries, a number of extensions are relatively straightforward. For instance, the Laplace operator can be extended to a second-order elliptic operator with appropriate coefficients, the piecewise smoothness of a boundary can often be relaxed [219, 329], and Euclidean domains can be replaced by Riemannian manifolds or weighted graphs [255]. The main emphasis is put on the geometrical structure of Laplacian eigenfunctions and on their relation to the shape of a domain. Although the bibliography contains more than 500 citations, it is far from complete, and readers are invited to refer to other reviews and books for further details and references.

The review is organized as follows. We start by recalling in section 2 general properties of the Laplace operator. Explicit representations of eigenvalues and eigenfunctions in simple domains are summarized in section 3. In section 4 we review the properties of eigenvalues and their relationship to the shape of a domain: Weyl's asymptotic law, isoperimetric inequalities and the related shape optimization problems, and Kac's inverse spectral problem. Although eigenfunctions are not involved at this step, valuable information can be learned about the domain from the eigenvalues alone. The next step consists in the analysis of nodal lines/surfaces or nodal domains in section 5. The nodal lines tell us how the zeros of eigenfunctions are spatially distributed, while their amplitudes are ignored. In section 6, several estimates for the amplitudes of the eigenfunctions are summarized. Most of these results were obtained during the last twenty years.

Section 7 is devoted to the property of eigenfunctions known as localization. We start by recalling the notion of localization in quantum mechanics: strong localization by a potential (see section 7.1), Anderson localization (see section 7.2), and trapped modes in infinite waveguides (see section 7.3). In all three cases, the eigenvalue problem is different from (1.1)–(1.2), due to either the presence of a potential or the unboundedness of a domain. Nevertheless, these cases are instructive, as similar effects may be observed for the eigenvalue problem (1.1)–(1.2). In particular, we discuss in section 7.4 an exponentially decaying upper bound for the norm of eigenfunctions in domains with branches of variable cross-sectional profiles. Section 7.5 reviews the properties of low-frequency eigenfunctions in "dumbbell" domains, in which two (or many) subdomains are connected by narrow channels. This situation is suitable for rigorous analysis as the width of channels plays the role of a small parameter [441]. A number of asymptotic results for eigenvalues and eigenfunctions have been derived for Dirichlet, Neumann, and Robin boundary conditions. A harder case of irregular or fractal domains is discussed in section 7.6. Here, it is difficult to identify a relevant small parameter to produce rigorous estimates. In spite of numerous numerical examples of localized eigenfunctions (for both Dirichlet and Neumann boundary conditions), a comprehensive theory of localization is still lacking. Section 7.7 is devoted to high-frequency localization and the related scarring problems in quantum billiards. We start by illustrating the classical whispering gallery modes, bouncing ball modes, and focusing modes in circular and elliptical domains. We also provide examples for the case without localization. A brief overview of quantum billiards is presented. In the final section 8, we mention some issues that could not be included in the review, e.g., numerical methods for the computation of eigenfunctions and their numerous applications.

2. Basic Properties.

We start by recalling basic properties of the Laplacian eigenvalues and eigenfunctions (see [71, 138, 422] or other standard textbooks).

(i) The eigenfunctions are infinitely differentiable inside the domain Ω. For any open subset $V \subset \Omega$, the restriction of u_m on V cannot be strictly 0 [300].

(ii) Multiplying (1.1) by u_m, integrating over Ω, and using Green's formula yields

$$\lambda_m = \frac{\int_\Omega dx\ |\nabla u_m|^2 - \int_{\partial\Omega} dx\ u_m \frac{\partial u_m}{\partial n}}{\int_\Omega dx\ u_m^2} = \frac{\|\nabla u_m\|^2_{L_2(\Omega)} + h\|u_m\|^2_{L_2(\partial\Omega)}}{\|u_m\|^2_{L_2(\Omega)}}, \tag{2.1}$$

where ∇ stands for the gradient operator, and we used Robin boundary condition (1.2) in the last equality; for Dirichlet or Neumann boundary conditions, the boundary integral (second term) vanishes. This formula ensures that all eigenvalues are nonnegative.

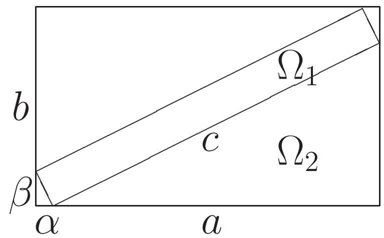


Fig. 2.1 *A counterexample for the property of domain monotonicity for the Neumann boundary condition. Although a smaller rectangle Ω_1 is inscribed inside a larger rectangle Ω_2 (i.e., $\Omega_1 \subset \Omega_2$), the second eigenvalue $\lambda_2(\Omega_1) = \pi^2/c^2$ is smaller than the second eigenvalue $\lambda_2(\Omega_2) = \pi^2/a^2$ (if $a > b$) when $c = \sqrt{(a-\alpha)^2 + (b-\beta)^2} > a$ (courtesy of N. Saito; see also* [238, sect. 1.3.2]*).*

(iii) A similar expression appears in the variational formulation of the eigenvalue problem, known as the minimax principle [138],

$$\lambda_m = \min\max \frac{\|\nabla v\|^2_{L_2(\Omega)} + h\|v\|^2_{L_2(\partial\Omega)}}{\|v\|^2_{L_2(\Omega)}}, \tag{2.2}$$

where the maximum is over all linear combinations of the form

$$v = a_1\phi_1 + \cdots + a_m\phi_m,$$

and the minimum is over all choices of m linearly independent continuous and piecewise-differentiable functions $\phi_1, \ldots, \phi_m$ (said to be in the Sobolev space $H^1(\Omega)$) [138, 238]. Note that the minimum is reached exactly on the eigenfunction u_m. For the Dirichlet eigenvalue problem, there is a supplementary condition $v = 0$ on the boundary $\partial\Omega$ so that the second term in (2.2) is canceled. For the Neumann eigenvalue problem, $h = 0$ and the second term vanishes again.

(iv) The minimax principle implies the monotonous increase of the eigenvalues λ_m with h, namely, if $h < h'$, then $\lambda_m(h) \leq \lambda_m(h')$. In particular, any eigenvalue $\lambda_m(h)$ of the Robin problem lies between the corresponding Neumann and Dirichlet eigenvalues.

(v) For the Dirichlet boundary condition, the minimax principle implies the property of domain monotonicity: eigenvalues monotonously decrease when the domain is enlarged, i.e., $\lambda_m(\Omega_1) \geq \lambda_m(\Omega_2)$ if $\Omega_1 \subset \Omega_2$. This property does not hold for Neumann or Robin boundary conditions, as illustrated by a simple counterexample in Figure 2.1.

(vi) The eigenvalues are invariant under translations and rotations of the domain. This is a key property for efficient image recognition and analysis [425, 437, 438]. When a domain is expanded by factor α, all the eigenvalues are rescaled by $1/\alpha^2$.

(vii) The first eigenfunction u_1 does not change the sign and can be chosen positive. Because of the orthogonality of eigenfunctions, u_1 is in fact the only eigenfunction not changing its sign.

(viii) The first eigenvalue λ_1 is simple and strictly positive for Dirichlet and Robin boundary conditions; for the Neumann boundary condition, $\lambda_1 = 0$ and u_1 is a constant.

(ix) The completeness of eigenfunctions in $L_2(\Omega)$ can be expressed as

$$\sum_m u_m(x)u_m^*(y) = \delta(x-y) \qquad (x,y\in\Omega), \tag{2.3}$$

where the asterisk denotes the complex conjugate, $\delta(x)$ is the Dirac distribution, and the eigenfunctions are L_2-normalized. Multiplying this relation by a function $f \in L_2(\Omega)$ and integrating over Ω yields the following decomposition of $f(x)$ over $u_m(x)$:

$$f(x) = \sum_m u_m(x) \int_\Omega dy\ f(y)\ u_m^*(y).$$

(x) The Green function $G(x,y)$ for the Laplace operator, which satisfies

$$-\Delta G(x,y) = \delta(x-y) \qquad (x,y\in\Omega) \tag{2.4}$$

(with an appropriate boundary condition), admits the following spectral decomposition over the L_2-normalized eigenfunctions:

$$G(x,y) = \sum_m \lambda_m^{-1} u_m(x)u_m^*(y). \tag{2.5}$$

(For the Neumann boundary condition, $\lambda_1 = 0$ has to be excluded; in that case, the Green function is defined up to an additive constant.)

Similarly, the heat kernel (or diffusion propagator) $G_t(x,y)$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t}G_t(x,y) - \Delta G_t(x,y) = 0 \qquad (x,y\in\Omega), \\ G_{t=0}(x,y) = \delta(x-y) \end{aligned} \tag{2.6}$$

(with an appropriate boundary condition) admits the spectral decomposition

$$G_t(x,y) = \sum_m e^{-\lambda_m t} u_m(x)u_m^*(y). \tag{2.7}$$

The Green function and heat kernel allow one to solve the standard boundary value and Cauchy problems for the Laplace and heat equations, respectively [118, 139]. The decompositions (2.5) and (2.7) are the major tools for finding explicit solutions in simple domains for which the eigenfunctions are known explicitly (see section 3). This representation is also important for the theory of diffusion due to the probabilistic interpretation of $G_t(x,y)dx$ as the conditional probability that Brownian motion started at y arrives in the dx vicinity of x after a time t [51, 52, 84, 177, 201, 249, 407, 421, 502]. Setting Dirichlet, Neumann, or Robin boundary conditions, one can, respectively, describe perfect absorptions, perfect reflections, and partial absorption/reflection on the boundary [212].

For Dirichlet boundary conditions, if $\Omega \subset \Omega'$, then $0 \le G_t^{(\Omega)}(x,y) \le G_t^{(\Omega')}(x,y)$ [494]. In particular, taking $\Omega' = \mathbb{R}^d$, one gets

$$0 \le G_t(x,y) \le (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right), \tag{2.8}$$

where the Gaussian heat kernel for free space is written on the right-hand side. The above domain monotonicity for heat kernels may not hold for Neumann boundary conditions [53].

(xi) For Dirichlet boundary conditions, the eigenvalues vary continuously under a "continuous" perturbation of the domain [138]. For Neumann boundary conditions, the situation is much more delicate. Continuity still holds when a bounded domain with a smooth boundary is deformed by a "continuously differentiable transformation," while in general this statement is false, with an explicit counterexample provided in [138]. Note that the continuity of the spectrum is important for numerical computations of the eigenvalues by finite element or other methods in which an irregular boundary is replaced by a suitable polygonal or piecewise smooth approximation. The underlying assumption that the eigenvalues are minimally affected by such domain perturbations holds in general for Dirichlet boundary conditions, but is much less evident for Neumann boundary conditions [105]. The spectral stability of elliptic operators under domain perturbations has been thoroughly investigated [105, 106, 107, 108, 109, 226, 240]. It is also worth stressing that the spectrum of the Laplace operator in a bounded domain with Neumann boundary conditions on an irregular boundary may not be discrete, with explicit counterexamples provided in [237].

3. Eigenbasis for Simple Domains.

We list here examples of "simple" domains, in which symmetries allow for variable separation and thus explicit representations of eigenfunctions in terms of elementary or special functions.

3.1. Intervals, Rectangles, and Parallelepipeds.

For rectangle-like domains $\Omega = [0,\ell_1]\times\cdots\times[0,\ell_d]\subset\mathbb{R}^d$ (with $\ell_i>0$), the natural variable separation yields

$$u_{n_1,\dots,n_d}(x_1,\dots,x_d) = u^{(1)}_{n_1}(x_1)\dots u^{(d)}_{n_d}(x_d), \qquad \lambda_{n_1,\dots,n_d} = \lambda^{(1)}_{n_1}+\cdots+\lambda^{(d)}_{n_d}, \tag{3.1}$$

where the multiple index $n_1,\dots,n_d$ is used instead of m, and $u^{(i)}_{n_i}(x_i)$ and $\lambda^{(i)}_{n_i}$ ($i = 1,\dots,d$) correspond to the one-dimensional problem on the interval $[0,\ell_i]$. Depending on the boundary condition, $u^{(i)}_n(x)$ are sines (Dirichlet), cosines (Neumann), or their combinations (Robin):

$$\begin{aligned}
&u^{(i)}_n(x) = \sin(\pi(n+1)x/\ell_i), \quad \lambda^{(i)}_n = \pi^2(n+1)^2/\ell_i^2 \quad &&\text{(Dirichlet)},\\
&u^{(i)}_n(x) = \cos(\pi n x/\ell_i), \quad \lambda^{(i)}_n = \pi^2 n^2/\ell_i^2 &&\text{(Neumann)},\\
&u^{(i)}_n(x) = \sin(\alpha_n x/\ell_i) + \frac{\alpha_n}{h\ell_i}\cos(\alpha_n x/\ell_i), \quad \lambda^{(i)}_n = \alpha_n^2/\ell_i^2 &&\text{(Robin)},
\end{aligned} \tag{3.2}$$

where $n = 0,1,2,\dots$ and the coefficients α_n depend on the parameter h and satisfy the equation obtained by imposing the Robin boundary condition in (1.2) at $x=\ell_i$:

$$\frac{2\alpha_n}{h\ell_i}\cos\alpha_n + \left(1-\frac{\alpha_n^2}{h^2\ell_i^2}\right)\sin\alpha_n = 0. \tag{3.3}$$

According to property (iv) of section 2, this equation has the unique solution α_n on each interval $[n\pi,(n+1)\pi]$ ($n=0,1,2,\dots$), which makes its numerical computation by the bisection (or any other) method easy and fast. All the eigenvalues $\lambda^{(i)}_n$ are simple (not degenerate), while

$$\|u^{(i)}_n(x)\|_{L_2((0,\ell_i))} = \left(\frac{\alpha_n^2 + 2h\ell_i + h^2\ell_i^2}{2h^2}\right)^{1/2}. \tag{3.4}$$

In turn, the eigenvalues $\lambda_{n_1,\dots,n_d}$ can be degenerate if there exists a rational ratio $(\ell_i/\ell_j)^2$ (with $i \neq j$). For instance, the first Dirichlet eigenvalues of the unit square are $2\pi^2$, $5\pi^2$, $5\pi^2$, $8\pi^2, \dots,$ with a twice degenerate second eigenvalue. An eigenfunction associated to a degenerate eigenvalue is a linear combination of the corresponding functions. For the above example, $u(x_1, x_2) = c_1 \sin(\pi x_1)\sin(2\pi x_2) + c_2 \sin(2\pi x_1)\sin(\pi x_2)$ with any c_1 and c_2 such that $c_1^2 + c_2^2 \neq 0$.

3.2. Disk, Sector, and Circular Annulus. The rotation symmetry of a circular annulus, $\Omega = \{x \in \mathbb{R}^2 \ : \ R_0 < |x| < R\}$, allows one to write the Laplace operator in polar coordinates,

$$\begin{cases} x_1 = r\cos\varphi, \\ x_2 = r\sin\varphi, \end{cases} \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}, \tag{3.5}$$

which leads to variable separation and the explicit representation of eigenfunctions

$$u_{nkl}(r,\varphi) = \big[J_n(\alpha_{nk} r/R) + c_{nk} Y_n(\alpha_{nk} r/R)\big] \times \begin{cases} \cos(n\varphi), & l = 1, \\ \sin(n\varphi), & l = 2 \ (n \neq 0), \end{cases} \tag{3.6}$$

where $J_n(z)$ and $Y_n(z)$ are the Bessel functions of the first and second kind [1, 86, 500] and the coefficients α_{nk} and c_{nk} are set by the boundary conditions at $r = R$ and $r = R_0$:

$$\begin{aligned} 0 &= \frac{\alpha_{nk}}{R}\left[J_n'(\alpha_{nk}) + c_{nk} Y_n'(\alpha_{nk})\right] + h\left[J_n(\alpha_{nk}) + c_{nk} Y_n(\alpha_{nk})\right], \\ 0 &= -\frac{\alpha_{nk}}{R}\left[J_n'\left(\alpha_{nk}\frac{R_0}{R}\right) + c_{nk} Y_n'\left(\alpha_{nk}\frac{R_0}{R}\right)\right] + h\left[J_n\left(\alpha_{nk}\frac{R_0}{R}\right) + c_{nk} Y_n\left(\alpha_{nk}\frac{R_0}{R}\right)\right]. \end{aligned} \tag{3.7}$$

For each $n = 0, 1, 2, \dots,$ this system of equations has infinitely many solutions α_{nk} which are enumerated by the index $k = 1, 2, 3, \dots$ [500]. The eigenfunctions are enumerated by the triple index nkl, with $n = 0, 1, 2, \dots$ counting the order of Bessel functions, $k = 1, 2, 3, \dots$ counting solutions of (3.7), and $l = 1, 2$. Since $u_{0k2}(r,\varphi)$ are trivially zero (as $\sin(n\varphi) = 0$ for $n = 0$), they are excluded. The eigenvalues $\lambda_{nk} = \alpha_{nk}^2/R^2$, which are independent of the final index l, are simple for $n = 0$ and twice degenerate for $n > 0$. In the latter case, an eigenfunction is any nontrivial linear combination of u_{nk1} and u_{nk2}. The squared L_2-norm of the eigenfunction is

$$\begin{aligned} \|u_{nkl}(r,\varphi)\|_2^2 = \frac{\pi(2 - \delta_{n,0})R^2}{2\alpha_{nk}^2} \Bigg[& \left(\alpha_{nk}^2 + h^2R^2 - n^2\right) v_{nk}^2(R) \\ & - \left((\alpha_{nk}^2 + h^2R^2)\frac{R_0^2}{R^2} - n^2\right) v_{nk}^2(R_0)\Bigg], \end{aligned} \tag{3.8}$$

where $v_{nk}(r) = J_n(\alpha_{nk} r/R) + c_{nk} Y_n(\alpha_{nk} r/R)$.

For the special case of a disk ($R_0 = 0$), all the coefficients c_{nk} in front of the Bessel functions $Y_n(z)$ (divergent at 0) are set to 0:

$$u_{nkl}(r,\varphi) = J_n(\alpha_{nk} r/R) \times \begin{cases} \cos(n\varphi), & l = 1, \\ \sin(n\varphi), & l = 2 \ (n \neq 0), \end{cases} \tag{3.9}$$

where α_{nk} are either the positive roots j_{nk} of the Bessel function $J_n(z)$ (Dirichlet), or the positive roots $\tilde{j}_{nk}$ of its derivative $J_n'(z)$ (Neumann), or the positive roots of their linear combination $J_n'(z)+hJ_n(z)$ (Robin). The asymptotic behavior of zeros of Bessel functions has been thoroughly investigated. For fixed k and large n, Olver's expansion holds, $j_{nk} \simeq n+\delta_k n^{1/3}+O(n^{-1/3})$ (with known coefficients δ_k) [166, 378, 379], while for fixed n and large k, McMahon's expansion holds, $j_{nk} \simeq \pi(k+n/2-1/4)+O(k^{-1})$ [500]. Similar asymptotic relations are applicable for Neumann and Robin boundary conditions.

For a circular sector of radius R and angle $\pi\beta$, the eigenfunctions are

$$u_{nk}(r,\varphi) = J_{n/\beta}(\alpha_{nk}r/R) \times \begin{cases} \sin(n\varphi/\beta) & \text{(Dirichlet)} \\ \cos(n\varphi/\beta) & \text{(Neumann)} \end{cases} \qquad (r<R,\ 0<\varphi<\pi\beta), \tag{3.10}$$

i.e., they are expressed in terms of Bessel functions of fractional order, and α_{nk} are the positive roots of $J_{n/\beta}(z)$ (Dirichlet) or $J_{n/\beta}'(z)$ (Neumann). The Robin boundary condition and a sector of a circular annulus can be treated similarly.

3.3. Sphere and Spherical Shell. The rotation symmetry of a spherical shell in three dimensions, $\Omega = \{x\in\mathbb{R}^3 \ :\ R_0 < |x| < R\}$, allows one to write the Laplace operator in spherical coordinates,

$$\begin{cases} x_1 = r\sin\theta\cos\varphi, \\ x_2 = r\sin\theta\sin\varphi, \\ x_3 = r\cos\theta, \end{cases} \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\varphi^2}\right), \tag{3.11}$$

which leads to variable separation and the explicit representation of eigenfunctions

$$u_{nkl}(r,\theta,\varphi) = \bigl[j_n(\alpha_{nk}r/R) + c_{nk}y_n(\alpha_{nk}r/R)\bigr]P_n^l(\cos\theta)e^{il\varphi}, \tag{3.12}$$

where $j_n(z)$ and $y_n(z)$ are the spherical Bessel functions of the first and second kind,

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z), \qquad y_n(z) = \sqrt{\frac{\pi}{2z}}Y_{n+1/2}(z), \tag{3.13}$$

and $P_n^l(z)$ are associated Legendre polynomials (note that the angular part, $P_n^l(\cos\theta)e^{il\varphi}$, is also called the spherical harmonic and denoted as $Y_{nl}(\theta,\varphi)$, up to a normalization factor). The coefficients α_{nk} and c_{nk} are set by the boundary conditions at $r=R$ and $r=R_0$, similar to (3.7). The eigenfunctions are enumerated by the triple index nkl, with $n=0,1,2,\dots$ counting the order of spherical Bessel functions, $k=1,2,3,\dots$ counting zeros, and $l=-n,-n+1,\dots,n$. The eigenvalues $\lambda_{nk}=\alpha_{nk}^2/R^2$, which are independent of the last index l, have the degeneracy $2n+1$. The squared L_2-norm of the eigenfunction is

$$\begin{aligned} \|u_{nkl}(r,\theta,\varphi)\|_2^2 = \frac{2\pi R^3}{(2n+1)\alpha_{nk}^2}\biggl[&\Bigl(\alpha_{nk}^2 + h^2R^2 - hR - n(n+1)\Bigr)v_{nk}^2(R) \\ &- \Bigl(\alpha_{nk}^2(R_0/R)^3 + h^2R_0^2 - hR_0 - n(n+1)R_0/R\Bigr)v_{nk}^2(R_0)\biggr], \end{aligned} \tag{3.14}$$

where $v_{nk}(r) = j_n(\alpha_{nk}r/R) + c_{nk}y_n(\alpha_{nk}r/R)$.

In the special case of a sphere ($R_0=0$), one has $c_{nk}=0$ and the equations are simplified. For balls and spherical shells in higher dimensions ($d>3$), the radial dependence of eigenfunctions is expressed through a linear combination of so-called ultraspherical Bessel functions $r^{1-d/2}J_{\frac{d}{2}-1+n}(\alpha_{nk}r/R)$ and $r^{1-d/2}Y_{\frac{d}{2}-1+n}(\alpha_{nk}r/R)$.

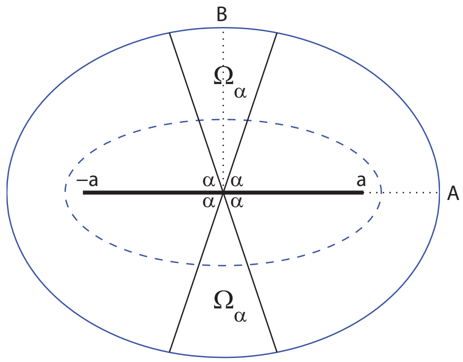


Fig. 3.1 *Two ellipses of radii $R = 0.5$ (dashed line) and $R = 1$ (solid line), with the focal distance $a = 1$. The major and minor semiaxes, $A = a\cosh R$ and $B = a\sinh R$, are shown by black dotted lines. The horizontal thick segment connects the foci.*

3.4. Ellipse and Elliptical Annulus.

In elliptic coordinates, the Laplace operator is

$$\begin{cases} x_1 = a\cosh r\cos\theta, \\ x_2 = a\sinh r\sin\theta, \end{cases} \qquad \Delta = \frac{1}{a^2(\sinh^2 r + \sin^2\theta)}\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial\theta^2}\right), \tag{3.15}$$

where $a > 0$ is the prescribed distance between the origin and the foci, and $r \geq 0$ and $0 \leq \theta < 2\pi$ are the radial and angular coordinates (see Figure 3.1). An ellipse is a curve of constant $r = R$ such that its points (x_1, x_2) satisfy $x_1^2/A^2 + x_2^2/B^2 = 1$, where R is the radius of the ellipse and $A = a\cosh R$ and $B = a\sinh R$ are the major and minor semiaxes. Note that the eccentricity $e = a/A = 1/\cosh R$ is strictly positive. A filled ellipse (i.e., the interior of an given ellipse) can be characterized in elliptic coordinates as $0 \leq r < R$ and $0 \leq \theta < 2\pi$. Similarly, an elliptical annulus (i.e., the interior between two ellipses with the same foci) is characterized by $R_0 < r < R$ and $0 \leq \theta < 2\pi$.

In elliptic coordinates, the variables can be separated, $u(r,\theta) = f(r)g(\theta)$, and then (1.1) becomes

$$\left(\frac{1}{f(r)}\frac{d^2 f}{dr^2} + \frac{\lambda a^2}{2}\cosh(2r)\right) = -\left(\frac{1}{g(\theta)}\frac{d^2 g}{d\theta^2} - \frac{\lambda a^2}{2}\cos(2\theta)\right),$$

so that both sides are equal to a constant (denoted c). As a consequence, the angular and radial parts, $g(\theta)$ and $f(r)$, are solutions of the Mathieu equation and the modified Mathieu equation, respectively [126, 353, 515],

$$g''(\theta) + (c - 2q\cos 2\theta)\, g(\theta) = 0, \qquad f''(r) - (c - 2q\cosh 2r)\, f(r) = 0,$$

where $q = \lambda a^2/4$ and the parameter c is called the characteristic value of Mathieu functions. Periodic solutions of the Mathieu equation are possible for specific values of c. They are denoted as $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ (with $n = 0, 1, 2, \dots$) and called the angular Mathieu functions of the first and second kind. Each function $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ corresponds to its own characteristic value c (the relation being implicit; see [353]).

For the radial part, there are two linearly independent solutions for each characteristic value c: two modified Mathieu functions $\mathrm{Mc}_n^{(1)}(r,q)$ and $\mathrm{Mc}_n^{(2)}(r,q)$ correspond to the same c as $\mathrm{ce}_n(\theta,q)$, and two modified Mathieu functions $\mathrm{Ms}_{n+1}^{(1)}(r,q)$ and $\mathrm{Ms}_{n+1}^{(2)}(r,q)$ correspond to the same c as $\mathrm{se}_{n+1}(\theta,q)$. As a consequence, there are four families of eigenfunctions (distinguished by the index $l=1,2,3,4$) in an elliptical domain,

$$\begin{aligned}
u_{nk1}(r,\theta) &= \mathrm{ce}_n(\theta,q_{nk1})\mathrm{Mc}_n^{(1)}(r,q_{nk1}),\\
u_{nk2}(r,\theta) &= \mathrm{ce}_n(\theta,q_{nk2})\mathrm{Mc}_n^{(2)}(r,q_{nk2}),\\
u_{nk3}(r,\theta) &= \mathrm{se}_{n+1}(\theta,q_{nk3})\mathrm{Ms}_{n+1}^{(1)}(r,q_{nk3}),\\
u_{nk4}(r,\theta) &= \mathrm{se}_{n+1}(\theta,q_{nk4})\mathrm{Ms}_{n+1}^{(2)}(r,q_{nk4}),
\end{aligned}$$

where the parameters q_{nkl} are determined by the boundary condition. For instance, for a filled ellipse of radius R with Dirichlet boundary conditions, there are four individual equations for the parameter q for each $n=0,1,2,\dots,$

$$\mathrm{Mc}_n^{(1)}(R,q_{nk1})=0,\quad \mathrm{Mc}_n^{(2)}(R,q_{nk2})=0,\quad \mathrm{Ms}_{n+1}^{(1)}(R,q_{nk3})=0,\quad \mathrm{Ms}_{n+1}^{(2)}(R,q_{nk4})=0,$$

each of them having infinitely many positive solutions q_{nkl} enumerated by $k=1,2,\dots$ [1, 353]. Finally, the associated eigenvalues are $\lambda_{nkl}=4q_{nkl}/a^2$.

3.5. Equilateral Triangle. Lamé discovered the Dirichlet eigenvalues and eigenfunctions of the equilateral triangle $\Omega=\{(x_1,x_2)\in\mathbb{R}^2\ : 0<x_1<1,\ 0<x_2<x_1\sqrt{3},\ x_2<\sqrt{3}(1-x_1)\}$ by using reflections and the related symmetries [301]:

$$\lambda_{mn}=\frac{16\pi^2}{27}\big(m^2+n^2-mn\big)\qquad (m,n\in\mathbb{Z}), \tag{3.16}$$

where 3 divides $m+n$, $m\neq 2n$, and $n\neq 2m$, and the associate eigenfunction is

$$u_{mn}(x_1,x_2)=\sum_{(m',n')}\pm\exp\left[\frac{2\pi i}{3}\left(m'x_1+(2n'-m')\frac{x_2}{\sqrt{3}}\right)\right], \tag{3.17}$$

where (m',n') runs over $(-n,m-n)$, $(-n,-m)$, $(n-m,-m)$, $(n-m,n)$, (m,n), and $(m,m-n)$ with the $\pm$ sign alternating (see also [339] for a basic introduction, as well as [316, 334]). Pinsky showed that this set of eigenfunctions is complete in $L_2(\Omega)$ [396, 397]. Note that the conditions $m\neq 2n$ and $n\neq 2m$ should be satisfied for all six pairs in the sum, which yields one additional condition: $m\neq -n$. The following relations hold: $u_{-m,-n}=u^*_{m,n}$, $u_{n,m}=-u^*_{m,n}$, and $u_{m,0}=u_{m,m}$. All symmetric eigenfunctions are enumerated by the index $(m,0)$. The eigenvalue λ_{mn} corresponds to a symmetric eigenfunction if and only if m is a multiple of 3 [396].

The eigenfunctions for the Neumann boundary condition are

$$u_{mn}(x_1,x_2)=\sum_{(m',n')}\exp\left[\frac{2\pi i}{3}\left(m'x_1+(2n'-m')\frac{x_2}{\sqrt{3}}\right)\right], \tag{3.18}$$

where the only condition is that $m+n$ are multiples of 3 (and no sign change). Further references and extensions (e.g., to Robin boundary conditions) can be found in a series of works by McCartin [343, 344, 345, 346, 348]. McCartin also developed a classification of all polygonal domains possessing a complete set of trigonometric eigenfunctions of the Laplace operator under either Dirichlet or Neumann boundary conditions [347].

4. Eigenvalues.

4.1. Weyl's Law. Weyl's law provided one of the first connections between the spectral properties of the Laplace operator and the geometrical structure of a bounded domain Ω. In 1911, Hermann Weyl derived the following asymptotic behavior of the Laplacian eigenvalues [503, 504]:

$$\lambda_m \propto \frac{4\pi^2}{(\omega_d \mu_d(\Omega))^{2/d}}\, m^{2/d} \qquad (m \to \infty), \tag{4.1}$$

where $\mu_d(\Omega)$ is the Lebesgue measure of Ω (its area in two dimensions and volume in three dimensions) and

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \tag{4.2}$$

is the volume of the unit ball in d dimensions ($\Gamma(z)$ being the Gamma function). As a consequence, plotting eigenvalues versus $m^{2/d}$ allows one to extract the area in two dimensions or the volume in three dimensions. This result can be written equivalently for the counting function $N(\lambda) = \#\{m \ : \ \lambda_m < \lambda\}$ (i.e., the number of eigenvalues smaller than λ):

$$N(\lambda) \propto \frac{\omega_d \mu_d(\Omega)}{(2\pi)^d}\, \lambda^{d/2} \qquad (\lambda \to \infty). \tag{4.3}$$

Weyl also conjectured the second asymptotic term, which in two and three dimensions is

$$N(\lambda) \propto \begin{cases} \dfrac{\mu_2(\Omega)}{4\pi}\, \lambda \mp \dfrac{\mu_1(\partial\Omega)}{4\pi}\sqrt{\lambda} & (d=2) \\ \dfrac{\mu_3(\Omega)}{6\pi^2}\, \lambda^{3/2} \mp \dfrac{\mu_2(\partial\Omega)}{16\pi}\lambda & (d=3) \end{cases} \qquad (\lambda \to \infty), \tag{4.4}$$

where $\mu_2(\Omega)$ and $\mu_1(\partial\Omega)$ are the area and perimeter of Ω in two dimensions, $\mu_3(\Omega)$ and $\mu_2(\partial\Omega)$ are the volume and surface area of Ω in three dimensions, and the sign "–" (resp., "+") refers to the Dirichlet (resp., Neumann) boundary condition. The correction terms, which yield information about the boundary of the domain, were justified under certain conditions on Ω (e.g., convexity) only in 1980 [252, 355] (see [11] for a historical review and further details).

Alternatively, one can study the heat trace (or partition function)

$$Z(t) \equiv \int_\Omega dx\; G_t(x,x) = \sum_{m=1}^{\infty} e^{-\lambda_m t} = \int_0^\infty e^{-\lambda t} dN(\lambda) \tag{4.5}$$

(here $G_t(x,y)$ is the heat kernel; cf. (2.6)), for which the following asymptotic expansion holds [87, 146, 149, 203, 352, 357, 413]:

$$Z(t) = (4\pi t)^{-d/2} \left(\sum_{k=0}^{K} c_k t^{k/2} + o(t^{(K+1)/2}) \right) \qquad (t \to 0), \tag{4.6}$$

where the coefficients c_k are again related to the geometrical characteristics of the domain:

$$c_0 = \mu_d(\Omega), \quad c_1 = -\frac{\sqrt{\pi}}{2}\mu_{d-1}(\partial\Omega), \quad \ldots \tag{4.7}$$

(see [425] for further discussion). Some estimates for the trace of the Dirichlet Laplacian were given by Davies [147] (see also [495] for the asymptotic behavior of the heat content).

A number of extensions have been proposed. Berry conjectured that for irregular boundaries, for which the Lebesgue measure in the correction term is infinite, the correction term should be $\lambda^{H/2}$ instead of $\lambda^{(d-1)/2}$, where H is the Hausdorff dimension of the boundary [62, 63]. However, Brossard and Carmona constructed a counterexample to this conjecture and suggested a modified version, in which the Hausdorff dimension was replaced by Minkowski dimension [88]. The modified Weyl–Berry conjecture was discussed at length by Lapidus, who proved it for $d = 1$ [303, 304] (see those references for further discussion). For dimension d higher than 1, this conjecture was disproved by Lapidus and Pomerance [307]. The correction term to Weyl's formula for domains with a rough boundary (in particular, for the Lipschitz class) was studied by Netrusov and Safarov [369]. Levitin and Vassiliev also considered the asymptotic formulas for iterated sets with fractal boundary [320]. Extensions to various manifolds and higher-order Laplacians were also discussed [154, 155].

The high-frequency Weyl's law and the related short-time asymptotics of the heat kernel have been thoroughly investigated [11]. The dependence of these asymptotic laws on the volume and surface of the domain has found applications in physics. For instance, diffusion-weighted nuclear magnetic resonance experiments were proposed and conducted to estimate the surface-to-volume ratio of mineral samples and biological tissues [213, 236, 250, 308, 309, 359, 360, 455].

The multiplicity of eigenvalues is a yet more difficult problem [364]. From basic properties (see section 2), the first eigenvalue λ_1 is simple. Cheng proved that the multiplicity $m(\lambda_2)$ of the second Dirichlet eigenvalue λ_2 is no greater than 3 [128]. This inequality is sharp since an example of a domain with $m(\lambda_2) = 3$ has been constructed. For $k \geq 3$, Hoffmann-Ostenhof et al. proved the inequality $m(\lambda_k) \leq 2k - 3$ [246, 247].

4.2. Isoperimetric Inequalities for Eigenvalues. In the low-frequency limit, the relationship between the shape of a domain and the associated eigenvalues manifests in the form of isoperimetric inequalities. Since there are many excellent reviews on this topic, we just provide a list of the best known inequalities, while further discussion and references can be found in [22, 28, 36, 58, 227, 238, 243, 300, 389, 406, 425].

(i) *The Rayleigh–Faber–Krahn inequality* states that the disk minimizes the first Dirichlet eigenvalue λ_1 among all planar domains of the same area $\mu_2(\Omega)$, i.e.,

$$\lambda_1^D \geq \frac{\pi}{\mu_2(\Omega)} (j_{0,1})^2, \tag{4.8}$$

where $j_{\nu,1}$ is the first positive zero of $J_\nu(z)$ (e.g., $j_{0,1} \approx 2.4048\ldots$). This inequality was conjectured by Lord Rayleigh and proven independently by Faber and Krahn [174, 289]. The corresponding isoperimetric inequality in d dimensions,

$$\lambda_1^D \geq \left(\frac{\omega_d}{\mu_d(\Omega)}\right)^{2/d} (j_{\frac{d}{2}-1,1})^2, \tag{4.9}$$

was proven by Krahn [290].

Another lower bound for the first Dirichlet eigenvalue for a simply connected planar domain was obtained by Makai [333] and later rediscovered (in a weaker form) by Hayman [231],

$$\lambda_1^D \geq \frac{\alpha}{\rho^2}, \tag{4.10}$$

where α is a constant, and

$$\rho = \max_{x\in\Omega} \min_{y\in\partial\Omega} \{|x-y|\} \tag{4.11}$$

is the inradius of Ω (i.e., the radius of the largest ball inscribed in Ω). The above inequality means that the lowest frequency (bass note) can be made arbitrarily small only if the domain includes an arbitrarily large circular drum (i.e., ρ goes to infinity). The constant α, which was equal to 1/4 in Makai's original proof (see also [381]) and to 1/900 in Hayman's proof, was gradually increased to the best value (currently) $\alpha = 0.6197\dots$ by Banuelos and Carroll [41]. For convex domains, the lower bound (4.10) with $\alpha = \pi^2/4 \approx 2.4674$ was derived much earlier by Hersch [241], with the equality holding if and only if Ω is an infinite strip (see also a historical overview in [28]).

An obvious upper bound for the first Dirichlet eigenvalue can be obtained from domain monotonicity (property (v) in section 2):

$$\lambda_1^D \leq \lambda_1^D(B_\rho) = \rho^{-2}\, j^2_{\frac{d}{2}-1,1}, \tag{4.12}$$

with the first Dirichlet eigenvalue $\lambda_1^D(B_\rho)$ for the largest ball B_ρ inscribed in Ω (ρ is the inradius). However, this upper bound is not accurate in general. Pólya and Szegő gave another upper bound for planar star-shaped domains [406]. Freitas and Krejčiřík extended their result to higher dimensions [192]: for a bounded strictly star-shaped domain $\Omega \subset \mathbb{R}^d$ with locally Lipschitz boundary, they proved

$$\lambda_1^D \leq \lambda_1^D(B_1) \frac{F(\Omega)}{d\ \mu_d(\Omega)}, \tag{4.13}$$

where the function $F(\Omega)$ is defined in [192]. From this inequality, they also deduced a weaker but more explicit upper bound which is applicable to any bounded convex domain in $\mathbb{R}^d$:

$$\lambda_1^D \leq \lambda_1^D(B_1)\ \frac{\mu_{d-1}(\partial\Omega)}{d\ \rho\ \mu_d(\Omega)}. \tag{4.14}$$

The second Dirichlet eigenvalue λ_2^D is minimized by the union of two identical balls,

$$\lambda_2^D \geq 2^{2/d} \left(\frac{\omega_d}{\mu_d(\Omega)}\right)^{2/d} (j_{\frac{d}{2}-1,1})^2. \tag{4.15}$$

This inequality, which can be deduced by looking at nodal domains for u_2 and using the Rayleigh–Faber–Krahn inequality (4.9) on each nodal domain, was first established by Krahn [290]. It is also sometimes attributed to Peter Szegő (see [404]). Note that finding the minimizer of λ_2^D among convex planar sets is still an open problem [239]. Bucur and Henrot proved the existence of a minimizer for the third eigenvalue in the family of domains in $\mathbb{R}^d$ of given volume, although its shape remains unknown [95]. The range of the first two eigenvalues was also investigated in [92, 507].

The first nontrivial Neumann eigenvalue λ_2^N (given that $\lambda_1^N = 0$) also satisfies the isoperimetric inequality

$$\lambda_2^N \leq \left(\frac{\omega_d}{\mu_d(\Omega)}\right)^{2/d} (\tilde{j}_{\frac{d}{2},1})^2, \tag{4.16}$$

which states that λ_2^N is maximized by a d-dimensional ball (here $\tilde{j}_{\nu,1}$ is the first positive zero of the function $\frac{d}{dz}[z^{1-d/2}J_{\frac{d}{2}-1+\nu}(z)]$, which reduces to $J'_\nu(z)$ and $\sqrt{2/\pi}\, j'_\nu(z)$ for $d=2$ and $d=3$, respectively). This inequality was proven for simply connected planar domains by Szegő [480] and in higher dimensions by Weinberger [501]. Pólya conjectured the following upper bound for all Neumann eigenvalues [403] in planar bounded regular domains (see also [451]):

$$\lambda_n^N \le \frac{4(n-1)\pi}{\mu_2(\Omega)} \qquad (n = 2, 3, 4, \dots) \tag{4.17}$$

(the domain is called regular if its Neumann eigenspectrum is discrete; see [204] for details). This inequality is true for all domains that tile the plane, e.g., for any triangle and any quadrilateral [405]. For $n=2$, the inequality (4.17) follows from (4.16). For $n \ge 3$, Pólya's conjecture is still open, although Kröger proved a weaker estimate $\lambda_n^N \le 8\pi(n-1)$ [292]. Recently, Girouard, Nadirashvili, and Polterovich obtained a sharp upper bound for the second nontrivial Neumann eigenvalue λ_3^N for a regular simply connected planar domain [204],

$$\lambda_3^N \le \frac{2\pi(\tilde{j}_{0,1})^2}{\mu_2(\Omega)}, \tag{4.18}$$

with the equality attained in the limit by a family of domains degenerating to a disjoint union of two identical disks.

Payne and Weinberger obtained the lower bound for the second Neumann eigenvalue in d dimensions [395]

$$\lambda_2^N \ge \frac{\pi^2}{\delta^2}, \tag{4.19}$$

where δ is the diameter of Ω:

$$\delta = \max_{x,y\in\partial\Omega}\{|x-y|\}. \tag{4.20}$$

This is the best bound that can be given in terms of the diameter alone in the sense that $\lambda_2^N\delta^2$ tends to π^2 for a parallelepiped all but one of whose dimensions shrink to zero.

Szegő and Weinberger noticed that Szegő's proof of the inequality (4.16) for planar simply connected domains extends to proving the bound

$$\frac{1}{\lambda_2^N} + \frac{1}{\lambda_3^N} \ge \frac{2\mu_2(\Omega)}{\pi(\tilde{j}_{1,1})^2}, \tag{4.21}$$

with equality if and only if Ω is a disk [480, 501]. Ashbaugh and Benguria derived another bound for an arbitrary bounded domain in $\mathbb{R}^d$ [26]:

$$\frac{1}{\lambda_2^N} + \cdots + \frac{1}{\lambda_{d+1}^N} \ge \frac{d}{d+2}\left(\frac{\mu_d(\Omega)}{\omega_d}\right)^{2/d}. \tag{4.22}$$

In particular, one gets $1/\lambda_2^N + 1/\lambda_3^N \ge \frac{\mu_2(\Omega)}{2\pi}$ for $d=2$ (see also extensions in [244, 508]).

(ii) *The Payne–Pólya–Weinberger inequality*, which is also called the *Ashbaugh–Benguria inequality*, concerns the ratio between the first two Dirichlet eigenvalues and states that

$$\frac{\lambda_2^D}{\lambda_1^D} \leq \left(\frac{j_{\frac{d}{2},1}}{j_{\frac{d}{2}-1,1}} \right)^2, \tag{4.23}$$

with equality if and only if Ω is the d-dimensional ball. This inequality (in two-dimensional form) was conjectured by Payne, Pólya, and Weinberger [391] and proved by Ashbaugh and Benguria in 1990 [24, 25, 26, 27]. A weaker estimate $\lambda_2^D/\lambda_1^D \leq 1 + 4/d$ was proved for $d = 2$ in the original paper by Payne, Pólya, and Weinberger [391].

(iii) Singer et al. derived upper and lower estimates for the spectral (or fundamental) gap between the first two Dirichlet eigenvalues for a smooth convex bounded domain Ω in $\mathbb{R}^d$ (in [466], a more general problem in the presence of a potential was considered),

$$\frac{d\pi^2}{\rho^2} \geq \lambda_2^D - \lambda_1^D \geq \frac{\pi^2}{4\delta^2}, \tag{4.24}$$

where δ is the diameter of Ω and ρ is the inradius [466]. For a convex planar domain, Donnelly proposed a sharper lower estimate [160]

$$\lambda_2^D - \lambda_1^D \geq \frac{3\pi^2}{\delta^2}. \tag{4.25}$$

However, Ashbaugh, Henrot, and Laugesen pointed out a flaw in the proof [30]. The estimate was later rigorously proved by Andrews and Clutterbuck for any bounded convex domain Ω in $\mathbb{R}^d$, even in the presence of a semiconvex potential [10] (for more background on the spectral gap, see notes by Ashbaugh [23]).

(iv) The isoperimetric inequalities for Robin eigenvalues are less known. Daners proved that among all bounded domains $\Omega \subset \mathbb{R}^d$ of the same volume, the ball B minimizes the first Robin eigenvalue [94, 144]

$$\lambda_1^R(\Omega) \geq \lambda_1^R(B). \tag{4.26}$$

Kennedy showed that among all bounded domains in $\mathbb{R}^d$, a domain B_2 composed of two disjoint balls minimizes the second Robin eigenvalue [278]

$$\lambda_2^R(\Omega) \geq \lambda_2^R(B_2). \tag{4.27}$$

(v) The minimax principle ensures that the Neumann eigenvalues are always smaller than the corresponding Dirichlet eigenvalues: $\lambda_n^N \leq \lambda_n^D$. Pólya proved $\lambda_2^N < \lambda_1^D$ [402], while Szegő found a sharper inequality $\lambda_2^N \leq c\lambda_1^D$ for a planar domain bounded by an analytic curve, where $c = (\tilde{j}_{1,1}/j_{0,1})^2 \approx 0.5862\ldots$ [480] (note that this result also follows from inequalities (4.8) and (4.16)). Payne derived a stronger inequality for a planar domain with a C^2 boundary: $\lambda_{n+2}^N < \lambda_n^D$ for all n [388]. Levine and Weinberger generalized this result for higher dimensions d and proved that $\lambda_{n+d}^N < \lambda_n^D$ for all n when Ω is smooth and convex, and that $\lambda_{n+d}^N \leq \lambda_n^D$ if Ω is merely convex [319]. Friedlander proved the inequality $\lambda_{n+1}^N \leq \lambda_n^D$ for a general bounded domain with a C^1 boundary [194]. Filonov found a simpler proof of this inequality in a more general situation (see [187] for details).

Many other inequalities can be found in several reviews [22, 28, 58]. It is worth noting that isoperimetric inequalities are related to shape optimization problems [6, 85, 93, 112, 113, 398, 469].

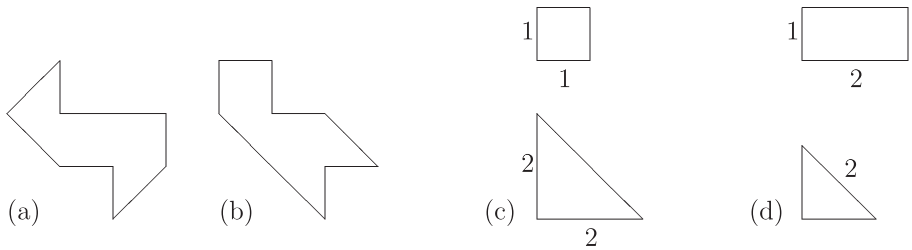


Fig. 4.1 *Two examples of nonisometric domains with identical Laplace operator eigenspectra (with Dirichlet or Neumann boundary conditions): the original example (shapes* (a) *and* (b)*) constructed by Gordon, Webb, and Wolpert* [208]*, and a simpler example with disconnected domains (shapes* (c) *and* (d)*) by Chapman* [124]*. In the latter case, the eigenspectrum is simply obtained as the union of the eigenspectra of two subdomains that are known explicitly. For instance, the Dirichlet eigenspectrum is* $\{\pi^2(m^2+n^2) \ : \ m,n \in \mathbb{N}\} \cup \{\pi^2((i/2)^2+(j/2)^2) \ : \ i,j \in \mathbb{N}, \ i>j\}$ *.*

4.3. Kac's Inverse Spectral Problem. The problem of finding relationships between the Laplacian eigenspectrum and the shape of a domain was formulated in Kac's famous question "Can one hear the shape of a drum?" [266]. In fact, a drum's frequencies are uniquely determined by the eigenvalues of the Laplace operator in the domain of its shape. By definition, the shape of the domain fully determines the Laplacian eigenspectrum. Is the opposite true, i.e., does the set of eigenvalues which appear as "fingerprints" of the shape uniquely identify the domain? The negative answer to this question for general planar domains was given by Gordon and coauthors [208], who constructed two different (nonisometric) planar polygons (see Figure 4.1a,b) with identical Laplacian eigenspectra, for both Dirichlet and Neumann boundary conditions (see also [59]). Their construction was based on Sunada's paper on isospectral manifolds [479]. An elementary proof, as well as many other examples of isospectral domains, were provided by Buser and coworkers [111] and by Chapman [124] (see Figure 4.1c,d). Experimental evidence for not "hearing the shape" of drums was produced by Sridhar and Kudrolli [472] (see also [135]). In all these examples, isospectral domains are either nonconvex or disjoint. Gordon and Webb addressed the question of existence of isospectral convex connected domains and answered this question positively (i.e., negatively to Kac's original question) for domains in Euclidean spaces of dimension $d \geq 4$ [207]. To the best of our knowledge, this question remains open for convex domains in two and three dimensions, as well as for domains with smooth boundaries. It is worth noting that a positive answer to Kac's question can be given for some classes of domains. For instance, Zelditch proved that for domains that possess the symmetry of an ellipse and satisfy some generic conditions on the boundary, the spectrum of the Dirichlet Laplacian uniquely determines the shape [511]. Later, he extended this result to real analytic planar domains with only one symmetry [512, 513].

A somewhat similar problem was recently formulated for domains in which one part of the boundary admits a Dirichlet boundary condition and the other a Neumann boundary condition. Does the spectrum of the Laplace operator uniquely determine which condition is imposed on which part? Jakobsen and coworkers gave a negative answer to this question by assigning Dirichlet and Neumann conditions to different parts of the boundary of the half-disk (and some other domains), in such a way as to produce the same eigenspectra [254].

Kac's inverse spectral problem can also be seen from a different point of view. For a given sequence $0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, does a domain Ω exist in $\mathbb{R}^d$ for which the Laplace operator with Dirichlet or Neumann boundary conditions has the spectrum given by this sequence? A similar problem can be formulated for a compact Riemannian manifold with arbitrary Riemannian metrics. Colin de Verdière studied these problems for finite sequences $\{\lambda_n\}_{n=1}^N$ and proved the existence of such domains or manifolds under certain restrictions [137]. We also mention the work of Sleeman, who discussed the relationship between inverse scattering theory (i.e., the Helmholtz equation for an exterior domain) and Kac's inverse spectral problem (i.e., for an interior domain) [467] (see [133] for further discussion on inverse eigenvalue problems).

5. Nodal Lines.

A first insight into the geometrical structure of eigenfunctions can be gained from their nodal lines. Kuttler and Sigillito gave a brief overview of the basic properties of nodal lines for Dirichlet eigenfunctions in two dimensions [300] that we partly reproduce here:

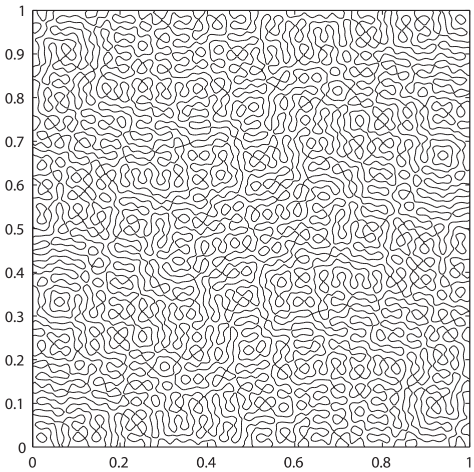
> The set of points in Ω where $u_m = 0$ is the nodal set of u_m. By the unique continuation property, it consists of curves that are C^∞ in the interior of Ω. Where nodal lines cross, they form equal angles [138]. Also, when nodal lines intersect a C^∞ portion of the boundary, they form equal angles. Thus, a single nodal line intersects the C^∞ boundary at right angles, two intersect it at 60° angles, and so forth. Courant's nodal line theorem [138] states that the nodal lines of the m-th eigenfunction divide Ω into no more than m subregions (called nodal domains): $\nu_m \le m$, ν_m being the number of nodal domains. In particular, u_1 has no interior nodes and so λ_1 is a simple eigenvalue (has multiplicity one).

It is worth noting that any eigenvalue λ_m of the Dirichlet–Laplace operator in Ω is the first eigenvalue for each of its nodal domains. This simple observation allows one to construct specific domains with a prescribed eigenvalue (see [300] for examples). Eigenfunctions with few nodal domains were constructed in [138, 321].

Even for a simple domain such as a square, the nodal lines and domains may have a complicated structure, especially for high-frequency eigenfunctions (see Figure 5.1). This is particularly true for degenerate eigenfunctions for which one can "tune" the coefficients of the corresponding linear combination to modify continuously the nodal lines.

Pleijel sharpened Courant's theorem by showing that the upper bound m for the number ν_m of nodal domains is attained only for a finite number of eigenfunctions [400]. Moreover, he obtained the upper limit $\overline{\lim}_{m\to\infty} \nu_m/m = 4/j_{0,1}^2 \approx 0.691\ldots$. Note that Lewy constructed spherical harmonics of degree n whose nodal sets have one component for odd n and two components for even n, implying that no nontrivial lower bound for ν_m is possible [321]. We also mention that the counting of nodal domains can be viewed as a partitioning of the domain into a fixed number of subdomains and minimization of an appropriate "energy" of the partition (e.g., the maximum of the ground state energies of the subdomains). When a partition corresponds to an eigenfunction, the ground state energies of all the nodal domains are the same, i.e., it is an equipartition [60].

Blum, Gnutzmann, and Smilansky considered the distribution of the (properly normalized) number of nodal domains of the Dirichlet Laplacian eigenfunctions in two-dimensional quantum billiards and showed the existence of the limiting distribution in the high-frequency limit (i.e., when $\lambda_m \to \infty$) [76]. These distributions were argued



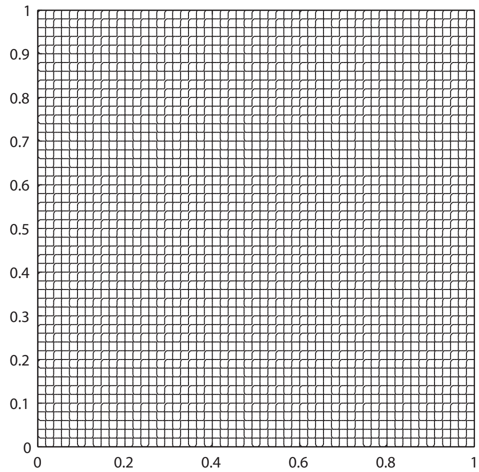


Fig. 5.1 *The nodal lines of a Dirichlet eigenfunction $u(x_1, x_2)$ on the unit square, with the associated eigenvalue $\lambda = 5525\pi^2$ of multiplicity 12. The eigenfunction was obtained as a linear combination of terms $\sin(\pi n_1 x_1)\sin(\pi n_2 x_2)$, with $n_1^2 + n_2^2 = 5525$ and randomly chosen coefficients. For comparison, another eigenfunction with the same eigenvalue, $\sin(50\pi x_1)\sin(55\pi x_2)$, is shown.*

to be universal for systems with integrable or chaotic classical dynamics, which allows one to distinguish them and thus provides a new criterion for quantum chaos (see section 7.7.4). It was also conjectured that the distribution of nodal domains for chaotic systems coincides with that for Gaussian random functions.

Bogomolny and Schmit proposed a percolation-like model to describe the nodal domains which permitted analytical calculations and agreed well with numerical simulations [80]. This model allows one to apply ideas and methods developed within percolation theory [473] to the field of quantum chaos. Using the analogy with Gaussian random functions, Bogomolny and Schmit found that the mean and variance of the number ν_m of nodal domains grow as m, with explicit formulas for the prefactors. Using percolation theory, the distribution of the area s of the connected nodal domains was conjectured to follow a power law, $n(s) \propto s^{-187/91}$, as confirmed by simulations [80]. In the particular case of random Gaussian spherical harmonics, Nazarov and Sodin rigorously derived the asymptotic behavior of the number ν_n of nodal domains of the harmonic of degree n [367]. They proved that as n grows to infinity, the mean of ν_n/n^2 tends to a positive constant, and that ν_n/n^2 exponentially concentrates around this constant (we recall that the associated eigenvalue is $n(n+1)$).

The geometrical structure of nodal lines and domains has been intensively studied (see [366, 401] for further discussion of the asymptotic nodal geometry). For instance, the length of the nodal line of an eigenfunction of the Laplace operator in two-dimensional Riemannian manifolds was separately investigated by Brüning, Yao, and Nadirashvili, who obtained its lower and upper bounds [90, 363, 509]. In addition, a number of conjectures about the properties of particular eigenfunctions have been discussed in the literature. We mention three of them:

(i) In 1967, Payne conjectured that the second Dirichlet eigenfunction u_2 cannot have a closed nodal line in a bounded planar domain [389, 390]. This conjecture was proved for convex domains [5, 354] and disproved for nonconvex domains [245]; see also [216, 256].

(ii) The hot spots conjecture formulated by J. Rauch in 1974 says that the maximum of the second Neumann eigenfunction is attained at a boundary point. This conjecture was proved by Banuelos and Burdzy for a class of planar domains [40], but in general the statement is wrong, as shown by several counterexamples [54, 101, 104, 257].

(iii) Liboff formulated several conjectures; one of them states that the nodal surface of the first excited state of a three-dimensional convex domain intersects its boundary in a single simple closed curve [324].

The analysis of nodal lines that describe zeros of eigenfunctions can be extended to other level sets. For instance, a level set of the first Dirichlet eigenfunction u_1 on a bounded convex domain $\Omega \in \mathbb{R}^d$ is itself convex [274]. Grieser and Jerison estimated the size of the first eigenfunction uniformly for all convex domains [217]. In particular, they located the place where u_1 achieves its maximum to within a distance comparable to the inradius, uniformly for arbitrarily large diameter. Other geometrical characteristics (e.g., the volume of a set on which an eigenfunction is positive) can also be analyzed [365].

6. Estimates for Laplacian Eigenfunctions.

The amplitudes of eigenfunctions can be characterized either globally by their L_p-norms

$$\|u\|_p \equiv \left(\int_\Omega dx\; |u(x)|^p\right)^{1/p} \qquad (p \geq 1) \tag{6.1}$$

or locally by pointwise estimates. Since eigenfunctions are defined up to a multiplicative constant, one often uses $L_2(\Omega)$-normalization, $\|u\|_2 = 1$. Note also the limiting case of L_∞-norm,

$$\|u\|_\infty \equiv \operatorname{ess\,sup}_{x\in\Omega} |u(x)| = \max_{x\in\Omega} |u(x)| \tag{6.2}$$

(the first equality is the general definition, while the second equality is applicable for eigenfunctions). It is worth recalling Hölder's inequality for any two measurable functions u and v and for any positive p, q such that $1/p + 1/q = 1$:

$$\|uv\|_1 \leq \|u\|_p \|v\|_q. \tag{6.3}$$

For a bounded domain $\Omega \subset \mathbb{R}^d$ (with a finite Lebesgue measure $\mu_d(\Omega)$), Hölder's inequality implies

$$\|u\|_p \leq [\mu_d(\Omega)]^{\frac{1}{p}-\frac{1}{p'}} \|u\|_{p'} \qquad (1 \leq p \leq p'). \tag{6.4}$$

We also mention Minkowski's inequality for two measurable functions and any $p \geq 1$:

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p. \tag{6.5}$$

6.1. First (Ground) Dirichlet Eigenfunction.

The Dirichlet eigenfunction u_1 associated with the first eigenvalue $\lambda_1 > 0$ does not change the sign in Ω and may be taken to be positive. It satisfies the following inequalities.

(i) Payne and Rayner showed in two dimensions that

$$\|u_1\|_2 \leq \frac{\sqrt{\lambda_1}}{\sqrt{4\pi}}\; \|u_1\|_1, \tag{6.6}$$

with equality if and only if Ω is a disk [392, 393]. Kohler-Jobin gave an extension of this inequality to higher dimensions [283] (see [132, 284, 393] for other extensions):

$$\|u_1\|_2 \leq \frac{\lambda_1^{d/4}}{\sqrt{2d\omega_d [j_{\frac{d}{2}-1,1}]^{d-2}}} \|u_1\|_1. \tag{6.7}$$

(ii) Payne and Stakgold derived two inequalities for a convex domain in two dimensions:

$$\frac{\pi}{2\mu_2(\Omega)} \|u_1\|_1 \leq \|u_1\|_\infty \tag{6.8}$$

and

$$u_1(x) \leq |x - \partial\Omega| \frac{\sqrt{\lambda_1}}{\mu_2(\Omega)} \|u_1\|_1 \qquad (x \in \Omega), \tag{6.9}$$

where $|x - \partial\Omega|$ is the distance from a point x in Ω to the boundary $\partial\Omega$ [394].

(iii) Van den Berg proved the following inequality for the L_2-normalized eigenfunction u_1 when Ω is an open, bounded, and connected set in $\mathbb{R}^d$ ($d = 2, 3, \dots$):

$$\|u_1\|_\infty \leq \frac{2^{\frac{2-d}{2}}}{\pi^{d/4}\sqrt{\Gamma(d/2)}} \frac{\left(j_{\frac{d-2}{2},1}\right)^{\frac{d-2}{2}}}{|J_{\frac{d}{2}}\left(j_{\frac{d-2}{2},1}\right)|} \rho^{-d/2}, \tag{6.10}$$

with equality if and only if Ω is a ball, where ρ is the inradius (see (4.11)) [492]. Van den Berg also conjectured the following stronger inequality for an open bounded convex domain $\Omega \subset \mathbb{R}^d$:

$$\|u_1\|_\infty \leq C_d \rho^{-d/2} (\rho/\delta)^{1/6}, \tag{6.11}$$

where δ is the diameter of Ω and C_d is a universal constant independent of Ω.

(iv) Kröger obtained another upper bound for $\|u_1\|_\infty$ for a convex domain $\Omega \subset \mathbb{R}^d$. Suppose that $\lambda_1(D) \geq \Lambda(\delta)$ for every convex subdomain $D \subset \Omega$ with $\mu_d(D) \leq \delta\mu_d(\Omega)$ and positive numbers δ and $\Lambda(\delta)$. The first eigenfunction u_1, which is normalized such that $\|u_1\|_2^2 = \mu_d(\Omega)$, satisfies

$$\|u_1\|_\infty \leq C_d \delta^{-1/2} \bigl[1 + \ln \|u_1\|_\infty - \ln(1 - \lambda_1/\Lambda(\delta))\bigr]^{d/2}, \tag{6.12}$$

with a universal positive constant C_d which depends only on the dimension d [293].

(v) Pang investigated how the first Dirichlet eigenvalue and eigenfunction change when the domain slightly shrinks [384, 385]. For a bounded simply connected open set $\Omega \subset \mathbb{R}^2$, let

$$\Omega_\epsilon \supseteq \{x \in \Omega : |x - \partial\Omega| \geq \epsilon\}$$

be its interior, i.e., Ω without an ϵ boundary layer. Let λ_m^ϵ and u_m^ϵ be the Dirichlet eigenvalues and L_2-normalized eigenfunctions in Ω_ϵ (with $\lambda_m^0 = \lambda_m$ and $u_m^0 = u_m$ referring to the original domain Ω). Then, for all $\epsilon \in (0, \rho/2)$,

$$\begin{aligned} |\lambda_1^\epsilon - \lambda_1| &\leq C_1 \epsilon^{1/2}, \\ \|u_1 - T_\epsilon u_1^\epsilon\|_{L_\infty(\Omega)} &\leq \Bigl[C_2 + C_3(\lambda_2 - \lambda_1)^{-1/2} + C_4(\lambda_2 - \lambda_1)^{-1}\Bigr] \epsilon^{1/2}, \end{aligned} \tag{6.13}$$

where ρ is the inradius of Ω (see (4.11)), T_ϵ is the extension operator from Ω_ϵ to Ω, and

$$\begin{aligned} C_1 &= \rho^{-3/2}\beta^{9/4}\ \frac{2^9\gamma_1^4}{3\pi^{9/4}}, \qquad C_2 = \rho^{-3/2}\beta^{13/4}\ \frac{2^{12}\gamma_1^5}{\pi^{15/4}}, \\ C_3 &= \rho^{-5/2}\beta^4\left(\frac{2^{15}\gamma_1^6\gamma_2}{3\sqrt{2\alpha}\pi^{9/2}}\right)\left[1+\frac{9\gamma_1}{\pi^{3/4}}\beta^{3/4}\right], \\ C_4 &= \rho^{-7/2}\beta^7\left(\frac{2^{26}\gamma_1^{10}\gamma_2^2}{81\sqrt{2}\ \alpha\ \pi^{15/2}}\right)\left[1+18\gamma_1\beta^{3/4}+\frac{81\gamma_1^2}{\pi^{3/2}}\beta^{3/2}\right], \end{aligned}$$

where $\beta = \mu_2(\Omega)/\rho^2$, α is the constant from (4.10) (for which one can use the best known estimate $\alpha = 0.6197\ldots$ from [41]), and γ_1 and γ_2 are the first and second Dirichlet eigenvalues for the unit disk, $\gamma_1 = j_{0,1}^2 \approx 5.7832$ and $\gamma_2 = j_{1,1}^2 \approx 14.6820$. Moreover, when Ω is the cardioid in $\mathbb{R}^2$, the term $\epsilon^{1/2}$ cannot be improved.[1]

In addition, Davies proved that for a bounded simply connected open set $\Omega \in \mathbb{R}^2$ and for any $\beta \in (0, 1/2)$, there exists $c_\beta \geq 1$ such that [148]

$$|\lambda_1^\epsilon - \lambda_1| \leq c_\beta \epsilon^\beta \tag{6.14}$$

for all sufficiently small $\epsilon > 0$. The estimate also holds for higher Dirichlet eigenvalues.

6.2. Estimates Applicable for All Eigenfunctions.

6.2.1. Estimates through the Green Function.

Using the spectral decomposition (2.5) of the Green function $G(x, y)$, one can rewrite (1.1) as

$$u_m(x) = \lambda_m \int_\Omega G(x, y)u_m(y)dy,$$

from which the Hölder inequality (6.3) yields a family of simple pointwise estimates

$$|u_m(x)| \leq \lambda_m \|u_m\|_{\frac{p}{p-1}} \left(\int_\Omega |G(x, y)|^p dy\right)^{1/p}, \tag{6.15}$$

with any $p \geq 1$. Here, a single function of x in the right-hand side bounds all the eigenfunctions. In particular, for $p = 1$, one gets

$$|u_m(x)| \leq \lambda_m \|u_m\|_\infty \int_\Omega |G(x, y)|dy. \tag{6.16}$$

For the Dirichlet boundary condition, $G(x, y)$ is positive everywhere in Ω so that

$$|u_m(x)| \leq \lambda_m \|u_m\|_\infty\ U(x), \qquad U(x) = \int_\Omega G(x, y)dy, \tag{6.17}$$

where $U(x)$ solves the boundary value problem

$$-\Delta U(x) = 1 \quad (x \in \Omega), \qquad U(x) = 0 \quad (x \in \partial\Omega). \tag{6.18}$$

The solution of this equation is known to be the mean first passage time to the boundary $\partial\Omega$ from an interior point x [421]. The inequalities (6.16)–(6.17) (or their

[1] In the original paper [385], the coefficient C_4 in (1.5) should be multiplied by the omitted prefactor $\sqrt{2}|\Omega|$ that follows from the derivation.

extensions) were reported by Moler and Payne [361] (see section 6.2.2) and were used by Filoche and Mayboroda to determine the geometrical structure of eigenfunctions [186] (see section 6.2.6). Note that the function $U(x)$ was also considered by Gorelick et al. for a reliable extraction of various shape properties of a silhouette, including part structure and rough skeleton, local orientation and aspect ratio of different parts, and convex and concave sections of the boundaries [209].

6.2.2. Bounds for Eigenvalues and Eigenfunctions of Symmetric Operators. Moler and Payne derived simple bounds for eigenvalues and eigenfunctions of symmetric operators by considering their extensions [361]. A typical example is a pair of operators A and A_*, in which a symmetric operator A is the Dirichlet–Laplace operator in a bounded domain Ω, and its extension A_* is the Laplace operator without boundary conditions. An approximation to an eigenvalue and eigenfunction of A can be obtained by solving a simpler eigenvalue problem $A_* u_* = \lambda_* u_*$ without a boundary condition. If there exists a function w such that $A_* w = 0$ and $w = u_*$ at the boundary of Ω, and if $\varepsilon = \frac{\|w\|_{L_2(\Omega)}}{\|u_*\|_{L_2(\Omega)}} < 1$, then there exists an eigenvalue λ_k of A satisfying

$$\frac{|\lambda_*|}{1+\varepsilon} \le |\lambda_k| \le \frac{|\lambda_*|}{1-\varepsilon}. \tag{6.19}$$

Moreover, if $\|u_*\|_{L_2(\Omega)} = 1$ and u_k is the L_2-normalized projection of u_* onto the eigenspace of λ_k, then

$$\|u_* - u_k\|_{L_2(\Omega)} \le \frac{\varepsilon}{\alpha}\left(1 + \frac{\varepsilon^2}{\alpha^2}\right)^{1/2}, \qquad \text{with} \quad \alpha = \min_{\lambda_n \ne \lambda_k} \frac{|\lambda_n - \lambda_*|}{|\lambda_n|}. \tag{6.20}$$

If u_* is a good approximation to an eigenfunction of the Dirichlet–Laplace operator, then it must be close to zero on the boundary of Ω, yielding small ε and thus accurate lower and upper bounds in (6.19). The accuracy of the bound (6.20) also depends on the separation α between eigenvalues.

In the same work, Moler and Payne also provided pointwise bounds for eigenfunctions that rely on Green's functions (an extension of section 6.2.1).

6.2.3. Estimates for L_p-Norms. Chiti extended the Payne–Rayner inequality (6.6) to the eigenfunctions of linear elliptic second-order operators in divergent form, with Dirichlet boundary conditions [132]. For the Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^d$, Chiti's inequality for any real numbers $q \ge p > 0$ states

$$\|u\|_q \le \|u\|_p \, (d\omega_d)^{\frac{1}{q}-\frac{1}{p}} \lambda^{\frac{q-p}{2pq}d} \frac{\left(\int_0^{j_{\frac{d}{2}-1,1}} dr \; r^{d-1+q(1-d/2)} [J_{\frac{d}{2}-1}(r)]^q\right)^{1/q}}{\left(\int_0^{j_{\frac{d}{2}-1,1}} dr \; r^{d-1+p(1-d/2)} [J_{\frac{d}{2}-1}(r)]^p\right)^{1/p}}, \tag{6.21}$$

where ω_d is given by (4.2).

6.2.4. Pointwise Bounds for Dirichlet Eigenfunctions. Banuelos derived a pointwise upper bound for L_2-normalized Dirichlet eigenfunctions [39],

$$|u_m(x)| \le \lambda_m^{d/4} \qquad (x \in \Omega). \tag{6.22}$$

Van den Berg and Bolthausen proved several estimates for L_2-normalized Dirichlet eigenfunctions [493]. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3, \dots$) be an open bounded domain with boundary $\partial\Omega$ which satisfies an α-uniform capacitary density condition with some $\alpha \in (0,1]$, i.e.,

$$\mathrm{Cap}\{\partial\Omega \cap B(x;r)\} \geq \alpha \mathrm{Cap}\{B(x;r)\}, \quad x \in \partial\Omega,\ 0 < r < \delta, \tag{6.23}$$

where $B(x,r)$ is the ball of radius r centered at x, δ is the diameter of Ω (see (4.20)), and Cap is the logarithmic capacity for $d = 2$ and the Newtonian (or harmonic) capacity for $d > 2$ (the harmonic capacity of an Euclidean domain presents a measure of its "size" through the total charge the domain can hold at a given potential energy [161]). The condition (6.23) guarantees that all points of $\partial\Omega$ are regular. The following estimates hold.

(i) In two dimensions ($d = 2$), for all $m = 1, 2, \ldots$ and all $x \in \Omega$ such that $|x - \partial\Omega|\sqrt{\lambda_m} < 1$, one has

$$|u_m(x)| \leq \left\{ \frac{6\lambda_m \ln(\alpha^{2\pi}/2)}{\ln\left(|x - \partial\Omega|\sqrt{\lambda_m}\right)} \right\}^{1/2}. \tag{6.24}$$

(ii) In higher dimensions ($d > 2$), for all $m = 1, 2, \ldots$ and all $x \in \Omega$ such that

$$|x - \partial\Omega|\sqrt{\lambda_m} \leq \left(\frac{\alpha^6}{2^{13}}\right)^{1+\gamma(d-1)/(d-2)}, \tag{6.25}$$

with $\gamma = \dfrac{3^{-d-1}\alpha}{\ln(2(2/\alpha)^{1/(d-2)})}$, one has

$$|u_m(x)| \leq 2\lambda_m^{d/4}\left(|x - \partial\Omega|\sqrt{\lambda_m}\right)^{\frac{1}{2}((1/\gamma)+(d-1)/(d-2))^{-1}}. \tag{6.26}$$

(iii) For a planar simply connected domain and all $m = 1, 2, \ldots,$

$$|u_m(x)| \leq m\ 2^{9/2}\pi^{1/4}\frac{(\mu_2(\Omega))^{1/4}}{\rho^2}\ |x - \partial\Omega|^{1/2} \qquad (x \in \Omega), \tag{6.27}$$

where ρ is the inradius of Ω (see (4.11)), and the inequality is sharp.

6.2.5. Upper and Lower Bounds for Normal Derivatives of Dirichlet Eigenfunctions. Suppose that M is a compact Riemannian manifold with boundary and u is an L_2-normalized Dirichlet eigenfunction with eigenvalue λ. Let ψ be its normal derivative at the boundary. A scaling argument suggests that the L_2-norm of ψ will grow as $\sqrt{\lambda}$ as $\lambda \to \infty$. Hassell and Tao proved that

$$c_M\sqrt{\lambda} \leq \|\psi\|_{L_2(\partial M)} \leq C_M\sqrt{\lambda}, \tag{6.28}$$

where the upper bound holds for any Riemannian manifold, while the lower bound is valid provided that M has no trapped geodesics [230]. The positive constants c_M and C_M depend on M, but not on λ.

6.2.6. Estimates for Restriction onto a Subdomain. For a bounded domain $\Omega \subset \mathbb{R}^d$, Filoche and Mayboroda obtained the following upper bound for the L_2-norm of a Dirichlet Laplacian eigenfunction u associated to λ, in any open subset $D \subset \Omega$ [186]:

$$\|u\|_{L_2(D)} \leq \left(1 + \frac{\lambda}{d_D(\lambda)}\right)\|v\|_{L_2(D)}, \tag{6.29}$$

where the function v solves the boundary value problem in D,

$$\Delta v = 0 \quad (x \in D), \qquad v = u \quad (x \in \partial D),$$

and $d_D(\lambda)$ is the distance from λ to the spectrum of the Dirichlet–Laplace operator in D. Note also that the above bound was proved for general self-adjoint elliptic operators [186]. When combined with (6.17), this inequality helps to investigate the spatial distribution of eigenfunctions because harmonic functions are in general much easier to analyze or estimate than eigenfunctions.

We complete the above estimate by a lower bound [370]

$$\|u\|_{L_2(D)} \geq \frac{\lambda_1(D)}{\lambda + \lambda_1(D)} \|v\|_{L_2(D)}, \tag{6.30}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of the subdomain D.

7. Localization of Eigenfunctions. Localization is defined in Webster's dictionary as an "act of localizing, or state of being localized." The notion of localization appears in various fields of science and often has different meanings. Throughout this review, a function u defined on a domain $\Omega \subset \mathbb{R}^d$ is called L_p-*localized* (for $p \geq 1$) if there exists a bounded subset $\Omega_0 \subset \Omega$ which supports most of the L_p-norm of u, i.e.,

$$\frac{\|u\|_{L_p(\Omega \backslash \Omega_0)}}{\|u\|_{L_p(\Omega)}} \ll 1 \qquad \text{and} \qquad \frac{\mu_d(\Omega_0)}{\mu_d(\Omega)} \ll 1. \tag{7.1}$$

Qualitatively, a localized function essentially "lives" on a small subset of the domain and takes small values on the remaining part. For instance, a Gaussian function $\exp(-x^2)$ on $\Omega = \mathbb{R}$ is L_p-localized for any $p \geq 1$, since one can choose $\Omega_0 = [-a, a]$ with large enough a that the ratio of L_p-norms can be made arbitrarily small, while the ratio of lengths $\mu_1(\Omega_0)/\mu_1(\Omega)$ is strictly 0. In turn, when $\Omega = [-A, A]$, the localization character of $\exp(-x^2)$ on Ω becomes dependent on A and thus conventional. A simple calculation shows that both ratios in (7.1) cannot be made simultaneously smaller than $1/(A+1)$ for any $p \geq 1$. For instance, if $A = 3$ and the "threshold" $1/4$ is viewed as small enough, then we are justified in calling $\exp(-x^2)$ localized on $[-A, A]$. This example illustrates that the above inequalities do not provide a universal quantitative criterion to distinguish localized from nonlocalized (or extended) functions. In this section, we will describe various kinds of localization for which some quantitative criteria can be formulated. We will also show that the choice of the norm (i.e., p) may be important.

Another "definition" of localization was given by Felix et al., who combined L_2- and L_4-normst define the existence area as [175]

$$S(u) = \frac{\|u\|^4_{L_2(\Omega)}}{\|u\|^4_{L_4(\Omega)}}. \tag{7.2}$$

They called a function u localized when its existence area $S(u)$ is much smaller than the area $\mu_2(\Omega)$ [175] (this definition trivially extends to other dimensions). In fact, if a function is small in a subdomain, the fourth power diminishes it more strongly than the second power. For instance, if $\Omega = (0, 1)$ and u is 1 on the subinterval $\Omega_0 = (1/4, 1/2)$ and 0 otherwise, one has $\|u\|_{L_2(\Omega)} = 1/2$ and $\|u\|_{L_4(\Omega)} = 1/\sqrt{2}$, so that $S(u) = 1/4$, i.e., the length of the subinterval Ω_0. This definition is still qualitative; e.g., in the above example, is the ratio $S(u)/\mu_1(\Omega) = 1/4$ small enough to call u localized? Note that a family of existence areas can be constructed by comparing L_p- and L_q-norms (with $p < q$),

$$S_{p,q}(u) = \left(\frac{\|u\|_{L_p(\Omega)}}{\|u\|_{L_q(\Omega)}} \right)^{\frac{1}{\frac{1}{p}-\frac{1}{q}}} . \tag{7.3}$$

7.1. Bound Quantum States in a Potential. The notion of bound, trapped, or localized quantum states has been known for a long time [74, 422]. The simplest "canonical" example is the quantum harmonic oscillator, i.e., a particle of mass m in a harmonic potential of frequency ω that is described by the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2, \tag{7.4}$$

where $\hat{p} = -i\hbar\partial_x$ is the momentum operator and $\hat{x} = x$ is the position operator ($\hbar$ being the Planck's constant). The eigenfunctions of this operator are well known:

$$\psi_n(x) = \sqrt{\frac{1}{2^n\ n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\big(\sqrt{m\omega/\hbar}x\big), \tag{7.5}$$

where $H_n(x)$ are the Hermite polynomials. All these functions are localized in a region around the minimum of the harmonic potential (here, $x = 0$) and rapidly decay outside this region. For this example, the definition (7.1) of localization is rigorous. In physical terms, the presence of a strong potential prevents the particle from traveling far from the origin, the size of the localization region being $\sqrt{\hbar/(m\omega)}$. This so-called strong localization has been thoroughly investigated in physics and mathematics [3, 337, 338, 380, 422, 450, 454, 461].

7.2. Anderson Localization. The previous example of a single quantum harmonic well is too idealized. A piece of matter contains an extremely large number of interacting atoms. Even if one focuses on a single atom in an effective potential, the form of this potential can be so complicated that the study of the underlying eigenfunctions would in general be intractable. In 1958, Anderson considered a lattice model for a charge carrier in a random potential and proved the localization of eigenfunctions under certain conditions [9]. The localization of charge carriers means no electric current through the medium (insulating state), in contrast to the metallic or conducting state when the charge carriers are not localized. The Anderson transition between insulating and conducting states is illustrated for the tight-binding model in Figure 7.1. The eigenfunctions shown were computed by Obuse for three disorder strengths W that correspond to metallic ($W < W_0$), critical ($W = W_0$), and insulating ($W > W_0$) states, $W_0 = 5.952$ being the critical disorder strength. The latter eigenfunction is strongly localized such that diffusion of charge carriers is prohibited (i.e., no electric current). Anderson localization, which explains the metal-insulator transitions in semiconductors, has been thoroughly investigated during the last fifty years (see [56, 170, 291, 315, 358, 429, 476, 477, 483] for details and references). Similar localization phenomena were observed for microwaves with two-dimensional random scattering [142], for light in a disordered medium [506] and in disordered photonic crystals [442, 453], for matter waves in a controlled disorder [70] and in a noninteracting Bose–Einstein condensate [430], and for ultrasound [248]. The multifractal structure of the eigenfunctions at the critical point (illustrated by Figure 7.1b) has

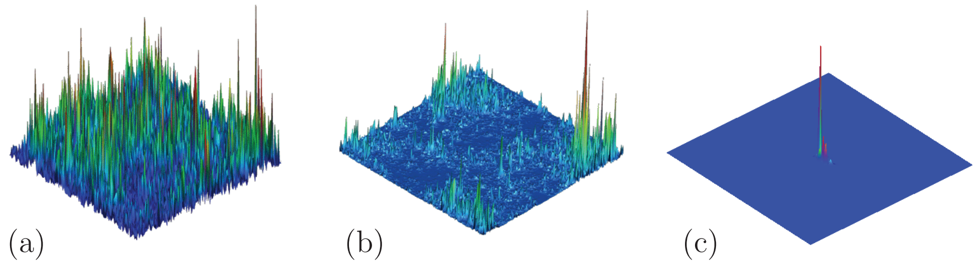


Fig. 7.1 *Illustration of the Anderson transition in a tight-binding model (or so-called $SU(2)$ model) in the two-dimensional symplectic class* [20, 21, 373, 374]. *The three eigenfunctions shown (with the energy close to* 1*) were computed for three disorder strengths W that correspond to* (a) *a metallic state ($W < W_0$),* (b) *a critical state ($W = W_0$), and* (c) *an insulating state ($W > W_0$), $W_0 = 5.952$ being the critical disorder strength. The latter eigenfunction is strongly localized, which prohibits diffusion of charge carriers (i.e., no electric current). The eigenfunctions were computed and provided by H. Obuse (previously unpublished).*

also been intensively investigated (see [170, 220] and references therein). Localization of eigenstates and transport phenomena in one-dimensional disordered systems are reviewed in [251]. An introduction to wave scattering and the related localization is given in [456].

7.3. Trapping in Infinite Waveguides. In both previous cases, localization of eigenfunctions was related to an external potential. In particular, if the potential was not strong enough, Anderson localization could disappear (see Figure 7.1a). Is the presence of a potential necessary for localization? The formal answer is positive because the eigenstates of the Laplace operator in the whole space $\mathbb{R}^d$ are simply $e^{i(k\cdot x)}$ (parameterized by the vector k), which are all extended in $\mathbb{R}^d$. These waves are called "resonances" (not eigenfunctions) of the Laplace operator, as their L_2-norm is infinite.

The situation is different for the Laplace operator in a bounded domain with Dirichlet boundary conditions. In quantum mechanics, such a boundary presents a "hard wall" that separates the interior of the domain with zero potential from the exterior of the domain with infinite potential. For instance, this "model" was employed by Crommie, Lutz, and Eigler to describe the confinement of electrons to quantum corrals on a metallic surface [140] (see also their Figure 2, which shows the experimental spatial structure of the electron's wavefunction). Although the physical interpretation of a boundary through an infinite potential is instructive, we will use mathematical terminology and speak about the eigenvalue problem for the Laplace operator in a bounded domain without potential.

For unbounded domains, the spectrum of the Laplace operator consists of two parts: (i) the discrete (or point-like) spectrum, with eigenfunctions of finite L_2-norm that are necessarily "trapped" or "localized" in a bounded region of the waveguide, and (ii) the continuous spectrum, with associated functions of infinite L_2-norm that are extended over the whole domain. The continuous spectrum may also contain embedded eigenvalues whose eigenfunctions have finite L_2-norm. A wave excited at the frequency of the trapped eigenmode remains in the localization region and does not propagate. In this case, the definition (7.1) of localization is again rigorous, as for any bounded subset Ω_0 of an unbounded domain Ω, one has $\mu_d(\Omega_0)/\mu_d(\Omega) = 0$, while the ratio of L_2-norms can be made arbitrarily small by expanding Ω_0.

This kind of localization in classical and quantum waveguides has been thoroughly investigated (see reviews [163, 328] and also references in [377]). In his seminal paper, Rellich proved the existence of a localized eigenfunction in a deformed infinite cylinder [423]. His results were significantly extended by Jones [264]. Ursell reported on the existence of trapped modes in surface water waves in channels [489, 490, 491], while Parker observed experimentally the trapped modes in locally perturbed acoustic waveguides [386, 387]. Exner and Seba considered an infinite bent strip of smooth curvature and showed the existence of trapped modes by reducing the problem to the Schrödinger operator in the straight strip, with the potential depending on the curvature [171]. Goldstone and Jaffe gave a variational proof that the wave equation subject to a Dirichlet boundary condition always has a localized eigenmode in an infinite tube of constant cross section in any dimension, provided that the tube is not exactly straight [206]. This result was further extended by Chenaud et al. to arbitrary dimension [127]. The problem of localization in acoustic waveguides with Neumann boundary conditions has also been investigated [167, 168]. For instance, Evans, Levitin, and Vassiliev considered a straight strip with an inclusion of arbitrary (but symmetric) shape [168] (see [150] for further extensions). Such an inclusion obstructs the propagation of waves and was shown to result in trapped modes. The effect of mixed Dirichlet, Neumann, and Robin boundary conditions on the localization has also been investigated (see [97, 157, 191, 377] and references therein). A mathematical analysis of guided water waves was developed by Bonnet-Ben Dhia and Joly [83] (see also [82]). Lower bounds for the eigenvalues below the threshold (for which the associated eigenfunctions are localized) were obtained by Ashbaugh and Exner for infinite thin tubes in two and three dimensions [29]. In addition, these authors derived an upper bound for the number of the trapped modes. Exner, Freitas, and Krejčiřík considered the Laplacian in finite-length curved tubes of arbitrary cross section, subject to Dirichlet boundary conditions on the cylindrical surface and Neumann conditions at the ends of the tube. They expressed a lower bound for the spectral threshold of the Laplacian through the lowest eigenvalue of the Dirichlet Laplacian in a torus determined by the geometry of the tube [172]. In a different work, Exner and coworkers investigated bound states and scattering in quantum waveguides coupled laterally through a boundary window [173].

Examples of waveguides with *numerous* localized states have been reported in the literature. For instance, Avishai et al. demonstrated the existence of many localized states for a sharp "broken strip," i.e., a waveguide made of two channels of equal width intersecting at a small angle θ [32]. Carini and coworkers reported an experimental confirmation of this prediction and its further extensions [116, 117, 331]. Bulgakov et al. considered two straight strips of the same width which cross at an angle $\theta \in (0, \pi/2)$ and showed that, for small θ, the number of localized states is greater than $(1 - 2^{-2/3})^{3/2}/\theta$ [96]. Even for the simple case of two strips crossed at a right angle $\theta = \pi/2$, Schult, Ravenhall, and Wyld showed the existence of two localized states, one lying below the cut-off frequency and the other embedded into the continuous spectrum [452].

7.4. Exponential Estimate for Eigenfunctions. Qualitatively, an eigenmode is trapped when it cannot "squeeze" outside the localization region through narrow channels or branches of the waveguide. This happens when typical spatial variations of the eigenmode, which are in the order of a wavelength $\pi\lambda^{-1/2}$, are larger than the size a of the narrow part, i.e., $\pi\lambda^{-1/2} \geq a$ or $\lambda \leq \pi^2/a^2$ [253]. This simplistic argument suggests that there exists a threshold value μ (which may eventually be 0), or so-called

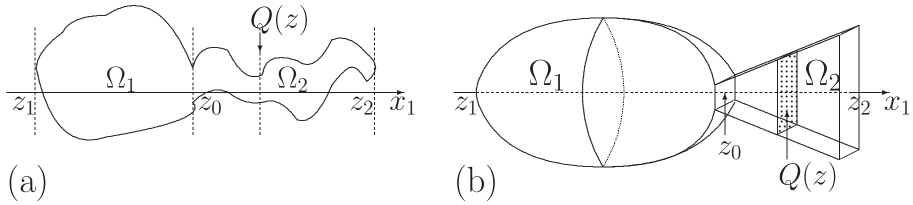


Fig. 7.2 *Two examples of a bounded domain $\Omega = \Omega_1 \cup \Omega_2$ with a branch of variable cross-sectional profile. When the eigenvalue λ is smaller than the threshold μ, the associated eigenfunction exponentially decays in the branch Ω_2 and is thus mainly localized in Ω_1. Note that the branch itself may even be increasing.*

cut-off $\sqrt{\mu}$ frequency, such that the eigenmodes with $\lambda \leq \mu$ are localized. Moreover, this qualitative geometrical interpretation is well adapted for both unbounded and bounded domains. While the former case of infinite waveguides has been thoroughly investigated, the existence of trapped or localized eigenmodes in bounded domains has attracted less attention. Even the definition of localization in bounded domains remains conventional because all eigenfunctions have finite L_2-norm.

This problem was studied by Delitsyn and coworkers for domains with branches of variable cross-sectional profiles [151]. More precisely, consider a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3, \dots)$ with a piecewise smooth boundary $\partial\Omega$ and denote $Q(z) = \Omega \cap \{x \in \mathbb{R}^d \ : \ x_1 = z\}$ the cross section of Ω at $x_1 = z \in \mathbb{R}$ by a hyperplane perpendicular to the coordinate axis x_1 (see Figure 7.2). Let

$$z_1 = \inf\{z \in \mathbb{R} \ : \ Q(z) \neq \emptyset\}, \qquad z_2 = \sup\{z \in \mathbb{R} \ : \ Q(z) \neq \emptyset\},$$

and we fix some z_0 such that $z_1 < z_0 < z_2$. Let $\mu(z)$ be the first eigenvalue of the Laplace operator in $Q(z)$, with Dirichlet boundary condition on $\partial Q(z)$, and $\mu = \inf_{z\in(z_0,z_2)} \mu(z)$. Let u be a Dirichlet Laplacian eigenfunction in Ω and λ the associated eigenvalue. If $\lambda < \mu$, then

$$\|u\|_{L_2(Q(z))} \leq \|u\|_{L_2(Q(z_0))} \exp(-\beta\sqrt{\mu - \lambda}\ (z - z_0)) \quad (z \geq z_0), \tag{7.6}$$

with $\beta = 1/\sqrt{2}$. Moreover, if $(e_1 \cdot n(x)) \geq 0$ for all $x \in \partial\Omega$ with $x_1 > z_0$, where e_1 is the unit vector $(1, 0, \dots, 0)$ in the direction x_1, and $n(x)$ is the normal vector at $x \in \partial\Omega$ directed outwards from the domain, then the above inequality holds with $\beta = 1$.

In this statement, a domain Ω is arbitrarily split into two subdomains, Ω_1 (with $x_1 < z_0$) and Ω_2 (with $x_1 > z_0$), by the hyperplane at $x_1 = z_0$ (the coordinate axis x_1 can be replaced by any straight line). Under the condition $\lambda < \mu$, the eigenfunction u exponentially decays in the subdomain Ω_2, which is loosely called "branch." Note that the choice of the splitting hyperplane (i.e., z_0) determines the threshold μ.

The theorem formalizes the notion of the cut-off frequency $\sqrt{\mu}$ for branches of variable cross-sectional profiles and provides a constructive way for its computation. For instance, if Ω_2 is a rectangular channel of width a, the first eigenvalue in all cross sections $Q(z)$ is π^2/a^2 (independent of z), so that $\mu = \pi^2/a^2$, as expected. The exponential estimate quantifies the "difficulty" of penetration, or "squeezing," into the branch Ω_2 and ensures the localization of the eigenfunction u in Ω_1. Since the cut-off frequency $\sqrt{\mu}$ is independent of the subdomain Ω_1, one can impose any boundary condition on $\partial\Omega_1$ (that still ensures the self-adjointness of the Laplace operator). In

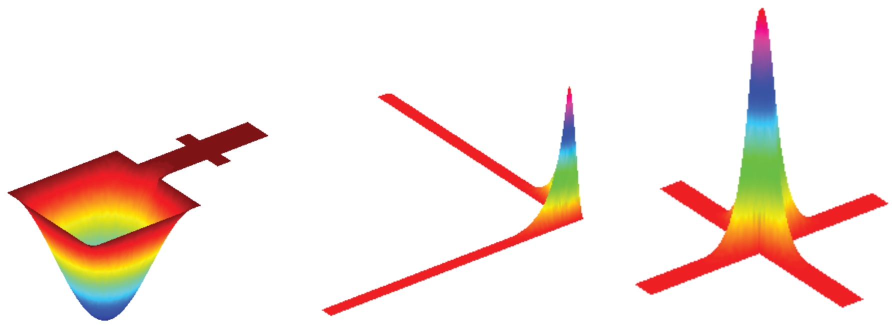


Fig. 7.3 *Examples of localized Dirichlet Laplacian eigenfunctions with an exponential decay: square with a branch (from* [151]*), L-shape, and crossing of two stripes (from* [152]*).*

turn, the Dirichlet boundary condition on the boundary of the branch Ω_2 is relevant, although some extensions were discussed in [151]. It is worth noting that the theorem also applies to infinite branches Ω_2, under supplementary condition $\mu(z) \to \infty$ to ensure the existence of the discrete spectrum.

According to this theorem, the L_2-norm of an eigenfunction with $\lambda < \mu$ in $\Omega(z) = \Omega \cap \{x \in \mathbb{R}^d \ : \ x_1 > z\}$ can be made exponentially small provided that the branch Ω_2 is long enough. Taking $\Omega_0 = \Omega \setminus \Omega(z)$, the ratio of L_2-norms in (7.1) can be made arbitrarily small. However, the second ratio may not necessarily be small. In fact, its smallness depends on the shape of the domain Ω. This is once again a manifestation of the conventional character of localization in bounded domains.

Figure 7.3 presents several examples of localized Dirichlet Laplacian eigenfunctions showing an exponential decay along the branches. Since an increase of branches diminishes the eigenvalue λ and thus further enhances the localization, the area of the localized region Ω_1 can be made arbitrarily small with respect to the total area (one can even consider infinite branches). Examples of an L-shape and a cross illustrate that the linear sizes of the localized region do not need to be large in comparison with the branch width (a sufficient condition for this kind of localization was proposed in [152]). It is worth noting that the separation into the localized region and branches is arbitrary. For instance, Figure 7.4 shows several localized eigenfunctions for elongated triangle and trapezoid, for which there is no explicit separation.

Localization and exponential decay of Laplacian eigenfunctions have been observed for various perturbations of cylindrical domains [115, 272, 368]. For instance, Kamotskii and Nazarov studied localization of eigenfunctions in a neighborhood of the lateral surface of a thin domain [272]. Nazarov and coworkers analyzed the behavior of eigenfunctions for thin cylinders with distorted ends [115, 368]. For a bounded domain $\omega \subset \mathbb{R}^{n-1}$ ($n \geq 2$) with a simple closed Lipschitz contour $\partial\omega$ and Lipschitz functions $H_\pm(\eta)$ in $\bar{\omega} = \omega \cup \partial\omega$, the thin cylinder with distorted ends is defined for a given small $\varepsilon > 0$ as

$$\Omega^\varepsilon = \{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} \ : \ y = \varepsilon\eta, \ \ -1 - \varepsilon H_-(\eta) < x < 1 + \varepsilon H_+(\eta), \ \eta \in \omega\}.$$

One can view this domain as a thin cylinder $[-1,1] \times (\varepsilon\omega)$ to which two distorted "cups" characterized by functions $H_\pm$ are attached (see Figure 7.5a). The Neumann

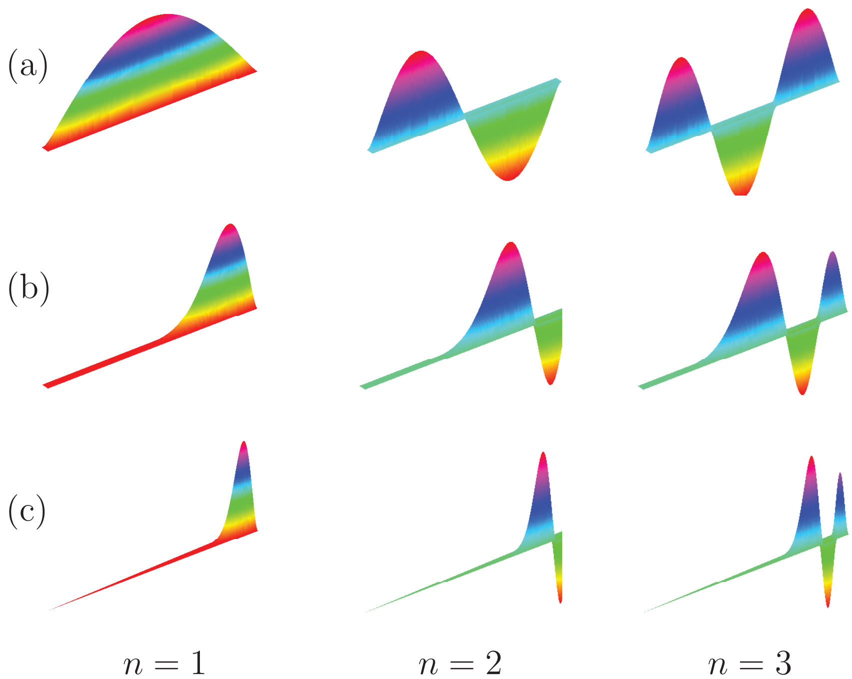


Fig. 7.4 *The first three Dirichlet Laplacian eigenfunctions for three elongated domains:* (a) *rectangle of size* 25×1*;* (b) *right trapezoid with bases* 1 *and* 0.9 *and height* 25*, which is very close to the above rectangle; and* (c) *right triangle with edges* 25 *and* 1 *(half of the rectangle). There is no localization for the first shape, while the first eigenfunctions for the second and third domains tend to be localized in the thicker end.*

boundary condition is imposed on the curved ends $\Gamma^\varepsilon_\pm$,

$$\Gamma^\varepsilon_\pm = \{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} \ : \ y = \varepsilon\eta, \ x = \pm 1 \pm \varepsilon H_\pm(\eta), \ \eta \in \omega\},$$

while the Dirichlet boundary condition is set on the remaining lateral side of the domain, $\Sigma^\varepsilon = \partial\Omega^\varepsilon \backslash (\overline{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_+})$. When the ends of the cylinder are straight ($H_\pm \equiv 0$), the eigenfunctions are factored as $u_{mn}(x,y) = \cos(\pi m(x+1)/2)\varphi_n(y)$, where $\varphi_n(y)$ are the eigenfunctions of the Laplace operator in the cross section ω with Dirichlet boundary condition. These eigenfunctions are extended over the whole cylinder, due to the cosine factor. Nazarov and coworkers showed that distortion of the ends (i.e., $H_\pm \neq 0$) may lead to localization of the ground eigenfunction at one (or both) ends, with an exponential decay toward the central part. In the limit $\varepsilon \to 0$, the thinning of the cylinder can be seen alternatively as its outstretching, allowing one to reduce the analysis to a semi-infinite cylinder with one distorted end (see Figure 7.5b), described by a single function $H(\eta)$:

$$\Omega = \{(x,\eta) \in \mathbb{R} \times \mathbb{R}^{n-1} \ : \ -H(\eta) < x, \ \eta \in \omega\}.$$

Two sufficient conditions producing the localized ground eigenfunction at the distorted end were proposed in [115]:

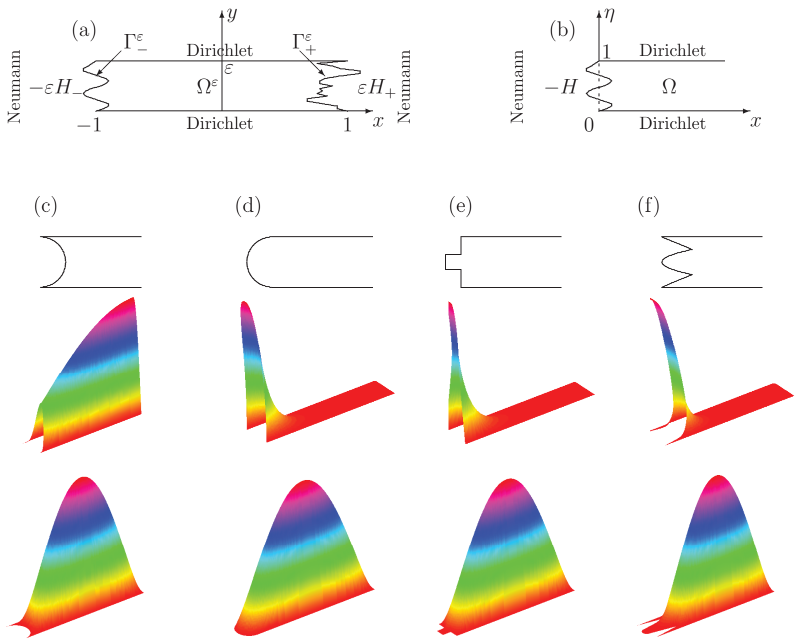


Fig. 7.5 (a) *A thin cylinder* $\Omega^\varepsilon \in \mathbb{R}^2$ *of width* ε *with two distorted ends defined by functions* $H_\pm(\eta)$, $\eta \in \omega = (0,1)$. (b) *In the limit* $\varepsilon \to 0$, *the analysis is reduced to a semi-infinite cylinder* Ω *with one distorted end.* (c)–(f) *Four semi-infinite cylinders with various distorted ends (top), the first eigenfunction for the Laplace operator in these shapes with mixed Dirichlet–Neumann boundary condition (middle), and the first eigenfunction for the Laplace operator with purely Dirichlet boundary conditions. According to the sufficient conditions* (7.7)–(7.8), *the first eigenfunction is localized near the distorted end for cases* (d), (e), *and* (f), *and not localized for the case* (c). *No localization happens when the Dirichlet boundary condition is set over the whole domain. For numerical computation, semi-infinite cylinders were truncated and an auxiliary Dirichlet boundary condition was set at the right straight end.*

(i) For $H \in C(\bar{\omega})$, the sufficient condition reads

$$\int_\omega d\eta\; H(\eta)\Big(|\nabla\varphi_1(\eta)|^2 - \mu_1[\varphi_1(\eta)]^2\Big) < 0, \tag{7.7}$$

where μ_1 is the smallest eigenvalue corresponding to φ_1 in the cross section ω (in two dimensions, when $\omega = [-1/2, 1/2]$, one has $\varphi_1(\eta) = \sin\pi(\eta + 1/2)$ and $\mu_1 = \pi^2$ so that this condition reads $\int_{-1/2}^{1/2} d\eta\; H(\eta)\cos(2\pi\eta) > 0$ [368]).

(ii) Under a stronger assumption $H \in C^2(\bar{\omega})$, the sufficient condition simplifies to

$$\int_\omega d\eta\; [\varphi_1(\eta)]^2 \Delta H(\eta) < 0. \tag{7.8}$$

This inequality becomes true for a subharmonic profile $-H$ (i.e., for $\Delta H(\eta) < 0$) but is false for superharmonic.[2]

[2] In the discussion after Theorem 3 in [115], the minus sign in front of H was omitted.

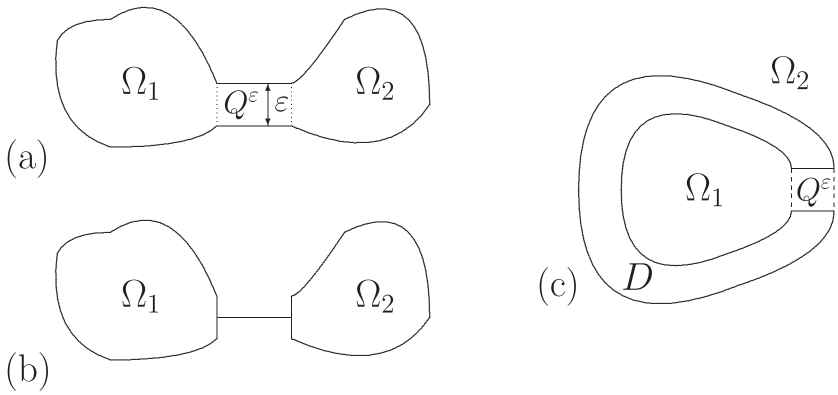


Fig. 7.6 (a) *A dumbbell domain Ω^ε is the union of two bounded domains Ω_1, Ω_2 and a narrow "connector" Q^ε of width ε.* (b) *In the limit $\varepsilon \to 0$, the connector degenerates to a curve (here, an interval) so that the subdomains Ω_1 and Ω_2 become disconnected.* (c) *In Beale's work, Ω^ε is composed of two components, a bounded domain Ω_1 and unbounded domain Ω_2, which are connected by a narrow channel Q^ε.*

Figure 7.5 shows several examples for which the sufficient condition is either satisfied (see Figure 7.5d,e,f), or not (see Figure 7.5c). Nazarov and coworkers showed that these results are applicable to bounded thin cylinders for small enough ε. In addition, they found a domain where the first eigenfunction concentrates at both ends simultaneously. Finally, they showed that no localization occurs in the case in which the mixed Dirichlet–Neumann boundary condition is replaced by the Dirichlet boundary condition on the whole boundary, as illustrated in Figure 7.5 (see [115] for further discussion and results).

Friedlander and Solomyak studied the spectrum of the Dirichlet Laplacian in a family of narrow strips of variable profile: $\Omega = \{(x,y) \in \mathbb{R}^2 : -a < x < b,\ 0 < y < \varepsilon h(x)\}$ [195, 196]. Their main assumption was that $x = 0$ is the only point of global maximum of the positive continuous function $h(x)$. In the limit $\varepsilon \to 0$, they found the two-term asymptotics of the eigenvalues and the one-term asymptotics of the corresponding eigenfunctions. The asymptotic formulas obtained involve the eigenvalues and eigenfunctions of an auxiliary ordinary differential equation on $\mathbb{R}$ that depends only on the behavior of $h(x)$ as $x \to 0$, i.e., in the vicinity of the widest cross section of the strip.

7.5. Dumbbell Domains. Yet another type of localization emerges for domains that can be split into two or several subdomains with narrow connections (of "width" ε) [418], a standard example being a dumbbell: $\Omega^\varepsilon = \Omega_1 \cup Q^\varepsilon \cup \Omega_2$ (see Figure 7.6a). The asymptotic behavior of eigenvalues and eigenfunctions in the limit $\varepsilon \to 0$ has been thoroughly investigated for both Dirichlet and Neumann boundary conditions [260]. We start by considering Dirichlet boundary condition.

In the limiting case of zero width connections, the subdomains Ω_i $(i = 1, \dots, N)$ become disconnected and the eigenvalue problem can be independently formulated for each subdomain. Let Λ_i be the set of eigenvalues for the subdomain Ω_i. Each eigenvalue λ^ε of the Dirichlet–Laplace operator in the domain Ω^ε approaches an eigenvalue λ^0 corresponding to one limiting subdomain $\Omega_i \subset \Omega^0$: $\lambda^0 \in \Lambda_i$ for certain i. Moreover, if

$$\Lambda_i \cap \Lambda_j = \emptyset \quad \forall\ i \neq j, \tag{7.9}$$

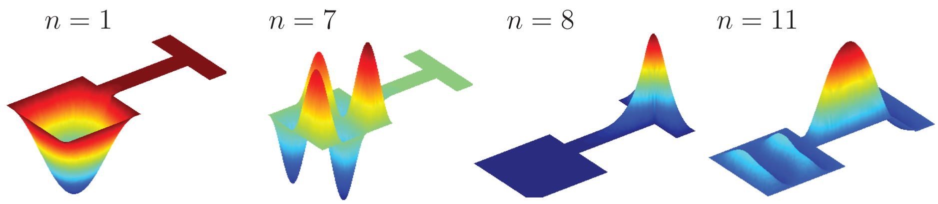


Fig. 7.7 *Several Dirichlet Laplacian eigenfunctions for a dumbbell domain which is composed of two rectangles connected by a third rectangle (from* [151]*). The first and seventh eigenfunctions are localized in the larger subdomain, the eighth eigenfunction is localized in the smaller subdomain, while the eleventh eigenfunction is not localized at all. Note that the width of connection is not small (*1/4 *of the width of both subdomains).*

the space of eigenfunctions in the limiting (disconnected) domain Ω^0 is the direct product of spaces of eigenfunctions for each subdomain Ω_i (see [143] for discussion of convergence and related issues). This is a basis for what we will call "bottleneck localization." In fact, each eigenfunction u_m^ε on the domain Ω^ε approaches an eigenfunction u_m^0 of the limiting domain Ω^0 which is fully localized in one subdomain Ω_i and zero in the others. For a small ε, the eigenfunction u_m^ε is therefore mainly localized in the corresponding ith subdomain Ω_i and is *almost* zero in the other subdomains. In other words, for any eigenfunction, one can take the width ε small enough to ensure that the L_2-norm of the eigenfunction in the subdomain Ω_i is arbitrarily close to that in the whole domain Ω^ε:

$$\forall\, m \geq 1 \quad \exists i \in \{1, \ldots, N\} \quad \forall\, \delta \in (0,1) \quad \exists\, \varepsilon > 0 \ :\ \|u_m^\varepsilon\|_{L_2(\Omega_i)} > (1-\delta)\|u_m^\varepsilon\|_{L_2(\Omega^\varepsilon)}. \tag{7.10}$$

This behavior is exemplified for a dumbbell domain which is composed of two rectangles connected by a third rectangle (see Figure 7.7). The first and seventh eigenfunctions are localized in the larger rectangle, the eighth eigenfunction is localized in the smaller rectangle, while the eleventh eigenfunction is not localized at all. Note that the width of connection is not small (1/4 of the width of both subdomains).

It is worth noting that, for a small fixed width ε and a small fixed threshold δ, there may be infinitely many high-frequency "nonlocalized" eigenfunctions for which the inequality (7.10) is not satisfied. In other words, for a given connected domain with a narrow connection, one can only expect to observe a finite number of low-frequency localized eigenfunctions. The condition (7.9) is important to ensure that limiting eigenfunctions are fully localized in their respective subdomains. Without this condition, a limiting eigenfunction may be a linear combination of eigenfunctions in different subdomains with the same eigenvalue that would destroy localization. The asymptotic behavior of eigenfunctions at the "junction" was studied by Felli and Terracini [178].

For the Neumann boundary condition, the situation is more complicated, as the eigenvalues and eigenfunctions may also approach the eigenvalues and eigenfunctions of the limiting connector (in the simplest case, the interval). Arrieta considered a planar dumbbell domain Ω_ε consisting of two disjoint domains Ω_1 and Ω_2 connected by a channel Q^ε of variable profile $g(x)$: $Q^\varepsilon = \{x \in \mathbb{R}^2 \ :\ 0 < x_1 < 1,\ 0 < x_2 < \varepsilon g(x_1)\}$, where $g \in C^1(0,1)$ and $g(x_1) > 0$ for all $x_1 \in [0,1]$. In the limit $\varepsilon \to 0$, each eigenvalue of the Laplace operator in Ω^ε with Neumann boundary condition was shown to converge either to an eigenvalue μ_k of the Neumann–Laplace operator in $\Omega_1 \cup \Omega_2$ or to an eigenvalue ν_k of the Sturm–Liouville operator $-\frac{1}{g}(gu_x)_x$ acting on a function u on $(0,1)$, with Dirichlet boundary condition [14, 15]. The first-order term in the small ε-asymptotic expansion was obtained. The special case of cylindrical channels (of constant profile) in higher dimensions was studied by Jimbo [258] (see also results by Hempel, Seco, and Simon [237]). Jimbo and Morita studied an N-dumbbell domain, i.e., a family of N pairwise disjoint domains connected by thin channels [261]. They proved that $\lambda_m^\varepsilon = C_m\varepsilon^{d-1} + o(\varepsilon^{d-1})$ as $\varepsilon \to 0$ for $m = 1, 2, \ldots, N$, while $\lambda_{N+1}^\varepsilon$ is uniformly bounded away from zero, where d is the dimension of the embedding space, and C_m are shape-dependent constants. Jimbo also analyzed the asymptotic behavior of the eigenvalues λ_m^ε with $m > N$ under the condition that the sets $\{\mu_k\}$ and $\{\nu_k\}$ do not intersect [259]. In particular, for an eigenvalue λ_m^ε that converges to an element of $\{\mu_k\}$, the asymptotic behavior is $\lambda_m^\varepsilon = \mu_k + C_m\varepsilon^{d-1} + o(\varepsilon^{d-1})$.

Brown and coworkers studied upper bounds for $|\lambda_m^\varepsilon - \lambda_m^0|$ and showed the following [89]:

(i) If $\lambda_m^0 \in \{\mu_k\} \setminus \{\nu_k\}$,

$$|\lambda_m^\varepsilon - \lambda_m^0| \le C|\ln\varepsilon|^{-1/2} \qquad (d = 2),$$
$$|\lambda_m^\varepsilon - \lambda_m^0| \le C\varepsilon^{(d-2)/d} \qquad (d \ge 3).$$

(ii) If $\lambda_m^0 \in \{\nu_k\} \setminus \{\mu_k\}$,

$$|\lambda_m^\varepsilon - \lambda_m^0| \le C\varepsilon^{1/2}|\ln\varepsilon| \qquad (d = 2),$$
$$|\lambda_m^\varepsilon - \lambda_m^0| \le C\varepsilon^{1/2} \qquad (d \ge 3).$$

For a dumbbell domain in $\mathbb{R}^d$ with a thin cylindrical channel of a smooth profile, Gadyl'shin obtained the complete small ε asymptotics of the Neumann Laplacian eigenvalues and eigenfunctions and explicit formulas for the first term of these asymptotics, including multiplicities [198, 199, 200].

Arrieta and Krejčiřík considered the problem of spectral convergence from another point of view [19]. They showed that if $\Omega_0 \subset \Omega_\varepsilon$ are bounded domains and if the eigenvalues and eigenfunctions of the Laplace operator with Neumann boundary conditions in Ω_ε converge to those in Ω_0, then necessarily $\mu_d(\Omega_\varepsilon\backslash\Omega_0) \to 0$ as $\varepsilon \to 0$, while it is not necessarily true that $\mathrm{dist}(\Omega_\varepsilon, \Omega_0) \to 0$. In fact, they constructed an example of a perturbation where the spectra behave continuously but $\mathrm{dist}(\Omega_\varepsilon, \Omega_0) \to \infty$ as $\varepsilon \to 0$.

A somewhat related problem of scattering frequencies of the wave equation associated to an exterior domain in $\mathbb{R}^3$ with an appropriate boundary condition was investigated by Beale [55] (for more general aspects of geometric scattering theory, see [356]). We recall that a scattering frequency $\sqrt{\lambda}$ of an unbounded domain Ω is a (complex) number for which there exists a nontrivial solution of $\Delta u + \lambda u = 0$ in Ω, subject to Dirichlet, Neumann, or Robin boundary conditions and to an "outgoing" condition at infinity. In Beale's work, a bounded cavity Ω_1 was connected by a thin channel to the exterior (unbounded) space Ω_2. More specifically, he considered a bounded domain D such that its complement in $\mathbb{R}^3$ has a bounded component Ω_1 and an unbounded component Ω_2. Then a thin hole Q^ε in D is introduced to connect both components (see Figure 7.6c). Beale showed that the joint domain $\Omega^\varepsilon = \Omega_1 \cup Q^\varepsilon \cup \Omega_2$ with Dirichlet boundary conditions has a scattering frequency which is arbitrarily close either to an eigenfrequency (i.e., the square root of the eigenvalue) of the Laplace operator in Ω_1 or to a scattering frequency in Ω_2, provided the channel Q^ε is narrow enough. The same result was extended to Robin boundary conditions of the form

$\partial u/\partial n + hu = 0$ on $\partial\Omega^\varepsilon$, where h is a function on $\partial\Omega^\varepsilon$ with a positive lower bound. In both cases, the method in his proof relies on the fact that the lowest eigenvalue of the channel tends to infinity as the channel narrows. However, this is no longer true for Neumann boundary conditions. In this case, with some restrictions on the shape of the channel, Beale proved that the scattering frequencies converge not only to the eigenfrequencies of Ω_1 and scattering frequencies of Ω_2, but also to the longitudinal frequencies of the channel. Similar results can be obtained in domains of space dimension other than 3.

There are other problems for partial differential equations in dumbbell domains which undergo a singular perturbation [225, 240, 341, 342]. For instance, in a series of articles, Arrieta et al. studied the behavior of the asymptotic nonlinear dynamics of a reaction-diffusion equation with Neumann boundary condition [16, 17, 18]. In this context, dumbbell domains appear naturally as the counterpart of convex domains for which the stable stationary solutions to a reaction-diffusion equation are necessarily spatially constant [123]. As explained in [16], one way to produce "patterns" (i.e., stable stationary solutions which are not spatially constant) is to consider domains which make it difficult to diffuse from one part of the domain to the other, creating a constriction in the domain. Kosugi studied the convergence of the solution to a semilinear elliptic equation in a thin network-shaped domain which degenerates into a geometric graph when a certain parameter tends to zero [288] (see also [295, 408, 431, 432]).

7.6. Localization in Irregularly Shaped Domains. As we have seen, a narrow connection between subdomains could lead to localization. How narrow should it be? A rigorous answer to this question is only known for several "tractable" cases such as dumbbell-like or cylindrical domains (see section 7.5). Sapoval and coworkers have formulated and studied the problem of localization in irregularly shaped or fractal domains through numerical simulations and experiments [169, 175, 224, 232, 434, 435, 444, 445, 446]. In their first publication, they monitored the vibrations of a prefractal "drum" (i.e., a thin membrane with a fixed boundary) excited at different frequencies [445]. Tuning the frequency allowed them to directly visualize different Dirichlet Laplacian eigenfunctions in a (prefractal) quadratic von Koch snowflake (an example is shown in Figure 7.8). For this and similar domains, certain eigenfunctions were found to be localized in a small region of the domain, for both Dirichlet and Neumann boundary conditions (see Figure 7.9). This effect was first attributed to the self-similar structure of the domain. However, similar effects were later observed in numerical simulations for nonfractal domains [175, 443], as illustrated by Figure 7.10. In the study of sound attenuation by noise-protective walls, Félix and coworkers have fur-

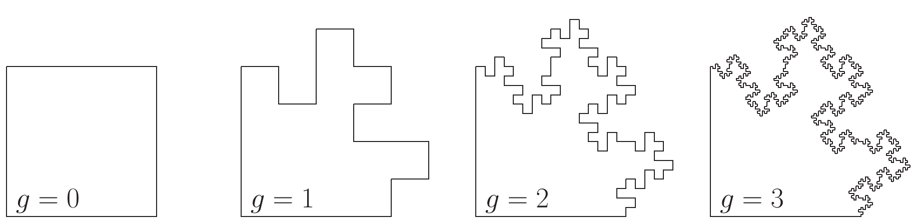


Fig. 7.8 *The unit square and three prefractal domains obtained iteratively, one from the other (two sides of these domains are finite generations of the von Koch curve of fractal dimension 3/2). These domains were intensively studied, both numerically and experimentally, by Sapoval and coworkers* [169, 224, 232, 434, 435, 444, 445, 446].

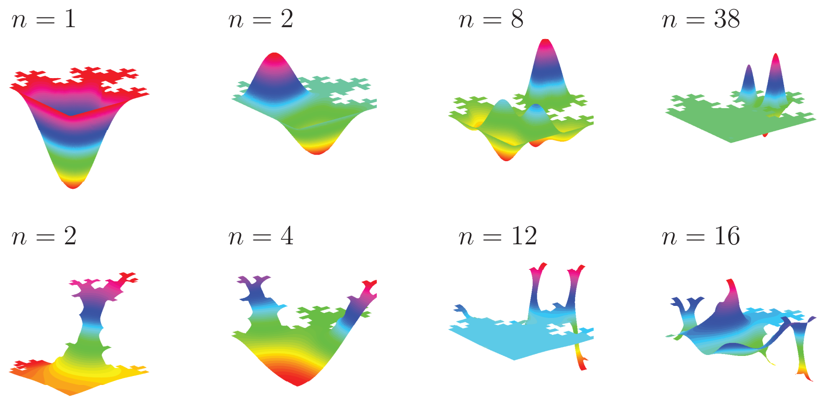


Fig. 7.9 *Several Dirichlet (top) and Neumann (bottom) eigenfunctions for the third domain in Figure 7.8 ($g = 2$). The 38th Dirichlet and the 12th Neumann eigenfunctions are localized in a small subdomain (located in the upper right corner of Figure 7.8), while the 1st/2nd Dirichlet and the 4th Neumann eigenfunctions are almost zero on this subdomain. Finally, the 8th Dirichlet and the 2nd Neumann eigenfunctions are examples of eigenfunctions extended over the whole domain.*

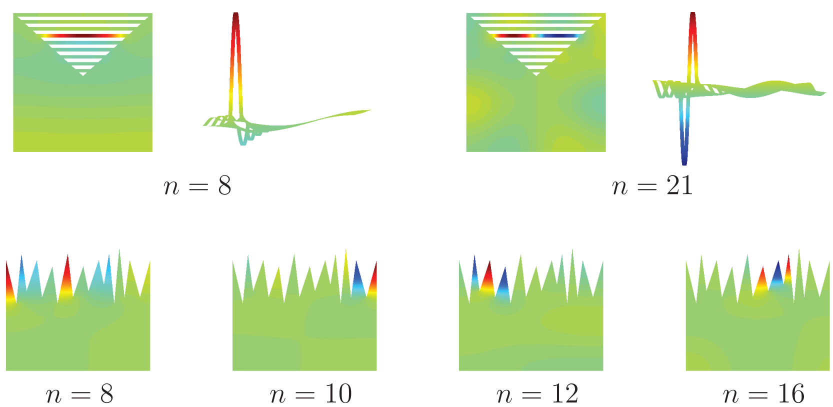


Fig. 7.10 *Examples of localized Neumann Laplacian eigenfunctions in two domains adapted from [175]: a square with many elongated holes (top) and random sawteeth (bottom). Colors represent the amplitude of eigenfunctions, from the most negative value (dark blue), through zero (green), to the largest positive value (dark red). Notice that the eigenfunctions on the top are not negligible outside the localization region. This is yet another illustration of the conventional character of localization in bounded domains.*

ther extended the analysis to the union of two domains with different refraction indices which are separated by an irregular boundary [175, 176, 443]. Many eigenfunctions of the related second-order elliptic operator were shown to be localized on this boundary (so-called astride localization). Other examples of localized eigenfunctions in planar regions can be found in [487]. A rigorous mathematical theory of these important phenomena is still lacking. Takeda et al. showed experimentally that, at a specific frequency, the electromagnetic field was confined in the central part of the third stage of three-dimensional fractals called the Menger sponge [481]. This localization was attributed to a singular photon density of states realized in the fractal structure.

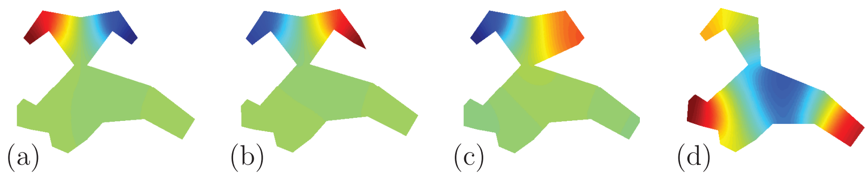


Fig. 7.11 *Neumann Laplacian eigenfunction u_4 in the original "cow" domain from* [233] (a) *and in three modified domains* (b)–(d)*, in which the reflection symmetry of the upper subdomain is broken. The fourth eigenfunction is localized for the first three domains* (a)–(c)*, while the last domain with the strongest modification shows no localization* (d)*. Colors represent the amplitude of an eigenfunction, from the most negative value (dark blue), through zero (green), to the largest positive value (dark red).*

A number of mathematical studies have been devoted to the theory of partial differential equations on fractals in general and to localization of Laplacian eigenfunctions in particular (see [280, 478] and references therein). For instance, the spectral properties of the Laplace operator on the Sierpinski gasket and its extensions were thoroughly investigated [42, 43, 44, 46, 75, 197, 457]. Barlow and Kigami studied the localized eigenfunctions of the Laplacian on a more general class of self-similar sets (so-called postcritically finite self-similar sets; see [281, 282] for details). They related the asymptotic behavior of the eigenvalue counting function to the existence of localized eigenfunctions and established a number of sufficient conditions for the existence of a localized eigenfunction in terms of the symmetries of a set [45].

Berry, Heilman, and Strichartz developed a new method to approximate the Neumann spectrum of the Laplacian on a planar fractal set Ω as a renormalized limit of the Neumann spectra of the standard Laplacian on a sequence of domains that approximate Ω from the outside [66]. They applied this method to compute the Neumann Laplacian eigenfunctions in several domains, including a sawtooth domain, Sierpinski gasket and carpet, as well as nonsymmetric and random carpets and the octagasket. In particular, they gave numerical evidence for the localized eigenfunctions for a sawtooth domain, in agreement with the earlier work by Félix et al. [175].

Heilman and Strichartz reported several numerical examples of localized Neumann Laplacian eigenfunctions in two domains [233], one of them illustrated in Figure 7.11a. Each of these domains consists of two subdomains with a narrow, but not too narrow, connection. This is a kind of dumbbell shape with a connector of zero length. Heilman and Strichartz argued that one subdomain must possess an axis of symmetry to obtain localized eigenfunctions. Since an antisymmetric eigenfunction vanishes on the axis of symmetry, it is necessarily small near the bottleneck that somehow prevents its extension to the other domain. Although the argument is plausible, we have to stress that such a symmetry is neither sufficient nor necessary for localization. It is obviously not sufficient because, even for symmetric domains, there exist plenty of extended eigenfunctions (including the trivial example of the ground eigenmode which is a constant over the whole domain). In order to illustrate that the reflection symmetry is not necessary, we plot in Figure 7.11b,c examples of localized eigenfunctions for modified domains for which the symmetry is broken. Although rendering the upper domain less and less symmetric gradually reduces or even fully destroys localization (see Figure 7.11d), its mechanism remains poorly understood. We also note that methods of section 7.4 are not applicable in this case because of the Neumann boundary condition.

Lapidus and Pang studied the boundary behavior of the Dirichlet Laplacian eigenfunctions and their gradients on a class of planar domains with fractal boundary, including the triangular and square von Koch snowflakes and their polygonal approximations [306]. Numerical evidence for the boundary behavior of eigenfunctions was reported in [305], with numerous pictures of eigenfunctions. Later, Daubert and Lapidus considered more specifically the localization character of eigenfunctions in von Koch domains [145]. In particular, different measures of localization were discussed.

Note also that Filoche and Mayboroda studied the problem of localization for the bi-Laplacian in rigid thin plates and discovered that clamping just one point inside such a plate not only perturbs its spectral properties, but essentially divides the plate into two independently vibrating regions [185].

7.7. High-Frequency Localization. A hundred years ago, Lord Rayleigh documented an interesting acoustical phenomenon in the whispering gallery under the dome of Saint Paul's Cathedral in London [420] (see also [416, 417]). A whisper of one person propagated along the curved wall to another person standing near the wall. Keller and Rubinow discussed the related "whispering gallery modes" and also "bouncing ball modes," and showed that these modes exist for a two-dimensional domain with an arbitrary smooth convex curve as its boundary [277]. A semiclassical approximation of Laplacian eigenfunctions in convex domains was developed by Lazutkin [33, 311, 312, 313, 314] (see also [13, 414, 415, 468]). Chen, Morris, and Zhou analyzed Mathieu and modified Mathieu functions and reported another type of localization called "focusing modes" [126]. All such eigenmodes become more and more localized in a small subdomain when the associated eigenvalue increases. This so-called high-frequency or high-energy limit was intensively studied for various domains, known as quantum billiards [222, 235, 255, 448, 475]. In quantum mechanics, this limit is known as the semiclassical approximation [65]. In optics, it corresponds to ray approximation of wave propagation, from which the properties of an optical, acoustical, or quantum system can often be reduced to the study of rays in classical billiards. Jakobson, Nadirashvili, and Toth gave an overview of many results on geometric properties of the Laplacian eigenfunctions on Riemannian manifolds, with special emphasis on the high-frequency limit (weak star limits, the rate of growth of L_p-norms, relationships between positive and negative parts of eigenfunctions, etc.) [255] (see also [4, 488]). Bearing in mind this comprehensive review, we start by illustrating high-frequency localization and related problems in simple domains such as the disk, ellipse, and rectangle for which explicit estimates are available. Subsequently, some results for quantum billiards are summarized.

7.7.1. Whispering Gallery Modes and Focusing Modes. The disk is the simplest shape to illustrate the whispering gallery modes and focusing modes. The explicit form (3.9) of eigenfunctions allows for accurate estimates and bounds, as shown below. When the index k is fixed and n increases, the Bessel functions $J_n(\alpha_{nk}r/R)$ are strongly attenuated near the origin (as $J_n(z) \sim (z/2)^n/n!$ at small z) and essentially localized near the boundary, yielding whispering gallery modes. In turn, when n is fixed and k increases, the Bessel functions rapidly oscillate, the amplitude of oscillations decreasing towards the boundary. In that case, the eigenfunctions are mainly localized at the origin, yielding focusing modes.

These qualitative arguments were rigorously formulated in [372]. For each eigenfunction u_{nk} on the unit disk Ω, one introduces the subdomain $\Omega_{nk} = \{x \in \mathbb{R}^2 \,:\, |x| <$

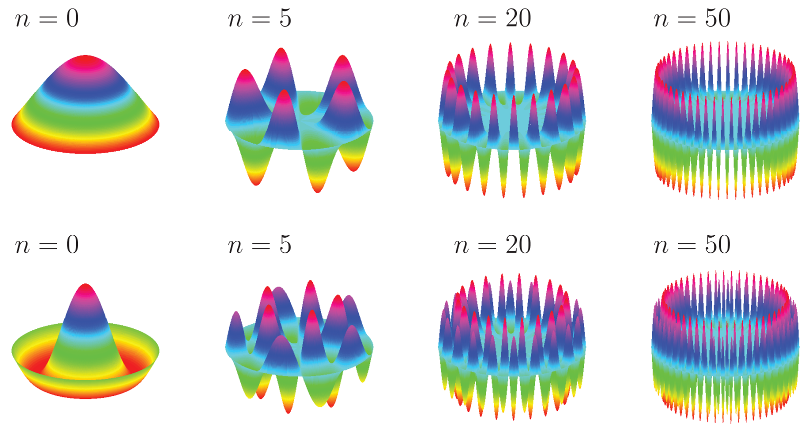


Fig. 7.12 *Formation of whispering gallery modes for the unit disk with Dirichlet boundary condition: for a fixed k ($k = 1$ for the top figures and $k = 2$ for the bottom figures), an increase in the index n leads to stronger localization of eigenfunctions near the boundary (from* [372]*).*

$d_n/\alpha_{nk}\} \subset \Omega$, where $d_n = n - n^{2/3}$, and α_{nk} are, depending on boundary conditions, the positive zeros of either $J_n(z)$ (Dirichlet), or $J_n'(z)$ (Neumann), or $J_n'(z) + hJ_n(z)$ for some $h > 0$ (Robin), with $n = 0, 1, 2, \ldots$ denoting the order of Bessel function $J_n(z)$ and $k = 1, 2, 3, \ldots$ counting zeros. Then, for any $p \geq 1$ (including $p = \infty$), there exists a universal constant $c_p > 0$ such that for any $k = 1, 2, 3, \ldots$ and any large enough n, the Laplacian eigenfunction u_{nk} for Dirichlet, Neumann, or Robin boundary conditions satisfies

$$\frac{\|u_{nk}\|_{L_p(\Omega_{nk})}}{\|u_{nk}\|_{L_p(\Omega)}} < c_p n^{\frac{1}{3}+\frac{2}{3p}} \exp(-n^{1/3}\ln(2)/3). \tag{7.11}$$

The definition of Ω_{nk} and the above estimate imply

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(\Omega_{nk})}}{\|u_{nk}\|_{L_p(\Omega)}} = 0, \qquad \lim_{n\to\infty} \frac{\mu_2(\Omega_{nk})}{\mu_2(\Omega)} = 1. \tag{7.12}$$

This result shows the existence of infinitely many Laplacian eigenmodes which are L_p-localized in a thin layer near the boundary $\partial\Omega$. In fact, for any prescribed thresholds for both ratios in (7.1), there exists n_0 such that for all $n > n_0$, the eigenfunctions u_{nk} are L_p-localized. These eigenfunctions are called whispering gallery modes and illustrated on Figure 7.12.

We outline a peculiar relation between high-frequency and low-frequency localization. The explicit form (3.9) of Dirichlet Laplacian eigenfunctions u_{nk} leads to their simple nodal structure, which is formed by $2n$ radial nodal lines and $k - 1$ circular nodal lines. The radial nodal lines split the disk into $2n$ circular sectors with Dirichlet boundary conditions. As a consequence, whispering gallery modes in the disk and the underlying exponential estimate (7.11) turn out to be related to the exponential decay of eigenfunctions in domains with branches (see section 7.4), as illustrated in Figure 7.4 for elongated triangles.

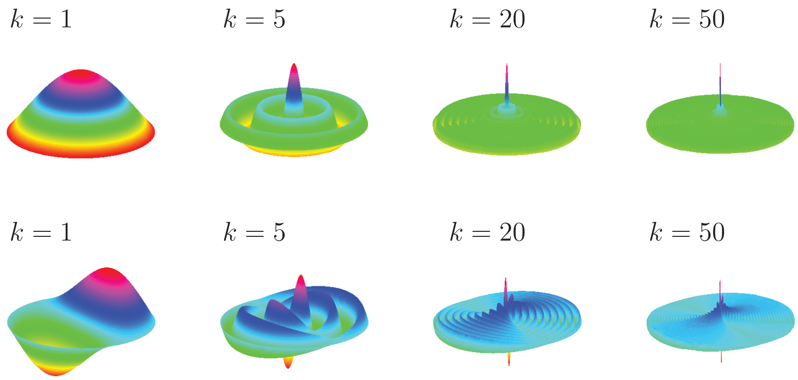


Fig. 7.13 *Formation of focusing modes for the unit disk with Dirichlet boundary condition: for a fixed n ($n = 0$ for the top figures and $n = 1$ for the bottom figures), an increase in the index k leads to stronger localization of eigenfunctions at the origin (from* [372]*).*

A simple consequence of the above result is that for any $p \geq 1$ and any open subset V compactly included in the unit disk Ω (i.e., $\bar{V} \cap \partial\Omega = \emptyset$),

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(V)}}{\|u_{nk}\|_{L_p(\Omega)}} = 0 \tag{7.13}$$

and

$$C_p(V) \equiv \inf_{nk} \left\{ \frac{\|u_{nk}\|_{L_p(V)}}{\|u_{nk}\|_{L_p(\Omega)}} \right\} = 0. \tag{7.14}$$

Qualitatively, for any subset V, there exists a sequence of eigenfunctions that progressively escape V.

The localization of focusing modes at the origin is revealed in the limit $k \to \infty$. For each $R \in (0,1)$, define an annulus $\Omega_R = \{x \in \mathbb{R}^2 \ : \ R < |x| < 1\} \subset \Omega$ of the unit disk Ω. Then, for any $n = 0, 1, 2, \ldots$, the Laplacian eigenfunction u_{nk} with Dirichlet, Neumann, or Robin boundary conditions satisfies

$$\lim_{k\to\infty} \frac{\|u_{nk}\|_{L_p(\Omega_R)}}{\|u_{nk}\|_{L_p(\Omega)}} = \begin{cases} (1 - R^{2-p/2})^{1/p} & (1 \leq p < 4), \\ 0 & (p > 4). \end{cases} \tag{7.15}$$

When the index k increases (with fixed n), the eigenfunctions u_{nk} become more and more L_p-localized near the origin when $p > 4$ [372]. These eigenfunctions are called focusing modes and are illustrated in Figure 7.13. The result shows that the definition of localization is sensitive to the norm: the focusing modes above are L_p-localized for $p > 4$, but they are not L_p-localized for $p < 4$. Similar results for whispering gallery modes and focusing modes hold for a ball in three dimensions [372].

7.7.2. Bouncing Ball Modes. Filled ellipses and elliptical annuli are simple domains for illustrating bouncing ball modes. For fixed foci (i.e., a given parameter a in the elliptic coordinates in (3.15)), these domains are characterized by two radii, R_0 ($R_0 = 0$ for filled ellipses) and R, as $\Omega = \{(r,\theta) \ : \ R_0 < r < R, \ 0 \leq \theta < 2\pi\}$, while

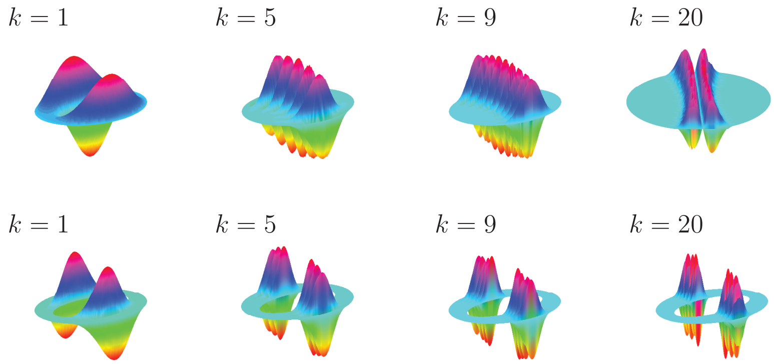


Fig. 7.14 *Formation of bouncing ball modes u_{nkl} in a filled ellipse of radius $R = 1$ (top) and an elliptical annulus of radii $R_0 = 0.5$ and $R = 1$ (bottom), with the focal distance $a = 1$ and Dirichlet boundary condition. For fixed $n = 1$ and $l = 1$, an increase in the index k leads to stronger localization of the eigenfunction near the vertical semiaxis (from* [372]*).*

the eigenfunctions u_{nkl} were defined in section 3.4. For each $\alpha \in \left(0, \frac{\pi}{2}\right)$, we consider an elliptical sector Ω_α inside an elliptical domain Ω (see Figure 3.1),

$$\Omega_\alpha = \{(r, \theta) \;:\; R_0 < r < R,\ \theta \in (\alpha, \pi - \alpha) \cup (\pi + \alpha, 2\pi - \alpha)\}\,.$$

For any $p \geq 1$, there exists $\Lambda_{\alpha,n} > 0$ such that for any eigenvalue $\lambda_{nkl} > \Lambda_{\alpha,n}$, the corresponding eigenfunction u_{nkl} satisfies [372] (see also [67])

(7.16)
$$\frac{\|u_{nkl}\|_{L_p(\Omega \backslash \Omega_\alpha)}}{\|u_{nkl}\|_{L_p(\Omega)}} < D_n \left(\frac{16\alpha}{\pi - \alpha/2}\right)^{1/p} \exp\left(-a\sqrt{\lambda_{nkl}}\left[\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \sin\alpha\right]\right),$$

where

$$D_n = 3\sqrt{\frac{1 + \sin\left(\frac{3\pi}{8} + \frac{\alpha}{4}\right)}{\left[\tan\left(\frac{\pi}{16} - \frac{\alpha}{8}\right)\right]^n}}.$$

Given that $\lambda_{nkl} \to \infty$ as k increases (for any fixed n and l), while the area of Ω_α can be made arbitrarily small by sending $\alpha \to \pi/2$, the estimate implies that there are infinitely many eigenfunctions u_{nkl} which are L_p-localized in the elliptical sector Ω_α:

$$\lim_{k\to\infty} \frac{\|u_{nkl}\|_{L_p(\Omega \backslash \Omega_\alpha)}}{\|u_{nkl}\|_{L_p(\Omega)}} = 0. \tag{7.17}$$

These eigenfunctions, which are localized near the minor axis, are called bouncing ball modes and are illustrated in Figure 7.14. The above estimate allows us to illustrate the bouncing ball modes which are known to emerge for any convex planar domain with smooth boundary [126, 277]. At the same time, the estimate is applicable to elliptical annuli, thus providing an example of bouncing ball modes for nonconvex domains.

7.7.3. Domains without Localization. The analysis of geometrical properties of eigenfunctions in rectangle-like domains $\Omega = (0, \ell_1) \times \cdots \times (0, \ell_d) \subset \mathbb{R}^d$ (with sizes $\ell_1 > 0, \ldots, \ell_d > 0$) may seem to be the simplest case because the eigenfunctions are expressed through sines (Dirichlet) and cosines (Neumann), as discussed in section 3.1. The situation is indeed elementary when all eigenvalues are simple, i.e., $(\ell_i/\ell_j)^2$ are not rational numbers for all $i \neq j$. For any $p \geq 1$ and any open subset $V \subset \Omega$, one can prove that [372]

$$C_p(V) \equiv \inf_{n_1,\ldots,n_d} \left\{ \frac{\|u_{n_1,\ldots,n_d}\|_{L_p(V)}}{\|u_{n_1,\ldots,n_d}\|_{L_p(\Omega)}} \right\} > 0. \tag{7.18}$$

This property is in sharp contrast to (7.14) for eigenfunctions in the unit disk (or ball). The fact that $C_p(V) > 0$ for any open subset V means that there is no eigenfunction that could fully avoid any location inside the domain, i.e., there is no L_p-localized eigenfunction. Since the set of rational numbers has zero Lebesgue measure, there are no L_p-localized eigenfunctions in almost any randomly chosen rectangle-like domain.

When at least one ratio $(\ell_i/\ell_j)^2$ is rational, certain eigenvalues are degenerate and their associated eigenfunctions are linear combinations of products of sines or cosines (see section 3.1). Although the computation is still elementary for each eigenfunction, it is unknown whether or not the infimum $C_p(V)$ from (7.18) is strictly positive for arbitrary rectangle-like domain Ω and any open subset V. For instance, the most general known result for a rectangle $\Omega = (0, \ell_1) \times (0, \ell_2)$ states that $C_2(V) > 0$ for any $V \subset \Omega$ of the form $V = (0, \ell_1) \times \omega$, where ω is any open subset of $(0, \ell_2)$ [110] (see also [335]). Even for the unit square, the statement $C_p(V) > 0$ for any open subset V is an open problem. More generally, one may wonder whether or not $C_p(V)$ is strictly positive for any open subset V in polygonal (convex) domains.

7.7.4. Quantum Billiards. The examples above of whispering gallery modes or bouncing ball modes illustrate that certain high-frequency eigenfunctions tend to be localized in specific regions of circular and elliptical domains. But what is the structure of a high-frequency eigenfunction in a general domain? What are these specific regions in which a sequence of eigenfunctions may be localized, and do localized eigenfunctions exist for a given domain? Answers to these and other related questions can be found by relating the high-frequency behavior of a quantum system (in our case, the structure of Laplacian eigenfunctions) to the classical dynamics in a billiard of the same shape [12, 129, 221, 271, 327]. This relation is also known as a semiclassical approximation in quantum mechanics and a ray approximation of wave propagation in optics, while the correspondence between classical and quantum systems can be shown using the WKB method, Eikonal equation, or Feynman path integrals [179, 180, 362]. For instance, the dynamics of a particle in a classical billiard is translated into a quantum mechanism through the stationary Schrödinger equation $Hu_n(x) = E_n u_n(x)$ with Dirichlet boundary condition, where the free Hamiltonian is $H = p^2/(2m) = -\hbar^2\Delta/(2m)$, and the energy E_n is related to the corresponding Laplacian eigenvalue $\lambda_n = 2mE_n/\hbar^2$. Since $|u_n(x)|^2$ is the probability density for finding a quantum particle at x, this density should resemble some classical trajectory of that particle in the (high-frequency) semiclassical limit ($\hbar \to 0$ or $m \to \infty$). In particular, some orbits of a particle moving in a classical billiard may appear as "scars" in the spatial structure of eigenfunctions in the related quantum billiard [31, 64, 164, 222, 223, 234, 235, 255, 267, 269, 447, 448, 474, 475]. This effect is illustrated in Figure 7.15 by Liu and coworkers, who investigated the localization

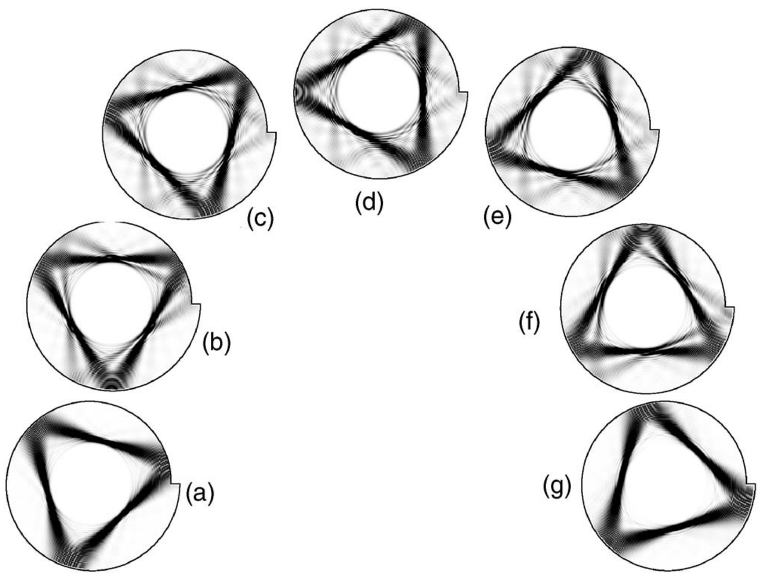


Fig. 7.15 *Several eigenstates with localization on periodic orbits for the spiral-shaped billiard with* $\epsilon = 0.1$*, from* [330]*. (Reprinted with permission from Liu et al., Physical Reviews E,* 74 *(*2006*),* 046214*. Copyright* (2006) *by the American Physical Society.)*

of Dirichlet Laplacian eigenfunctions on classical periodic orbits in a spiral-shaped billiard [330] (see also [317]).

In classical dynamics, one can distinguish domains with regular, integrable, and chaotic dynamics. In particular, for a bounded domain Ω with an ergodic billiard flow [465], Shnirelman's theorem (also known as the quantum ergodicity theorem [136, 510, 514]) states that among the set of L_2-normalized Dirichlet (or Neumann) Laplacian eigenfunctions, there is a sequence u_{j_k} of density 1 (i.e., $\lim_{k\to\infty} j_k/k = 1$), such that for any open subset $V \subset \Omega$, one has [460]

$$\lim_{k\to\infty} \int_V |u_{j_k}(x)|^2 dx = \frac{\mu_d(V)}{\mu_d(\Omega)} \tag{7.19}$$

(this version of the theorem was formulated in [110]). Marklof and Rudnick extended this theorem to rational polygons (i.e., simple plane polygons whose interior is connected and simply connected and whose vertex angles are all rational multiples of π) [335]. Loosely speaking, $\{u_{j_k}\}$ is a sequence of nonlocalized eigenfunctions which become more and more uniformly distributed over the domain (see [110, 202, 255] for further discussion and references). At the same time, this theorem does not prevent the existence of localized eigenfunctions. How large might the excluded subsequence of (localized) eigenfunctions be? In the special case of arithmetic hyperbolic manifolds, Rudnick and Sarnak proved that there is no such excluded subsequence [433]. This statement is known as quantum unique ergodicity (QUE). Its validity for other dynamical systems (in particular, ergodic billiards) remains under investigation [47, 159, 228]. The related notion of weak quantum ergodicity was discussed by Kaplan and Heller [268]. A classification of eigenstates as regular and irregular was thoroughly discussed (see [412, 497] and references therein).

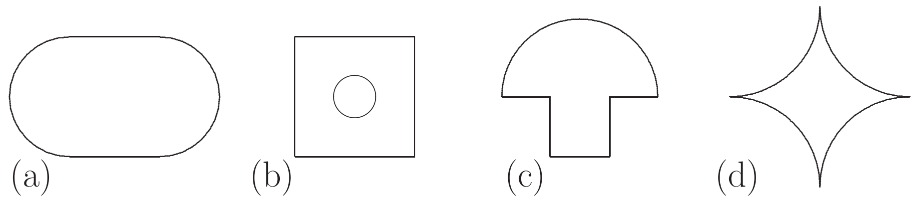


Fig. 7.16 *Examples of chaotic billiards:* (a) *Bunimovich stadium (union of a square and two half-disks)* [69, 98, 99, 131, 234, 484, 485]*;* (b) *Sinai's billiard* [463, 464]*;* (c) *mushroom billiard* [49, 100]*; and* (d) *hyperbolic billiard* [2]*. Many other examples are given in* [99].

There have been numerous studies of Laplacian eigenfunctions in chaotic domains such as, e.g., the Bunimovich stadium [69, 98, 99, 131, 234, 376, 484, 485], Sinai's billiard [463, 464], the mushroom billiard [49, 100], and hyperbolic billiard [2], illustrated in Figure 7.16. The literature on quantum billiards is vast, and we mention only selected works on the spatial structure of high-frequency eigenfunctions. McDonald and Kaufman studied the Bunimovich stadium billiard and reported a random structure of nodal lines of eigenfunctions and Wigner-type distribution for eigenvalue spacings [349, 350]. Bohigas, Gianoni, and Schmit analyzed eigenvalue spacings for Sinai's billiard and also obtained a Wigner-type distribution [81], which means that eigenvalue spacings for these chaotic billiards obey the same distribution as that for random matrices from the Gaussian orthogonal ensemble. This is in sharp contrast to regular billiards, for which eigenvalue spacings generally follow a Poisson distribution. The problem of circular-sector and related billiards was studied, for example, in [323].

Polygon billiards have attracted a lot of attention, especially the class of rational polygons for which all the vertex angles are rational multiples of π [325, 326, 428]. As the dynamics in rational polygons is neither integrable nor ergodic (except for several classical integrable cases such as rectangles, equilateral triangles, and right triangles with an acute vertex angle $\pi/3$ or $\pi/4$), they are often called *pseudointegrable* systems. Bellomo and Uzer studied scarring states in a pseudointegrable triangular billiard and detected scars in regions which contain no periodic orbits [57]. Amar, Pauri, and Scotti gave a complete characterization of the polygons for which a Dirichlet eigenfunction can be found in terms of a finite superposition of plane waves [7, 8] (see also [310] for an experimental study). Biswas and Jain investigated in detail the $\pi/3$-rhombus billiard which presents an example of the simplest pseudointegrable system [72]. Hassell, Hillairet, and Marzuola proved for an arbitrary polygonal billiard that eigenfunction mass cannot concentrate away from the vertices [229] (see also [336]). The level spacing properties of rational and irrational polygons were studied numerically by Shimizu and Shudo [458], who also analyzed the structure of the related eigenfunctions [459].

Bäcker, Schubert, and Stifter analyzed the number of bouncing ball modes in a class of two-dimensional quantized billiards with two parallel walls [35]. Bunimovich introduced a family of simple billiards (called "mushrooms") that demonstrate a continuous transition from a completely chaotic system (stadium) to a completely integrable one (circle) [100]. Barnett and Betcke reported the first large-scale statistical study of very high frequency eigenfunctions in these billiards [49]. Using nonstandard numerical techniques [47], Barnett also studied the rate of equidistribution for a uniformly hyperbolic, Sinai-type, planar Euclidean billiard with Dirichlet boundary

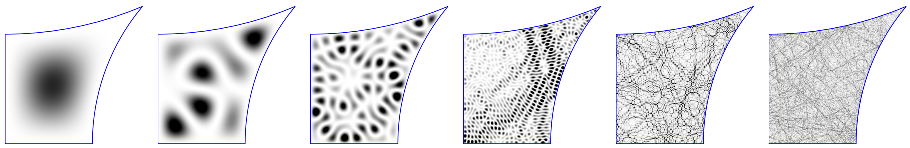


Fig. 7.17 *Illustration of the spatial distribution of the Dirichlet Laplacian eigenfunction* $|u_m|^2$ *(shown as density plots: larger values are darker) with* $m = 1, 10, 100, 1000$ *and* $m \approx 50000$ *(by A. Barnett, with permission).*

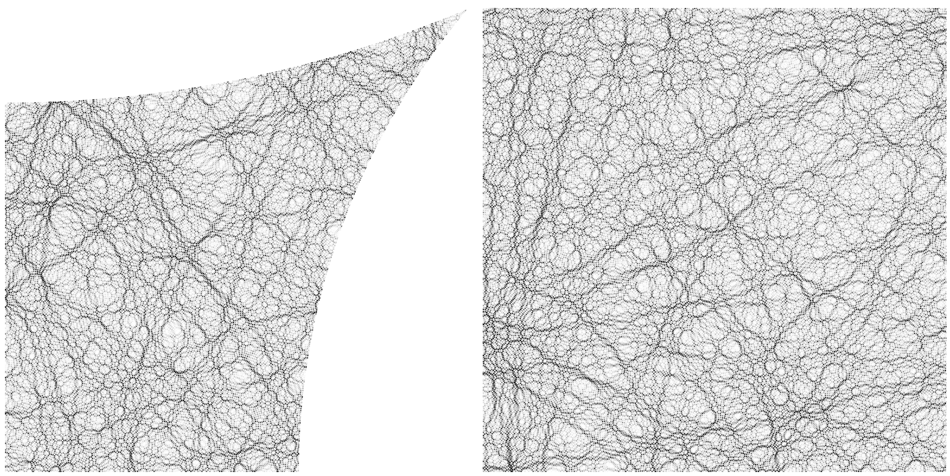


Fig. 7.18 *(Left) Density plot of a Dirichlet Laplacian eigenfunction* $|u_m|^2$ *for* $m \approx 50000$ *with the eigenvalue* $\lambda_m \approx 10^6$*. There are about 225 wavelengths across the diagonal. (Right) Density plot of one sample from the ensemble of random plane waves with the same wavenumber and mean intensity, shown in a square region of space (with no boundary conditions) (by A. Barnett, with permission).*

conditions, as illustrated in Figure 7.17. This study gave strong numerical evidence for the QUE in this system. The spatial structure of high-frequency eigenfunctions shown in Figure 7.17 looks somewhat random. This observation goes back to Berry, who conjectured that high-frequency eigenfunctions in domains with ergodic flow should look locally like a random superposition of plane waves with a fixed wavenumber [61]. This analogy is illustrated in Figure 7.18. O'Connor, Gehlen, and Heller analyzed the random pattern of ridges in a random superposition of plane waves [375].

Pseudointegrable barrier billiards were intensively studied in a series of theoretical, numerical, and experimental works by Bogomolny et al. [77, 79]. They reported on the emergence of scarring eigenstates which are related to families of classical periodic orbits that do not disappear at large quantum numbers, in contrast to the case of chaotic systems. These so-called superscars were observed experimentally in a flat microwave billiard with a barrier inside [77]. Wiersig performed an extensive numerical study of nearest-neighbor spacing distributions, next-to-nearest spacing distributions, number variances, spectral form factors, and level dynamics [505]. Dietz and coworkers analyzed the number of nodal domains in a barrier billiard [156].

Tomsovic and Heller verified the remarkable accuracy of the semiclassical approximation that relates classical and quantum billiards [484, 485]. In some cases, eigenfunctions can therefore be constructed by purely semiclassical calculations. Li, Reichl, and Wu studied the spatial distribution of eigenstates of a rippled billiard with sinusoidal walls [322]. For this type of ripple billiards, a Hamiltonian matrix can be found exactly, in terms of elementary functions, that greatly improves computation efficiency. They found both localized and extended eigenfunctions, as well as peculiar hexagon and circle-like pattern formations. Frahm and Shepelyansky considered almost circular billiards with a rough boundary which they realized as a random curve with some finite correlation length. At first glance, it may seem that such a rough boundary in a circular billiard would destroy the conservation of angular momentum and lead to ergodic eigenstates and the level statistics predicted by random matrix theory. They showed, however, that there is a region of roughness in which the classical dynamics is chaotic but the eigenstates are localized [190]. Bogomolny, Giraud, and Schmit presented the exact computation of the nearest-neighbor spacing distribution for a rectangular billiard with a point-like scatterer inside [78].

Prosen computed numerically very high-lying energy spectra for a generic chaotic three-dimensional quantum billiard (a smooth deformation of a unit sphere) and analyzed Weyl's asymptotic formula and the nearest-neighbor level spacing distribution. He found significant deviations from the Gaussian orthogonal ensemble statistics that were explained in terms of localization of eigenfunctions on lower-dimensional classically invariant manifolds [410]. He also found that the majority of eigenstates were more or less uniformly extended over the entire energy surface, except for a fraction of strongly localized scarred eigenstates [411]. An extensive study of the three-dimensional Sinai billiard was reported by Primack and Smilansky [409]. Deviations from a semiclassical description were discussed by Tanner [482]. Casati and coworkers investigated how the interplay between quantum localization and the rich structure of the classical phase space influences the quantum dynamics, with applications to hydrogen atoms under microwave fields [119, 120, 121, 122] (see also references therein).

A large number of physical experiments have been performed with classical and quantum billiards. For instance, Gräf and coworkers measured more than a thousand first eigenmodes in a quasi-two-dimensional superconducting microwave stadium billiard with chaotic dynamics [210]. Sridhar and coworkers performed a series of experiments in microwave cavities in the shape of Sinai's billiard [470, 471]. In particular, they observed bouncing ball modes and modes with quasi-rectangular or quasi-circular symmetry, which are associated with nonisolated periodic orbits (which avoid the central disk). Some scarring eigenstates, which are associated with isolated periodic orbits (which hit the central disk; see Figure 7.16b), were also observed. Kudrolli et al. investigated the signatures of classical chaos and the role of periodic orbits in the eigenvalue spectra of two-dimensional billiards through experiments in microwave cavities [297, 298]. The eigenvalue spectra were analyzed using the nearest-neighbor spacing distribution for short-range correlations and spectral rigidity for longer-range correlations. The density correlation function was used to study the spatial structure of eigenstates. The role of disorder was also investigated. Chinnery and Humphrey experimentally visualized acoustic resonances within a stadium-shaped cavity [131]. Bittner et al. performed double-slit experiments with regular and chaotic microwave billiards [73]. Chaotic resonators were also employed to find specific properties of lasers (e.g., high-power directional emission or Fresnel filtering) [205, 426].

8. Further Points and Concluding Remarks. This review has focused on the geometrical properties of Laplacian eigenfunctions in Euclidean domains. We started from the basic properties of the Laplace operator and explicit representations of its eigenfunctions in simple domains. Subsequently, the properties of the eigenvalues and their relationship to the shape of a domain were briefly summarized, including Weyl's asymptotic behavior, isoperimetric inequalities, and Kac's inverse spectral problem. The structure of nodal domains and various estimates for the norms of eigenfunctions were then presented. The main section, section 7, was devoted to the spatial structure of eigenfunctions, with special emphasis on their localization in small subsets of a domain. One of the major difficulties in the study of localization is that localization is a property of an individual eigenfunction. For the same domain, two consecutive eigenfunctions with very close eigenvalues may have drastically different geometrical structures (e.g., one is localized and the other is extended). Therefore, fine analytical tools that operate differently with localized and nonlocalized eigenfunctions are needed. In this review, we distinguished two types of localization for low-frequency and high-frequency eigenfunctions.

In the former case, an eigenfunction remains localized in a subset due to a geometric constraint that prohibits its extension to other parts of the domain. A standard example is a dumbbell (two domains connected by a narrow channel), for which an eigenfunction may be localized in one domain if its wavelength is larger than the width of the channel (meaning that an eigenfunction cannot "squeeze" through the channel). Such "expulsion" from a channel is quite generic, as the analysis is applicable to domains with branches of variable cross-sectional profiles. It is important to note that a geometric constraint does not need to be strong (e.g., two domains may be separated by a cloud of point-like obstacles of zero measure). Another example is an elongated triangle, in which there are no "obstacles" at all. Low-frequency localization was found numerically in many irregularly shaped domains, for both Dirichlet and Neumann boundary conditions. From a practical point of view, low-frequency localization may find various applications in, e.g., the theories of quantum, optical, and acoustical waveguides and microelectronic devices, as well as for analysis and engineering of highly reflecting or absorbing materials (noise protective barriers, antiradar coatings, etc.).

High-frequency localization manifests in quantum billiards when a sequence of eigenfunctions tends to concentrate on some orbits of the associated classical billiard. In this regime, the asymptotic properties of eigenvalues and eigenfunctions are strongly related to the underlying classical dynamics (e.g., regular, integrable, or chaotic). For instance, the ergodic character of the classical system is reflected in the spatial structure of the eigenfunctions. Working on simple domains, we illustrated several kinds of localized eigenfunctions which emerge for a large class of domains. We also provided examples of rectangular domains without localization. Although a number of rigorous and numerical results were obtained (e.g., the quantum ergodicity theorem for ergodic billiards), many questions about the spatial structure of high-frequency eigenfunctions remain open, even for very simple domains (e.g., a square).

Although this review counts more than 500 citations, it is far from complete. As already mentioned, we have focused on the Laplace operator in bounded Euclidean domains and have mostly omitted technical details, in order to keep the review at a level accessible to scientists from various fields. Many other issues also had to be omitted:

(i) Many important results for Laplacians on Riemannian manifolds or weighted graphs could not be included. In addition, we did not discuss the spectral properties of domains with "holes" [130, 188, 287, 302, 340, 341, 342, 382, 383, 440, 449, 499], nor their consequences for diffusion in domains with static traps [211, 275, 276, 371, 486]. The behavior of the eigenvalues and eigenfunctions under deformations of a domain was partly considered in sections 6.1 and 7.5, while many significant results were not included (see [91, 270, 279, 424, 462] and references therein).

(ii) There have been important developments of numerical techniques for computing the Laplacian eigenbasis. In fact, standard finite difference or finite element methods rely on a regular or adapted discretization of a domain that reduces the continuous eigenvalue problem to a finite set of linear equations [134, 141, 193, 218, 242, 299, 436]. Since finding the eigenbasis of the resulting matrix is still an expensive computational task, various hints and tricks are often implemented. For instance, for planar polygonal domains, one can exploit the behavior of eigenfunctions at corners through radial basis functions in polar coordinates and the integration of related Fourier–Bessel functions on subdomains [153, 162, 399]. Another "trick" is conformal mapping of planar polygonal domains onto the unit disk, for which the modified eigenvalue problem can be efficiently solved [37, 38]. Yet another approach, known as the method of particular solutions, was suggested by Fox, Henrici, and Moler [189] and later progressively improved [34, 48, 68]. The main idea is to consider various solutions of the eigenvalue equation for a given value of λ and to vary λ until a linear combination of such solutions satisfies the boundary condition at a number of sample points along the boundary. We also mention a stochastic method from Lejay and Maire for computing the principal eigenvalue [318]. The eigenvalue problem can also be reformulated in terms of boundary integral equations that reduce the dimensionality and allow for rapid computation of eigenvalues [332]. Kaufman, Kosztin, and Schulten proposed a simple expansion method in which wave functions inside a two-dimensional quantum billiard are expressed in terms of an expansion of a complete set of orthonormal functions defined in a surrounding rectangle for which Dirichlet boundary conditions apply, while approximating the billiard boundary by a potential energy step of a sufficiently large size [273]. Vergini and Saraceno proposed a scaling method for computing high-frequency eigenmodes [498]. This method was later improved by Barnett and Hassell [50] (this reference also contains a good review of numerical methods for high-frequency Dirichlet Laplacian eigenvalues).

(iii) We did not discuss various applications of Laplacian eigenfunctions, which nowadays range from pure and applied mathematics to physics, chemistry, biology, and computer sciences. Examples include manifold parameterizations by Laplacian eigenfunctions and heat kernels [265], the use of Laplacian spectra as a diagnostic tool for network structure and dynamics [351], efficient image recognition and analysis [425, 427, 437, 438], shape optimization and spectral partition problems [6, 93, 112, 114, 398, 469, 496], computation and analysis of diffusion-weighted NMR signals [213, 214, 215], localization in heterogeneous materials (e.g., photonic crystals), and the related optimization problem [158, 181, 182, 183, 184, 262, 263, 285, 286, 294, 296, 439].

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