ANOMALOUS DIFFUSION IN DISORDERED MEDIA: STATISTICAL MECHANISMS, MODELS AND PHYSICAL APPLICATIONS

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Abstract

The subject of this paper is the evolution of Brownian particles in disordered environments The "Ariadne's clew" we follow is understanding of the general statistical mechanisms which may generate "anomalous" (non-Brownian) diffusion laws, this allows us to develop simple arguments to obtain a qualitative (but often quite accurate) picture of most situations Several analytical techniques – such as the Green function formalism and renormalization group methods – are also exposed Care is devoted to the problem of sample to sample fluctuations, particularly acute here We consider the specific effects of a bias on anomalous diffusion, and discuss the generalizations of Einstein's relation in the presence of disorder An effort is made to illustrate the theoretical models by describing many physical situations where anomalous diffusion laws have been – or could be – observed

Introduction

This article deals with Brownian motion (classical diffusion) in inhomogeneous media It mainly concentrates on cases in which the inhomogeneities can be modelled as a quenched disorder on the local hopping rates ("geometrical disorder" arising in fractal structures as the percolation network is also briefly considered, in chapter 6)

Besides the observation that Brownian motion is one of the classical and best understood problems of physics and that studying its behaviour in the presence of disorder is quite natural, there are several physical phenomena motivating this study. These are mainly *transport* processes, e g. the diffusion of a tracer in inhomogeneous hydrodynamic flows under the combined action of molecular diffusion and convection along the flow lines (porous media, section 5 6, turbulent flow, section 4 1 2 1, array of convection rolls section 1.2.3 4), or the diffusion of a charge carrier (electron, hole, ion) in a conductor with impurities in regimes where conduction can be modelled as a classical process involving independent carriers (sections 5.4.2 and 5.4 3)

Another type of motivation, perhaps more remote, has to do with *relaxation* properties of disordered systems (e.g random field magnets or spin glasses) This is for at least two different reasons First, because important processes contributing to the overall dynamics—like domain wall or defect motion—are diffusion processes (see section 3.3.7). Second, because relaxation can be traced back to the diffusion properties in the configuration space of the system, which is indeed known to have a complicated structure for, e g , spin glasses. While the study of a random walk in a disordered energy landscape (section 4 1.2.2) is certainly a very crude caricature of such a process, it can be a first, question raising attempt to deal with the effect of disorder on dynamical properties

While this article is primarily a theoretical one, an effort will be made to motivate the models by discussing in some detail several physical situations and experiments

The disorder can have two kinds of effects on diffusion properties·

- it may affect only the *value* of the transport coefficients (velocity, diffusion constant, etc.) as compared to the ordered system,
- or it may alter in various ways the very *laws* of Brownian motion (e g. the mean-square position may no longer increase linearly in time at large times).

We shall be mainly concerned here with "anomalous diffusion" phenomena, where the second kind of effect takes place Since the usual laws of Brownian motion result from the central limit theorem of probability theory, the usual form of the latter has to fail whenever anomalous diffusion occurs. As will be illustrated at length in chapter 1, this can be due to the presence of either "broad distributions" (with diverging first or second moment) or of "long-range" correlations These statistical mechanisms can be present a priori in the problem at hand for some underlying physical reason (e g long-range correlations in the velocity field of a turbulent flow) or, most interestingly, they can be induced by the dynamics itself (see, e.g., section 3 3). It is one of the main themes of this article to unveil the presence of one (or both) of these mechanisms in all the models and physical situations described Identifying these statistical mechanisms provides us with a unified framework; this is indeed most useful both for model building (when looking for the physical origin of some observed anomalous diffusion) and for analyzing models without resorting to ponderous theoretical apparatus.

Such situations, in which the disorder has such strong consequences, also require specific techniques. In particular, all mean-field types of approach (e g effective-medium approximations) which attempt to replace the disordered system by an equivalent ordered one following some averaging technique, can only fail when anomalous diffusion takes place Indeed, viewing in the usual way the large-time limit of

a random walk as a critical phenomenon, it appears that anomalous diffusion does correspond to departure from mean-field behaviour· a non-trivial fixed point appears, corresponding to a non-Gaussian central limit theorem (sections 4 2 2 3 and 4.3.1). Besides, as for any physical effect specific of disorder, attention should be paid to the question of ensemble averaging and of sample to sample fluctuations, random walks in random media provide a remarkable example of problems in which both fixed environment and disorder-averaged properties have a direct physical meaning (corresponding to the study of an initial distribution of walkers initially concentrated or spread out over the sample, respectively) For these reasons, among others, diffusion in disordered media provides an interesting test field for methods in the theory of disordered systems

In addition to the general tools of probability theory, several more specific analytic techniques will be reviewed in some detail in this article This encompasses mainly Green function and steady-state methods in one dimension (where a number of exact results can be obtained) and renormalization group methods in higher-dimensional cases The probabilistic meaning of the latter will be emphasized.

As the list of references testifies, diffusion in inhomogeneous media has been the subject of numerous works in the past ten years and is still quite an active field Two review articles have recently appeared [G12, G13] and the present one is meant to be complementary to both, with very limited overlap The one by Havlin and Ben Avraham [G12] concentrates on fractal structures rather than on quenched disorder (with relative weights roughly inverted as compared to the present paper), and the article by Haus and Kehr [G13] focusses on situations in which diffusion is normal. An older review article by Alexander et al [G10] dealt only with the case of symmetric random barriers in one dimension Most analytical methods (like the Green function and steady-state techniques of chapter 3 for the general asymmetric case in one dimension, and the renormalization group methods of chapter 4) have never been reviewed. Besides, the present article contains new results which have not been published elsewhere It is also intended to discuss several physical motivations and experiments, which has not been attempted before

The present paper should thus not be considered as a review article in the usual sense It is intended to provide an original view of the field in a unified language, and as such certainly suffers from numerous omissions The following topics in particular will deliberately not be addressed:

- most situations in which the inhomogeneities are neither modelled as quenched disorder nor as fractal-type (geometrical) disorder; this encompasses mainly the cases with "*deterministic disorder*" like diffusion on hierarchical structures or with hierarchical transition rates (see, e.g , [Hub85, Gro85, Pal85, Tei85, Bah87] and references therein), or diffusion in the phase space of dynamical systems (with the exception of the Lorentz gas of section 1 2 2 1 and the intermittent system of section 1.2.3 3); these are subjects which by themselves would deserve an independent paper,
- interactions among the diffusing particles are always neglected,
- all problems in which the total probability is *not conserved*, like, e.g., diffusion among infinitely deep traps (see [Don79, Gra82, Nie89]) or in the presence of chemical reactions;
- the effects of possible *quantum mechanical* couplings between the diffusing particle and the impurities, leading to entirely new physics,
- finally, the paper is restricted to the diffusion of *pointlike particles* (maybe with the exception of dislocation motion in section 3.3.7 3); it would probably be premature to attempt a review of the still growing number of studies of the diffusion of lines (polymer reptation, DNA chain in a gel, etc) or interfaces (domain wall motion, etc.) in the presence of disorder and of the new effects (pinning, etc.) induced by the internal structure.

Let us end this introduction by providing some guide to the reader through this article. It is divided in six chapters, the detailed contents of which can be found in the table of contents. It should be clear

from this table of contents that many sections can (and must) be bypassed in a first reading. Indeed, the paper is organized as a main "backbone", along which numerous illustrative physical examples branch off Smaller characters have been used to mark these sections Chapter 1 in particular is meant to be a general introduction to the main probabilistic concepts and mechanisms· the numerous examples it contains deal with anomalous diffusion but not with media with quenched disorder. The study of the latter begins with chapter 2 For the sake of convenience, an index can be found at the end of the article

1. Statistical mechanisms. Broad distributions and long-range correlations

> "In fact, all epistemologic value of the theory of probability is based on this that large-scale random phenomena in their collective action create strict, non random regularity "
>
> B V Gnedenko and A N Kolmogorov, Limit Distributions for Sums of Independent Random Variables [G2]

This chapter should be considered as a general introduction to the probabilistic concepts that will be used throughout this article It is meant to illustrate the idea which pertains to all the problems discussed in this article, namely that anomalous diffusion phenomena correspond to physical instances in which *the central limit theorem (CLT) no longer holds in its usual form.* This idea is indeed useful in practice: Once the statistical mechanisms responsible for this unusual circumstance are identified, very simple statistical arguments (or even precise mathematical results) can often be used in order to understand how the resulting diffusion behaviour is modified. Several illustrations will be given in this chapter Most of them do not deal, strictly speaking, with disordered media (except section 1 3 2); this is the subject of the following chapters (2–6).

In section 1.1, very elementary properties of random walks and Brownian motion are discussed These properties follow from the CLT in its usual form (see, e g , ref. [G1]), which applies in an overwhelming majority of cases, that is, provided the two following conditions are satisfied (stated here in a somewhat loose way, to be made precise later on)

(i) The distribution of the summed random variables must not be "too broad" (a sufficient condition is, in particular, the finiteness of its second moment)

(ii) These random variables must not be "long-range correlated".

We study in the two following sections the situations in which one of these conditions is not satisfied, (i) in section 1.2, (ii) in section 1.3, making use, whenever possible, of simple statistical reasonings. The first case, involving "broad" distributions, is well understood: The works of P. Levy and A Khintchine in the 1930s [G1–G3] have shown how to extend the CLT to such situations A summary of these mathematical results is given in appendix B, while several physical illustrations are presented in the text No such general understanding is available in the case of strongly correlated random variables (section 1 3) This is indeed a very difficult question· a self-avoiding walk (polymer) or the physics of critical phenomena are examples of such a situation, as will be made clear in sections 1 3 3 and 1 3 4

1.1. Random walks, normal diffusion and the central limit theorem

Let us first consider the very simple one-dimensional example of a walker performing at each time n

a jump of length l_n independently chosen at each time according to a given distribution $p(l)$ Its position after N steps, i e at time $t = N\tau$, is the sum of N independent displacements,

$$X_t = \sum_{n=1}^{N} l_n \tag{1 1}$$

Provided the first two moments $\langle l \rangle$ and $\langle l^2 \rangle$ of $p(l)$ are finite, the mean and the variance of the position depend linearly on time,

$$\bar{X}_t = Vt\,, \quad \overline{X_t^2} - \bar{X}_t^2 = 2Dt\,, \tag{1 2}$$

where the velocity and the diffusion constant are given by

$$V = \langle l \rangle / \tau\,, \qquad D = (2\tau)^{-1}[\langle l^2 \rangle - \langle l \rangle^2] \tag{1 3}$$

This is the well-known behaviour of (biased) Brownian motion, which we shall call "normal diffusion" throughout this article We shall speak of "anomalous diffusion" (of "anomalous drift") whenever the variance of the position X_t (its average) no longer increases linearly with time at large time

A more precise characterization of the random walk (1.1) is provided by the CLT, which states that, when the above conditions are satisfied, the distribution of the position X_t ("diffusion front") takes at large times a Gaussian (or "normal") form,

$$\text{Probability}\{u_1 \le (X_t - Vt)/2\sqrt{Dt} \le u_2\} \xrightarrow[t\to\infty]{} \frac{1}{\sqrt{\pi}} \int_{u_1}^{u_2} \mathrm{e}^{-\xi^2}\, \mathrm{d}\xi \tag{1 4}$$

Let us sketch an elementary derivation of this classical result in probability theory (see, for example, ref. [G1]) which, in this form, goes back to Laplace (1812) [G9] Up to a translation of the reference frame, one can always suppose that $\langle l \rangle = 0$ X_t then behaves typically as $N^{1/2}$, and it is the distribution of the variable $X_t/\sqrt{N}$ which admits a limiting form This distribution reads

$$\int \cdots \int \prod_{n=1}^{N} p(l_n)\, \mathrm{d}l_n\, \delta\Big(\frac{X_t}{\sqrt{N}} - N^{-1/2} \sum_{n=1}^{N} l_n \Big) \tag{1 5}$$

Using an integral representation of the δ-function, this can be written as

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kX_t/\sqrt{N}} \Big(\int \mathrm{d}l\, p(l)\, \mathrm{e}^{-\mathrm{i}kl/\sqrt{N}} \Big)^N \tag{1 6}$$

This expression involves the Fourier transform (or "characteristic function") $\hat{p}(k)$ of $p(l)$ At large times, only the behaviour of $p(k)$ close to $k = 0$ matters, since $|p(k)| \le 1$ Since

$$\hat{p}(k/\sqrt{N})^N = [1 - \tfrac{1}{2}\langle l^2 \rangle k^2/N + \mathrm{O}(N^{-3/2})]^N \xrightarrow[N\to\infty]{} \mathrm{e}^{-\langle l^2 \rangle k^2/2}\,, \tag{1 7}$$

the integration over k leads to the Gaussian form (1 4)

This derivation calls for several important remarks:

(i) At large times, the limit distribution (1 4) involves only the first two moments $\langle l \rangle$ and $\langle l^2 \rangle$ All further details of the structure of $p(l)$ (higher-order moments, etc) only contribute outside the *scaling region* in which (1 4) holds. Indeed,

(ii) the CLT only specifies the limit distribution within the *scaling region*, where the variable $(X_t - Vt)/\sqrt{Dt}$ takes finite values, corresponding to *typical values* of X_t. Outside this region, the limit distribution $P(X, t)$ will generally have *tails* which are not described by (1 4). These tails can be called "non-universal", since they depend on all the details of the original distribution $p(l)$. (The case where the jumps take the two values $l_n = \pm 1$ with equal probability is a simple illustration of this point in this case the *exact* distribution is a binomial law). An explicit characterization of the difference between the limit distribution and a normal law for large but finite time is provided by the Chebyshev expansion in powers of $1/\sqrt{t}$, which involves the higher-order moments of $p(l)$ and is described in appendix B

(iii) A natural question to be asked is to characterize *all* the distributions $p(l)$ for which the limit distribution of the sum X_t conveniently rescaled is the normal law or, in other words, to identify the "attraction basin" of the normal law. Theorem 1 of appendix B answers this question completely. The finiteness of the second moment $\langle l^2 \rangle$ is in fact not a necessary condition As an example, the attraction basin of the normal law contains the distributions $p(l)$ decreasing as l^{-3} for large l, for which $\langle l^2 \rangle = +\infty$ X_t^2 is then typically of order $t \ln t$ We shall see in the next section (1 2.2.1) a physical application of this result

(iv) Let us finally note that the large-time limit for a random walk can be seen as a critical phenomenon. This is clear from the above remarks: small number of relevant parameters, universal scaling law for the limit distribution within a certain attraction basin, etc It is the inverse of the time t (or more precisely E, its conjugate parameter by Laplace transform) which plays the role of the distance $T - T_c$ to the critical temperature. The typical position X_t is to be interpreted as a correlation length $\xi(t)$ (diverging for normal diffusion with the mean-field exponent $\nu = 1/2$), and the distribution $P(X, t)$ as a correlation function $\langle S(0)S(X) \rangle_T$ (see section 4 2.2.3 and table 4.1 for elaboration on this equivalence)

The CLT can be readily generalized to random walks on regular lattices of arbitrary dimensionality (see appendix A) Since only the vicinity of $\boldsymbol{k} = 0$ is of any importance, as in the above derivation, the asymptotic form of $P(X, t)$ keeps no trace of the underlying lattice structure In a suitably normalized basis of unit vectors, this asymptotic form is a simple d-dimensional generalization of (1.4),

$$P(X, t) \to (4\pi Dt)^{-d/2} \exp[-(X - Vt)^2/4Dt] . \tag{1.8}$$

Some properties of random walks (distribution of first passage times, number of different sites visited, etc) will play an important role in the following – as well as some analytical tools needed to derive them (master equation, generating functions, etc.) Since these are standard topics which can be found in several excellent books and review articles (e.g. refs [G5–G9]), they will not be further detailed here. Rather, for the sake of convenience, some of these properties and techniques are briefly presented in appendix A.

The normal form of the CLT presented in this section applies provided the random variables which are summed satisfy the two conditions already mentioned at the beginning of this section (not too broad distribution or too long-ranged correlations). Anomalous diffusion arises whenever one of these conditions is not satisfied. This may of course result from some "ad hoc" hypothesis of the model at hand. But the physically most interesting situations are those in which such "pathologies" are induced

by the dynamics itself This is the case for almost all physical situations dealt with in the following. It is amusing, for example, to realize that a broad distribution is in fact induced in such a familiar problem as the one-dimensional unbiased Brownian motion. Indeed, the distribution of the first return time to the origin decays as (cf appendix A)

$$P_1(t) \sim t^{-3/2} \quad (\text{for } V=0)\,, \tag{1 9}$$

and P_1 has an infinite mean value! This is at the origin of geometry-induced anomalous diffusion on comb-like structures, as explained in section 1.2.3 (see also section 6 3 3)

1 2 The central limit theorem for broad distributions and physical applications

> "All these distribution laws, called stable, ( ) deserve the most serious attention It is probable that the scope of applied problems in which they play an essential role will become in due course rather wide "
>
> B V Gnedenko and A N Kolmogorov (op cit)

1.2 1. Sums of independent random variables; stable laws

1 2 1 1 General theory The problem considered in this section is to characterize the limit distribution of the sum of *independent* random variables,

$$X_N = \sum_{n=1}^{N} l_n\,, \tag{1 10}$$

when the distribution $p(l)$ is "broad" (that is, decreases more slowly than l^{-3} for large l)

This question is a classic in the theory of probability It has been answered by Levy and Khintchine [G3], who gave an exhaustive classification of the possible limit distributions, based on the requirement of stability under convolution This is beautifully reviewed and developed in Gnedenko and Kolmogorov's book [G2] A summary of the most useful mathematical results is given in appendix B of this article. We give here a qualitative presentation of their meaning based on simple statistical arguments.

Let us suppose that $p(l)$ decreases for large l as $l^{-(1+\mu)}$ (with $\mu > 0$ to allow normalization) Then:

- For $0 < \mu < 1$, X_N behaves typically as $N^{1/\mu}$ (or as $N \ln N$ if $\mu = 1$). Note that $\langle l \rangle$ is infinite in this case, and so is $\bar{X}_N$
- For $1 < \mu < 2$, $\langle l \rangle$ is finite and $\bar{X}_N = \langle l \rangle N$, while the difference between X_N and $\bar{X}_N$ again behaves typically as $N^{1/\mu}$ (or as $\sqrt{N \ln N}$ for $\mu = 2$). Note that the variance $\overline{X_N^2} - \bar{X}_N^2$ is still infinite
- For $\mu > 2$, $\langle l^2 \rangle$ is finite and this leads back to the situation of the previous section

Stated more precisely the variable $Z_N = X_N / N^{1/\mu}$ for $0 < \mu < 1$ [or $(X_N - \langle l \rangle N)/N^{1/\mu}$ for $2 > \mu > 1$] has a limit distribution when $N \to \infty$, in the same sense as eq (1 4), namely

$$\text{Prob}(u_1 < Z_N < u_2) \xrightarrow[N\to\infty]{} \int_{u_1}^{u_2} L_{\mu\,\beta}(u)\,\mathrm{d}u \tag{1 11}$$

The limit distributions $L_{\mu\,\beta}$ are defined by their characteristic functions [eq (B 8) of appendix B], they are called *Levy (or "stable") laws of index* μ Up to translations and dilatations, they are fully

characterized by the two parameters μ and β ($0<\mu<2$, $-1\le\beta\le1$). The latter characterizes its degree of asymmetry [which depends on the relative frequency of occurrence of large positive and negative increments in the sum (1.10)]. More precisely,

$$\frac{1-\beta}{1+\beta}=\lim_{L\to\infty}\frac{R(-L)}{1-R(L)}\,,\quad R(L)=\int_{-\infty}^{L}p(l)\,\mathrm{d}l \tag{1.12}$$

The value $\beta=0$ is obtained when large positive and negative values of l_n occur with equal frequencies (which is the case in particular if $p(l)$ is even) It corresponds to an even distribution $L_{\mu,0}(Z)$, which has a simple expression as a Fourier transform,

$$L_{\mu\,0}(Z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}kZ-C|k|^{\mu}}\,. \tag{1.13}$$

The normal distribution (1.4) is recovered for $\mu=2$, while $\mu=1$ leads to the Cauchy law,

$$L_{1\,0}(Z)=\frac{C}{\pi}\frac{1}{C^2+Z^2}\,. \tag{1 14}$$

At the opposite, when the l_n only take positive values, the value $\beta=+1$ is obtained, and $L_{\mu\,1}$ is conveniently expressed as a Laplace transform,

$$L_{\mu\,1}(Z)\equiv L_{\mu}(Z)=\frac{1}{2\pi\mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty}\mathrm{d}s\,\mathrm{e}^{sZ-C's^{\mu}} \tag{1 15}$$

($L_{\mu,1}$ vanishes outside $[0,+\infty[$ for $0<\mu<1$).

The stable laws $L_{\mu,\beta}(Z)$ decrease as $Z^{-(1+\mu)}$ for large values of Z and their moments of order larger than or equal to μ are thus infinite. Their "basin of attraction" consists of those distributions $p(l)$ that for large values of l behave in a way similar to $L_{\mu,\beta}(Z)$ itself. (Theorem 5a of appendix B characterizes this basin in a precise way) This is to be contrasted with the normal law, which attracts all the distributions $p(l)$ decreasing at least as fast as l^{-3}. Its attraction basin is thus very much larger; this is of course the deep reason for its ubiquity in nature. Figure 1 1 intends to give a pictorial view of the structure of these attraction basins.

Several properties of stable laws (moments, asymptotic behaviour, series expansions, etc.) are summarized in the third section of appendix B.

1 2 1 2 Statistical interpretation The behaviour of the sum X_N given above can be qualitatively understood in a very simple way. Let us ask what is the largest value $l_c(N)$ encountered among the N terms of the sum (1.10). $l_c(N)$ can be estimated by writing

$$N\int_{l_c(N)}^{\infty}p(l)\,\mathrm{d}l\simeq1\,. \tag{1 16}$$

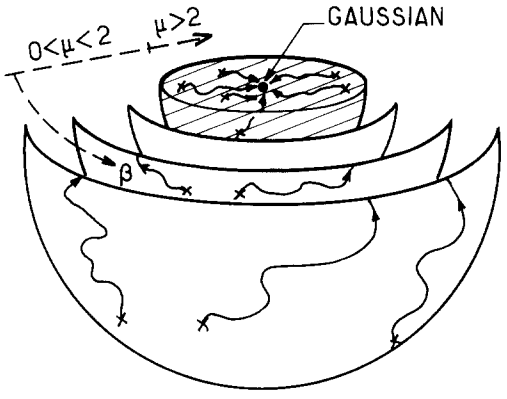


Fig 1 1 An artist's view of the partition of the "space of probability distributions" into basins of attraction of the different Levy laws $L_{\mu\,\beta}$ Note the predominance of the Gaussian ($\mu = 2$) basin of attraction

This criterion*) means that a value larger than $l_c(N)$ occurred at most once in N trials This leads to

$$l_c(N) \sim N^{1/\mu} , \quad N \to \infty \tag{1 17}$$

For a large but finite number N of trials, the value of X_N is insensitive to events with $l \gg l_c(N)$; the full distribution $p(l)$ can thus be cut off at $l \sim l_c(N)$ [since the region $l \gg l_c(N)$ is not sampled] Thus
– For $0 < \mu \leq 1$, the typical value of X_N can be estimated by computing the mean value of the sum (1.10) with this cut off "effective" distribution,

$$X_N \sim N \int^{l_c} l p(l)\, \mathrm{d}l \sim \begin{cases} N(N^{1/\mu})^{1-\mu} = N^{1/\mu} & (\mu < 1) , \\ N \ln N & (\mu = 1) \end{cases} \tag{1 18}$$

– For $1 < \mu < 2$, the typical value of the difference $X_N - \bar{X}_N$ is estimated by

$$(X_N - \bar{X}_N)^2 \sim N \int^{l_c} (l - \langle l \rangle)^2 p(l)\, \mathrm{d}l \sim \begin{cases} N(N^{1/\mu})^{2-\mu} = N^{2/\mu} & (\mu < 2) , \\ N \ln N & (\mu = 2) \end{cases} \tag{1 19}$$

– For $\mu > 2$, the integral in (1 19) converges when $l_c \to \infty$, and one recovers a purely linear dependence on N

From this simple reasoning, one sees that, when $\mu < 2$, the typical value of the sum X_N is dominated by its largest term $l_c(N)$. The sum X_N thus has a manifest *self-similar nature* (the whole sum resembles one single term)

1 2 2 "Levy flights" and physical applications

When the l_n denote the lengths of successive jumps performed at time steps $t = n\tau$, (1 10) provides a natural generalization of Brownian motion (see, e g , refs [G9, Blu86, Shl87] and references therein) The typical time dependence of the position of such "Levy flights" is faster than for normal Brownian motion (and even than for a ballistic motion for $\mu < 1$) From the above

*) More precisely the probability that the largest number chosen in N trials is l_c reads $Np(l_c)[\int_0^{l_c} p(l)\, \mathrm{d}l]^{N-1}$, which is maximal ($N \to \infty$) for $l_c \sim N^{1/\mu}$

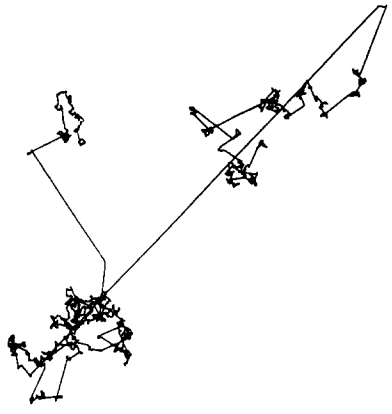


Fig 1 2 A typical Levy flight (from [Blu89]) Note that longer and longer steps are encountered as time goes on, so that the set of visited points is a fractal

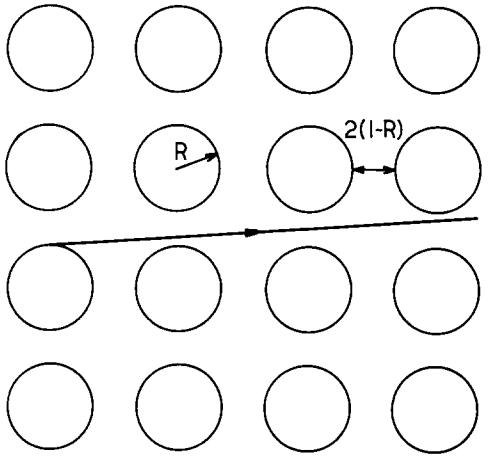


Fig 1 3 Sinai's billiard on a square lattice A classical particle evolves among perfectly reflecting circles This dynamical system is highly chaotic, nevertheless, due to the nearly tangential trajectories such as the one depicted above, the velocity autocorrelation function decays algebraically

statistical discussion, one sees that longer and longer jumps are encountered as time evolves, as depicted in fig 1 2 Since the volume occupied by the walk is related to its "mass" (i e the number of steps) by $R^{d_f} = N$, a Levy flight is a fractal of dimension μ (see [Man82]) We now describe some physical applications of Levy flights, in which the broadness of the distribution $p(l)$ is dictated by the physical nature of the problem at hand For a very recent application of Levy flights to diffusion in elongated micelles, see [Ott90]

1 2 2 1 Geometrically induced anomalous diffusion in some Lorentz gases Let us consider a Lorentz gas in which a particle is reflected by spherical obstacles centred at the nodes of a hypercubic lattice ("Sinai's billiard" [Sin70]) As depicted in fig 1 3, there exist arbitrarily long paths along which the particle can move freely, without collisions (the "horizon" is infinite) The distribution of the length of these paths can be estimated on simple geometrical grounds [Bou85, Zac86],

$$p(l) \sim (1 - R)/l^3 , \quad l \to \infty , \tag{1 20}$$

where R is the radius of the spheres, the lattice spacing being normalized to 2 (for $R = 1$ the spheres come into contact) Diffusion in this billiard can thus be modelled*) as a Levy flight with the "marginal" value of $\mu = 2$, which leads to a typical time dependence of the squared position $X_t^2 \sim t \ln t$ This behaviour was first suggested by Le Doussal and one of the authors [Bou85] and Zacherl et al [Zac86] and appears to be compatible with numerical simulations This Levy flight analysis also suggests that the diffusion front should remain Gaussian for this problem

In fact, for very small obstacles ($R \to 0$), the distribution (1 20) only applies to very long paths $l \gg l^*$ with $l^* \sim R^{-1}$ For shorter paths, one gets [Bou85]

$$p(l) \sim l^{-5/2} , \quad l \lesssim l^* \tag{1 21}$$

A transient regime with $X_t^2 \sim t^{4/3}$ should thus be observed for $t \lesssim t_c \sim R^{-3/2}$ before the asymptotic regime $X_t^2 \sim t \ln t$ is reached

If the lattice structure is changed, for example to a triangular one, with sufficiently large radius R, these long collisionless trajectories no longer exist (the horizon becomes finite), and it has been proven in this case that diffusion becomes normal [Sin80] One should note, however, that there exist billiards with finite horizon in which diffusion is nevertheless anomalous, due to complex long-range correlations between successive jumps (see, for example, [Dou87] for a discussion and references on the diffusion properties of billiards)

1 2 2 2 Polymer adsorption and self-avoiding Levy flights The structure of an adsorbed polymer is shown in fig 1 4 it is made of points in direct contact with an attractive wall, separated by large loops in the bulk The main point is that the

*) This is of course only a rough approximation, since complicated correlations between successive jumps exist in this deterministic motion They are, however, very likely to be short ranged and thus will not change the time dependence of X_t (see section 1 3)

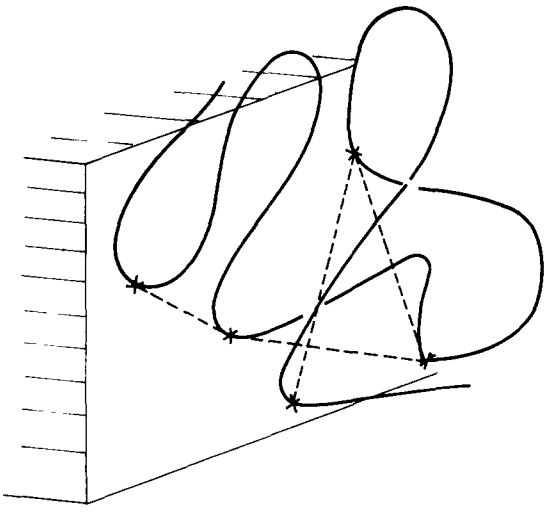


Fig 1 4 Typical configuration of an adsorbed polymer right at the adsorption transition long loops dive into the bulk and induce a broad distribution of distances between (singly) occupied surface sites

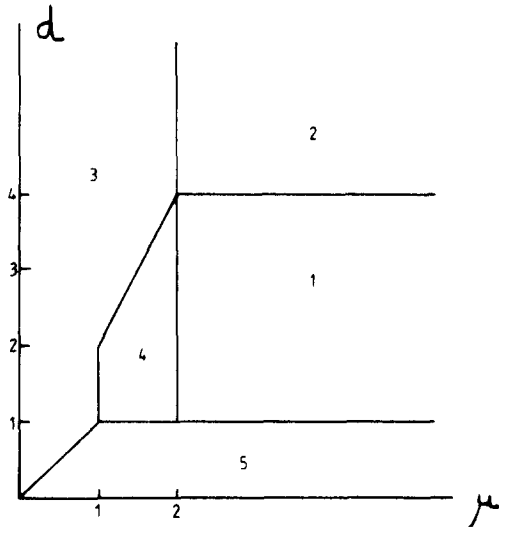


Fig 1 5 The "phase diagram" of a self-avoiding Levy flight (SALF) (from [Bou87e] in Flory's approximation, see section 1 3 3) in the (d, μ) plane [d is the dimension of the wall, μ is the exponent of the broad step size distribution $\psi(l) \simeq l^{-(1+\mu)}$] (1) Usual SAW, (2) free random walk, (3) free Levy flight, (4) SALF, (5) collapsed The fact that the chain is still isotropic at the transition with N^{ϕ} monomers on the wall, together with the value $\mu = \phi/\nu$, means that one is in domain (3), hence the bound (1 24)

distribution of the size of these loops *decays as a power law* [dGe82], inducing a broad distribution of distances between two consecutive adsorbed monomers In other words, as emphasized in [Bou87e, 88b], the projection of the chain's conformation on the wall *is a self-avoiding Levy flight**) (two adsorbed monomers cannot occupy the same site) Just at the *adsorption threshold* (i e , when the attraction due to the wall is just sufficient to bind the chain), the step length distribution reads

$$p(l) \sim l^{-(1+\mu)} , \tag{1 22}$$

for $l \lesssim N^{\nu}$ and $\mu = \phi/\nu$ ν is the exponent characterizing the end to end distance of a non-adsorbed chain ($R \sim N^{\nu}$, see section 1 3 3 and table 1 2), and ϕ the "crossover" exponent, defined, e g , by the number of adsorbed monomers *at threshold*, N^{ϕ}

The phase diagram in the plane (μ, d) (where d is the dimension of the wall) for a self-avoiding Levy flight (SALF) is shown in fig 1 5, where the diffusion exponent ν_{SALF} is calculated within a Flory theory (see section 1 3 3 and table 1 2) As the chain at threshold is still isotropic, one has the self-consistent relation

$$R_{\parallel} \sim N^{\nu} \sim (N^{\phi})^{\nu_{\mathrm{SALF}}} , \tag{1 23}$$

leading to $\nu_{\mathrm{SALF}} = \nu/\phi = 1/\mu$ Checking on figure 1 5 this leads to an upper bound on ϕ, which reads in three dimensions $(3 = d + 1)$

$$\phi \leq \nu \tag{1 24}$$

Numerical determinations [Eis82, Ish82, 83] of ϕ yield

$$\phi \simeq \nu \simeq 0\ 59$$

(For a review on this problem, see, e g , [Bou88c])

*) More precisely, it is a "node" avoiding Levy flight and not a "path" avoiding Levy flight, see [Hal85b, Lee87]

1.2.3. Anomalous diffusion due to long waiting times

1 2.3.1. Continuous time random walks In this section we consider a random walk on a regular lattice, such that the particle has to wait for a time τ on each site before performing the next jump. This waiting time is a random variable independently chosen at each new jump according to a distribution $\psi(\tau)$ (for simplicity τ is not correlated to the length of the jump l, which is distributed according to $p(l)$; see appendix A for the general case). One can think of this problem as a diffusion among traps, but without forgetting that a given trapping time is not associated forever with a given site, but changes at each new visit (fig 1 6); disorder is thus introduced in this model in an "annealed" way [through $\psi(\tau)$] The corresponding "quenched" problem, in which to each site is associated a *given* trapping time (random from site to site), is more difficult and will be studied in chapter 2 (section 2 4.1).

The "annealed" problem dealt with here is called "continuous time random walk" (CTRW) in the literature. The theory of CTRWs for arbitrary $\psi(\tau)$ has been extensively developed in the literature (see, e.g., [Mon65, G1, G7, G9]) A number of quantities [including $P(X, t)$] can be calculated exactly using generating function methods This is reviewed, for example, in ref [G13], and also, very briefly, in appendix A (section A 3) We show here how some of these results can be recovered through simple statistical reasoning, with emphasis on the anomalous diffusion behaviour arising when $\psi(\tau)$ is "broad"

Diffusion behaviour Let N be the number of steps performed by the walker during the time t; the average of the αth component of its position is thus given by

$$\overline{X_\alpha^2} = \langle l_\alpha^2 \rangle N \quad (N \to \infty), \tag{1.25}$$

where $\langle l_\alpha^2 \rangle$ is the mean squared length of a jump,

$$\langle l_\alpha^2 \rangle \equiv \int l_\alpha^2 p(l)\, \mathrm{d}^d l \tag{1.26}$$

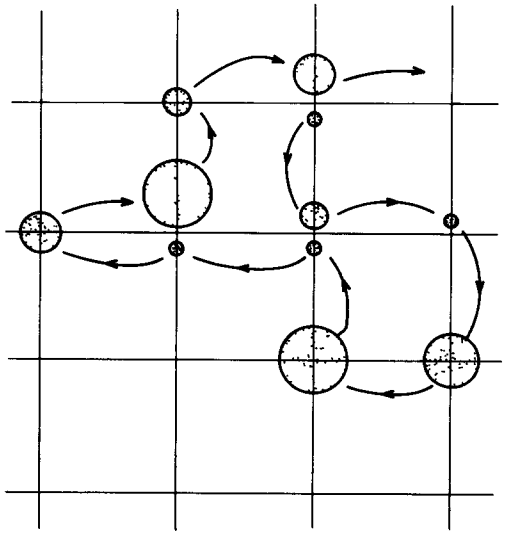


Fig 1 6 Pictorial representation of a continuous time random walk (annealed disorder on waiting times)

($=a^2/d$ on a hypercubic lattice with nearest-neighbour jumps) The total time t is simply the sum of the N waiting times encountered,

$$t = \sum_{n=1}^{N} \tau_n \tag{1 27}$$

It is thus a sum of independent random variables to which the results of section 1.2 1 can be applied One is thus led to distinguish two cases

(i) *If* $\langle \tau \rangle$ *is finite*, then t behaves typically as $t \sim N\langle t \rangle$, and diffusion is thus normal at large times,

$$\overline{X_\alpha^2} = 2D_{\alpha\alpha} t \quad \text{with } D_{\alpha\alpha} = \langle l_\alpha^2 \rangle / 2\langle \tau \rangle \tag{1 28}$$

Comparing this expression with eq (1 3), one sees that the diffusion constant takes the same value as if the successive jumps occurred at regularly separated times $t = n\langle \tau \rangle$ Indeed, the *renewal theorem* [G1] states that, for this process, successive jumps occur, on average, at a constant rate $\langle \tau \rangle^{-1}$ *at large times.*

(ii) If, on the other hand, $\psi(\tau)$ is a "broad" distribution,

$$\psi(\tau) \simeq \tau_0^\mu \tau^{-(1+\mu)} \quad (\tau \to \infty) , \tag{1.29}$$

with $0 < \mu \leq 1$ then $\langle \tau \rangle = +\infty$ and t behaves as

$$t \sim \tau_0 N^{1/\mu} \tag{1 30}$$

This leads to *subdiffusive* behaviour [Mon73],

$$\overline{X_\alpha^2} \sim \begin{cases} \langle l_\alpha^2 \rangle (t/\tau_0)^\mu & (0 < \mu < 1) , \\ \langle l_\alpha^2 \rangle t / [\tau_0 \ln(t/\tau_0)] & (\mu = 1) \end{cases} \tag{1 31}$$

Note that, when ψ behaves as (1 29) with $1 < \mu < 2$, this induces anomalous corrections to the normal large-time behaviour (1 28), namely,

$$\overline{X_\alpha^2} \sim \begin{cases} 2D_{\alpha\alpha} t + c t^{1/\mu} & (1 < \mu < 2) , \\ 2D_{\alpha\alpha} t + c t \ln t & (\mu = 2) , \end{cases} \tag{1 32}$$

while $\overline{X_\alpha^2} \sim (\langle l_\alpha^2 \rangle / \langle \tau \rangle) t + c t^{1/2}$ for $\mu > 2$

Diffusion front An explicit expression for the probability density of the position at time t, $P(\boldsymbol{X}, t)$ can be obtained in the large-time limit [with the initial condition $P(\boldsymbol{X}, t = 0) = \delta(\boldsymbol{X})$] This can be obtained [Bal87] from the general expression of its Fourier–Laplace transform for general $\psi(\tau)$ (see appendix A) We present here an alternative derivation, closer in spirit to the above statistical analysis Indeed, $P(\boldsymbol{X}, t)$ can be expressed as a sum over all possible numbers of jumps,

$$P(\boldsymbol{X}, t) = \sum_{N=0}^{\infty} P(\boldsymbol{X}, N) P(N, t) , \tag{1 33}$$

where $P(\boldsymbol{X}, N)$ stands for the probability that the position of the particle is $\boldsymbol{X}$, N jumps having been performed, and $P(N, t)$ stands for the probability distribution of the number N of jumps at a fixed time t When N is large, $P(\boldsymbol{X}, N)$ takes the Gaussian form (assuming isotropy for simplicity $\langle l_\alpha^2 \rangle \equiv l^2$),

$$P(\boldsymbol{X}, N) = (2\pi l^2 N)^{-d/2} \, \mathrm{e}^{-X^2/2l^2N} \tag{1 34}$$

When $\langle\tau\rangle$ is finite, the limiting form of the distribution $P(N, t)$ for large N is centred around $N = t/\langle\tau\rangle$, therefore the main contribution to the sum (1 33) for X and t simultaneously large comes from the vicinity of $N = t/\langle\tau\rangle$ and the asymptotic form of $P(X, t)$ is thus simply obtained by substituting this value into (1 34) Thus, when $\langle\tau\rangle$ is finite, not only the diffusion behaviour is normal but also the full limiting form of the diffusion front,

$$P(X, t) \to (4\pi Dt)^{-d/2} \, \mathrm{e}^{-X^2/4Dt} \,, \quad D = l^2/2\langle\tau\rangle \tag{1 35}$$

This no longer applies when $\mu < 1$, where the precise form of $P(N, t)$ has to be used t being a sum of independent positive random variables, its limit distribution is obtained from section 1 2 1, as

$$P(t, N) \underset{N\to\infty}{\longrightarrow} \frac{1}{\tau_0 N^{1/\mu}} \, L_\mu(t/\tau_0 N^{1/\mu}) \,, \tag{1 36}$$

where L_μ is a stable law of index μ with $\beta = +1$ Changing variables from $Z = t/N^{1/\mu}$ to $u = N/t^\mu = Z^{-\mu}$, one obtains

$$P(N, t) = (\tau_0/t)^\mu f_\mu(N(\tau_0/t)^\mu) \,, \quad f_\mu(u) \equiv \frac{1}{\mu} \, u^{-(1+1/\mu)} L_\mu(u^{-1/\mu}) \tag{1 37}$$

Using the representation (B 22) of appendix B, one thus obtains the Laplace transform of $P(N, t)$ in the form

$$P(N, E) = \tau_0^\mu E^{\mu-1} \, \mathrm{e}^{-N(\tau_0 E)^\mu} \tag{1 38}$$

(with a redefinition of τ_0 to absorb constant factors) In the long-time limit, the sum (1 33) can be replaced by an integral, using (1 38), the Laplace transform $P(X, E)$ of $P(X, t)$ thus reads

$$P(X, E) = \frac{\tau_0^\mu E^{\mu-1}}{(2\pi l^2)^{d/2}} \int_0^\infty \frac{\mathrm{d}N}{N^{d/2}} \exp[-N(\tau_0 E)^\mu - X^2/2l^2N] \tag{1 39}$$

This integral can be expressed in terms of the modified Bessel function $K_{1-d/2}$, leading to the final expression*) [Bal87]

$$P(X, E) = \tau_0(\pi l^2)^{-d/2}[(\sqrt{2}/l)|X|(\tau_0 E)^{\mu/2}]^{1-d/2}(\tau_0 E)^{\mu d/2-1} K_{1-d/2}[(\sqrt{2}|X|/l)(\tau_0 E)^{\mu/2}] \,, \quad \mu < 1 \tag{1 40}$$

The diffusion front is thus *no longer a Gaussian* for this problem when diffusion is anomalous Nevertheless, a generalized central limit theorem still applies in the asymptotic limit Indeed, (1 40) means that $P(X, t)$ takes the scaling form,

$$P(X, t) \to l^{-d}(\tau_0/t)^{\nu d} f[(\tau_0/l)|X|/t^\nu] \,, \tag{1 41}$$

where $\nu = \mu/2$ is the diffusion exponent [the precise meaning being similar to eq (1 4) or (1 11)] The scaling function $f(u)$ decays for large u ($|X| \gg t^\nu$) as

$$f(u) \underset{u\gg 1}{\sim} \mathrm{e}^{-u^\delta} \,, \quad \delta = 1/(1-\nu) \tag{1 42}$$

(Power law prefactors have been omitted, this behaviour can be derived either from the explicit expression (1 40) or directly by the steepest descent method on the integral representation (1 33), using the behaviour of $L_\mu(Z)$ near $Z = 0$) This stretched exponential decay of $P(X, t)$ for $|X| \gg t^\nu$ with $\delta = 1/(1-\nu)$ will turn out to arise in a number of situations considered in the following (see chapter 5 for a general discussion of diffusion fronts) In fact, it stems from the fact that, within the analogy with critical phenomena discussed above, $P(X, E)$ should be thought of as a correlation function, (1 42) then corresponds to a simple exponential decay of $P(X, E)$ for $|X|E^\nu \gg 1$

An explicit expression can be given for the scaling function f in one dimension, where $K_{1-d/2} = K_{1/2}$ has a simple form Then f is easily found to be

$$f(u) = \frac{1}{\nu} \, |u|^{-(1+1/\nu)} L_\nu(|u|^{-1/\nu}) \equiv f_\nu(|u|) \,, \quad d = 1 \,, \tag{1 43}$$

where L_ν is a stable law of index $\nu = \mu/2$, $\beta = +1$ (One checks, using eq (B 24), that a Gaussian is recovered for $\nu = 1/2$)

*) This has recently been derived in [Bal87], following a different method Note, however, some misprints in the final result of [Bal87], which does not have the correct scaling form (1 41)

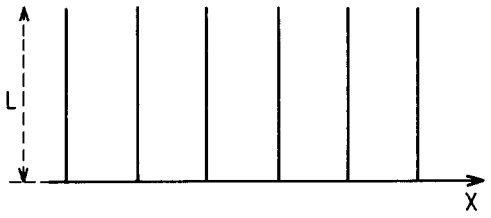


Fig 1 7 "Comb-like" structure the spikes behave as traps with a broad distribution of release time, proportional to the probability of first return of a one-dimensional random walk, indeed, the walker must return to the entry point, and then has a finite probability to leave the spike

We now turn to three examples which can be analyzed as CTRWs with a broad distribution of waiting times induced by the motion itself Several other physical applications will be described in the course of this article; in particular it will appear that random walks in a *quenched* random medium (which are a priori much more complex problems) can in some cases be "renormalized" at large scales onto a much simpler CTRW model [Mac85, Bro89a,b]

1.2 3 2 Diffusion on comb-like structures Let us consider the diffusion on the comb-like structure depicted in fig 1 7 The teeth of this comb behave as traps in which the particle stays for some time before continuing its random motion along the X-axis. Thus, for infinitely deep teeth ($L = +\infty$) the waiting time distribution $\psi(\tau)$ is simply the distribution of the first return time at the origin (the entrance of a tooth) of a one-dimensional Brownian motion As already mentioned (cf also appendix A), it decays for large τ as

$$\psi(\tau) = P_1(\tau) \sim \tau^{-3/2} \tag{1 44}$$

Thus, the motion along X can be described as a CTRW with $\mu = \frac{1}{2}$. The comb-like geometry thus induces *anomalous diffusion* along X,

$$\overline{X_t^2} \sim t^{1/2} \tag{1 45}$$

This result, obtained here in a very simple way, has been recently derived by Weiss and Havlin [Wei86] using somewhat more sophisticated methods (see also ref [G12] for detailed studies of diffusion on combs, and ref [Bal87] for a numerical confirmation of the shape (1 43) of the diffusion front).

Consider now the case of teeth with a finite depth L. The time required to explore a given tooth is of order $\tau_c \sim L^2/D_0$ (D_0 is the "bare" diffusion coefficient along each tooth). For $\tau \gg \tau_c$, an exponential decay of $P_1(\tau)$ is recovered, and the average time spent in a tooth is thus

$$\langle \tau \rangle \sim \int^{\tau_c} \tau \, \mathrm{d}\tau \, \tau^{-3/2} \sim L$$

Hence, for $t \gg \tau_c$ normal diffusion is recovered, with a modified diffusion constant depending on L as $D \sim 1/L$

1 2 3 3 Anomalous diffusion in an intermittent dynamical system [Man80, Gei84] It is well known that purely deterministic dynamical systems can give rise to diffusive motion, as a result of the chaotic nature of their dynamics A very simple example with a single degree of freedom X_t is provided by a mapping of the form

$$X_{t+1} = f(X_t)\,, \tag{1 46}$$

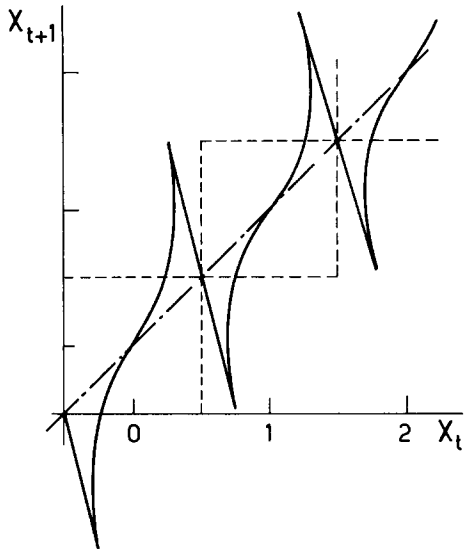


Fig 1 8 Intermittent periodic mapping $X_{t+1} = f(X_t)$ The representative point remains trapped in the "laminar" regions (channels) during a long period, which can be taken as random and distributed according to (1 52) if the reinjecting regions are sufficiently chaotic

where f has a discrete translation invariance (fig 1 8),

$$f(n + x) = n + f(x), \quad \text{for integer } n, \qquad f(-x) = -f(x) \tag{1 47}$$

X_t then follows a diffusion process between elementary cells of unit length centred on each integer n A Josephson junction subjected to a periodic potential $V(\varphi)$ is an example of a physical system following (1 47), there X_t stands for the phase φ evolving according to

$$d\varphi/dt = \Gamma V(\varphi) \tag{1 48}$$

in the limit of large friction The function $f(\varphi) = \varphi + T\,\Delta t\,V(\varphi)$ thus satisfies (1 47) (chaos in Josephson junctions has been observed, e g , in [Hub80]) Geisel and Thomae observed in [Gei84] that anomalous diffusion can arise as a result of the *intermittent* nature [Pom80] of the mapping f in the vicinity of the fixed points $x = \pm n$ Indeed, the dynamics of X_t then consists of long laminar sequences in a given cell, interrupted by chaotic bursts associated with transfer from one cell to another These laminar sequences can induce a broad distribution of waiting times in each cell The mapping considered in [Gei84] is sketched in fig 1 8 In the neighbourhood of each integer, f is characterized by an exponent α,

$$X_{t+1} = X_t + aX_t^{\alpha} + \tag{1 49}$$

(here, for $X_t \to 0^+$) Each integer is thus a marginally stable fixed point, and the larger the exponent α is the larger is the time spent by a trajectory in the narrow "vertex" close to the centre of each cell The detailed shape of f far from these regions is of no importance for the following, and must only insure ergodic reinjection in the vertex *) Following [Gei84] the motion close to each fixed point can be approximated by the continuous time solution of (1 49),

$$X_t = [X_0^{1-\alpha} - a(\alpha - 1)t]^{-1/(\alpha-1)}, \tag{1 50}$$

from which one deduces the waiting time τ as a function of the position where reinjection takes place,

$$\tau(X_0) = (X_0^{1-\alpha} - 2^{\alpha-1})/[a(\alpha - 1)] \tag{1 51}$$

This allows one to deduce the waiting time distribution $\psi(\tau)$ from the (unknown) distribution $\rho(X_0)$ of reinjection points, the

*) Note that the particle can be reinjected on both sides of the vertex, one deals here with "type III" intermittency

behaviour of $\psi(\tau)$ for large τ only depends on the one of ρ close to $X_0 = 0$, assuming $\rho(X_0 = 0) \neq 0, \infty$, (1 51) leads to

$$\psi(\tau) \sim \tau^{-(1+\mu)}, \quad \mu \equiv 1/(\alpha - 1) \tag{1 52}$$

Intermittency has thus induced a broad distribution of waiting times, and the resulting diffusion behaviour reads

$$\overline{X_t^2} \sim \begin{cases} t, & 1 < \alpha < 2, \\ t/\ln t, & \alpha = 2, \\ t^{2\nu}, & 2\nu = 1/(\alpha - 1), \ \alpha > 2 \end{cases} \tag{1 53}$$

1 2 3 4 Experiments anomalous diffusion in a linear array of convection rolls [Pom89, Car88] Another physical example of anomalous diffusion due to trapping has been proposed recently by Pomeau, Pumir and Young [Pom89] (see also [Shr87, Guy89]), in connection with an experiment of Cardoso and Tabeling [Car88]

The problem of interest is the motion of a tracer particle in a one-dimensional array of convection rolls (see fig 1 9) The particle is both convected along the flow lines and experiences molecular diffusion (D_0), which allows "jumps" between flow lines The basic observation, which will be discussed below, is that each roll *acts as a trap with a release time distribution* decaying as $\psi(\tau) \sim \tau^{-(1+\mu)}$ for $\tau \lesssim L^2/D_0$ (L is the diameter of the roll) Since different visits to a given roll lead to different diffusion histories, the total time t will again be the sum of independent, broadly distributed variables The number of visited rolls after a time t thus reads (from section 1 2 3 1)

$$N \sim t^{\mu/2}, \quad \text{for } \mu < 1 \tag{1 54}$$

In order to understand the origin of the broad distribution and the value of μ, it is convenient to view the flow lines as a discrete set, numbered from $i = 1$ (corresponding to the "edge" of the roll) to $i = S$ (see fig 1 10) The particle thus makes a one-dimensional random walk "in i space", onto which the circular convective motion is superimposed The probability of leaving the roll is proportional to the probability of first return to the site $i = 1$, which, after m steps and for $m^2 \ll S$ reads $P_1(m) \sim m^{-3/2}$ (see above, section 1 2 3 2) Now, if θ_i represents the time needed to make a closed loop on the ith line, the total time spent by the particle in the cell will be

$$\tau = \sum_{j=1}^{m} \theta_{i(j)} \tag{1 55}$$

Here, the boundary conditions imposed by the horizontal plates must be specified We shall assume that the velocity behaves, for small distances from the wall, as $V(z) \sim V_0(z/L)^\beta$ $\beta = 0$ corresponds to "free" boundaries while $\beta = 1$ describes a solid, "no slip" boundary*) Flux conservation then imposes that the ith flow line is at a distance $(i/S)^{1/(\beta+1)}L$ from the horizontal plate, and

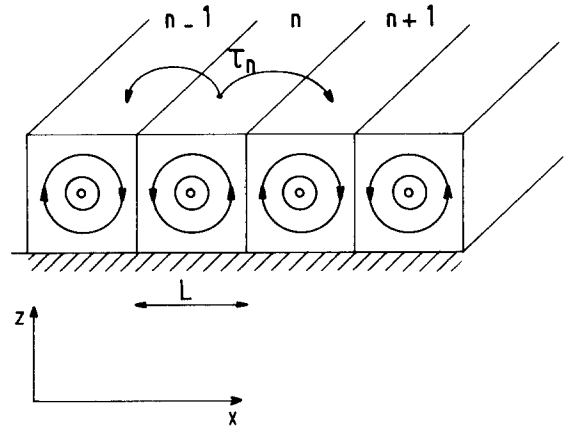


Fig 1 9 One-dimensional array of convection rolls, defining a one-dimensional discrete lattice on which the tracer evolves

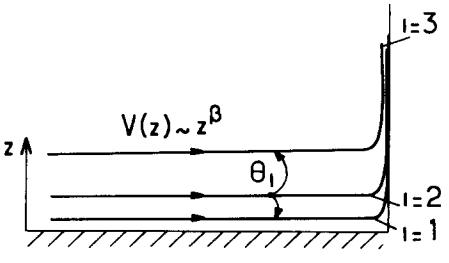


Fig 1 10 Blow up of a part of a convection roll, showing the deformation of the flow lines (which are taken as discrete here) near the solid boundary Note that due to flux conservation the flow lines expand when the velocity is lower

*) Other values of β might be considered, for example, if the plate is covered by a self-similar profile of adsorbed polymers [dGe85]

is thus characterized by a transit time $\theta_\iota \sim (L/V_0)(S/\iota)^{\beta/\beta+1}$ Now, the probability to be on the ιth line after $\jmath$ steps is simply

$$P(\iota, \jmath) \sim \jmath^{-1/2} \mathrm{e}^{-\iota^2/\jmath} \tag{1 56}$$

The average transit time $\bar{\theta}_\jmath$ hence *depends on* $\jmath$,

$$\bar{\theta}_\jmath \equiv \sum_\iota \theta_\iota P(\iota, \jmath) \sim \jmath^{-\beta/2(\beta+1)}, \tag{1 57}$$

and thus*)

$$\tau = \sum_{\jmath=1}^{m} \bar{\theta}_\jmath \sim m^{(\beta+2)/2(\beta+1)} \tag{1 58}$$

This is the basic physical result time and number of steps are *not* proportional to one another, except when $\beta = 0$, indeed, the first "loops" take a very long time since the particle gets close to the wall where the velocity is vanishing To complete the argument and obtain μ, one only needs to write

$$\psi(\tau)\, \mathrm{d}\tau = P_1(m)\, \mathrm{d}m ,$$

which yields $\mu = (1+\beta)/(2+\beta)$

From this picture, not only the time dependence of the number of "invaded" rolls follows, but also the full diffusion front in terms of Levy laws The calculation was performed in section 1 2 3 1, where we have shown that (eq 1 43)

$$P(X, t) = \frac{1}{t^{\mu/2}} f_{\mu/2}(|X|/t^{\mu/2}) \tag{1 59}$$

In particular, for $\beta = 1$, $\mu/2 = 1/3$, in this case (see appendix B), the Levy law can be expressed in terms of a Bessel function and one finally obtains

$$P(X, t) \simeq \sqrt{|X|/t}\, K_{1/3}[|X|^{3/2}/t^{1/2}] , \tag{1 60}$$

which is the result of [Pom89] (see also [Shr87, Guy89] [Note that (1 60) is an Airy function] This expression reproduces quite well the experimental data of Cardoso and Tabeling (see fig 1 11)

Note that it would be interesting to bias the tracer's motion by an external field, in order to probe the *non-linear response* properties that should occur in this case (see chapter 5, in particular section 5 4)

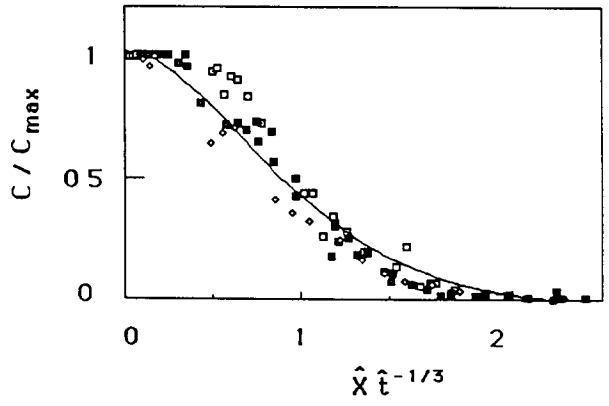


Fig 1 11 Rescaled experimental results [Car88] for the diffusion front compared with the theoretical prediction (1 59) $C/C_{\max} = f_{1/3}(Xt^{-1/3})$ Note that there is no adjustable parameter to obtain the above fit

*) Note that there is a physical upper bound on τ $\tau_{\max} \sim (L/V_0)[V_0 L/D_0]^{\beta/(2+\beta)}$

1 3 Long-range correlations

1 3 1 Generalization of the CLT to sums of correlated random variables

When the second condition needed to establish the usual CLT is not fulfilled, that is, when the summed random variables *are not independent*, the analysis cannot be developed as generally and precisely as in the previous section (for recent results, see, e g , [Ebe86, Dav89]).

It is a priori quite obvious that *long-range* correlations can modify the "normal" behaviour (think of the extreme case in which the N variables are identical!), but that very short-range correlations should safely be ignored What we do need is thus a criterion of the relevance of the correlations

1 3.1.1 Relevance of the correlations Let us consider the sum of a set of random variables (l_k), the correlation function of which, defined as

$$C(n) = \langle l_k l_{k+n} \rangle - \langle l_k \rangle \langle l_{k+n} \rangle ,$$

only depends upon the difference n (stationary process) *) Assuming for simplicity $\langle l_k \rangle = 0$, the variance of $X_N = \Sigma_{k=1}^{N} l_k$ reads

$$\overline{X_N^2} = NC(0) + 2 \sum_{k=1}^{N} (N-k)C(k) \tag{1 61}$$

Two cases must thus be considered.

(i) $\Sigma_{n=1}^{N} C(n)$ *converges* when $N \to \infty$, that is, $C(n)$ *decays more rapidly than* n^{-1} *for large* n In this case

$$\overline{X_N^2} \sim N\left(2 \sum_{n=1}^{\infty} C(n) + C(0)\right) \quad (N \to \infty) , \tag{1.62}$$

and thus X_N still behaves typically as $\sqrt{N}$; only the prefactor (the "diffusion constant") is modified by the correlations In this sense "short-range" correlations are *irrelevant* This is of course the case when the correlations have a *finite range* [for example when $C(n)$ decays exponentially] but the criterion is much less restrictive Introduction of finite-range correlations in the Brownian motion has been considered, for example, as an improved model of ideal polymer chains (Orr's model, see ref [G16], pp 31, 32). The range of those correlations is called the *persistence length* of the chain

(ii) If on the contrary correlations are "long ranged", i.e , when $C(n)$ decays as $1/n$ or more slowly [for example, if $C(n) \sim n^{-\gamma}$, $\gamma < 1$], then the typical behaviour of X_N is *modified* by those correlations From eq (1.6) one obtains (replacing, for large N, the sum by an integral)

$$\overline{X_N^2} \sim N \int^{N} C(n)\, \mathrm{d}n \sim \begin{cases} N^{2-\gamma} & (\gamma < 1) , \\ N \ln N & (\gamma = 1) \end{cases} \tag{1 63}$$

"Diffusion" is thus *enhanced* by correlations. the typical value of X_N is much larger than $\sqrt{N}$ The extreme case of perfect correlations (when all variables are equal) is described by $\gamma = 0$, which gives back the expected "ballistic" result $X_N \sim N$.

*) We assume that $C(n)$ does not display asymptotic oscillations which could invalidate the following discussion

1.3.1.2. Statistical interpretation: effective number of independent variables The previous criterion can be understood in simple statistical terms, thereby providing us with a heuristic tool which turns out to be quite useful to understand qualitatively many physical situations, as will be illustrated below

The integral of the correlation function

$$N_{id}(N) = \sum_{n=1}^{N} C(n)/(\langle l^2 \rangle - \langle l \rangle^2) \tag{1.64}$$

is an estimate of the number of variables which, among the set of N chosen variables, statistically "resemble" a given one l_0 This is because $(\langle l_0 l_k \rangle - \langle l_0 \rangle \langle l_k \rangle)/(\langle l^2 \rangle - \langle l \rangle^2)$ measures the probability for l_k to be "close" to l_0 One can thus rearrange the set $\{l_k, 0 \le k \le N\}$ into "families" of $N_{id}(N)$ similar variables. There are obviously

$$N_{eff}(N) \sim N/N_{id}(N)$$

such families, which can be considered as *effectively statistically independent* This amounts to decomposing X_N as

$$X_N \sim N_{id}(N) \sum_{k=1}^{N_{eff}} \hat{l}_k \,, \tag{1.65}$$

where the new sum now has "normal" fluctuations. In other words, one "integrates out" the correlations by redefining new random variables as "blocks" of old ones. From (1.64), one obtains the typical behaviour

$$X_N \sim N_{id}\sqrt{N_{eff}} \sim \sqrt{N N_{id}} \,, \tag{1.66}$$

which gives back (1 62) and (1.63) since

$$N_{id}(N) \begin{cases} \sim \text{const} & \text{if } y > 1 \,, \\ \sim \ln(N) & \text{if } y = 1 \,, \\ \sim N^{1-y} & \text{if } y < 1 \end{cases} \tag{1 67}$$

1 3.1.3. Shape of the limit distribution. When correlations are "short range" [with meaning (i) above], it is possible to prove that the limit distribution of the sum remains Gaussian, see, e g , [Ma85] This follows from the existence of a k^2 term when expanding the characteristic function in the proof of the CLT given above In the case of Markov processes, or more generally of processes with a *finite*-range memory, transfer matrix techniques can be used to establish this result and study related questions (see, e g , refs [G1, G17]).

However, long-range correlations do lead in general to limit distributions *deviating from the usual Gaussian shape* Examples are provided by the polymer problem (section 1 3 3), or the distribution of the magnetisation of a spin model at the critical point (section 1 3 4); see also the examples given in [Dav89]. In such cases, the limit distribution is expected to depend on the detailed structure of the correlations An elementary remark in this respect is that the limit distribution indeed remains Gaussian

if the correlations have a Gaussian structure, i.e , when the distribution of X_1, , X_N reads

$$P(X_1, \quad , X_N) = (\det C)^{1/2} \exp\Big(-\sum_{ij} X_i C_{ij}^{-1} X_j\Big) \tag{1.68}$$

This immediately follows from the properties of Gaussian integrals In the general case, we are not aware of any mathematical work providing a classification of the possible limit distributions

Some universal features of the *tails* of the limit distributions of the position in a number of diffusion problems considered in the following, will be pointed out in section 5 3 2 They are closely related to the response of the walk to a small bias

1 3 2. Geometrical correlations and anomalous diffusion: the example of a layered medium

In this and the following two subsections, we shall present physical illustrations of the above statistical mechanism In the situations we consider, correlations are not introduced by hand into the model, but rather are *dynamically generated* by the process itself

A simple example in which long-range temporal correlations induced by the dynamics lead to anomalous diffusion is that of a bidimensional layered medium represented in figure 1.12a Within each layer the velocity $V(Z)$ is constant and directed along the layer In the following, $V(Z)$ will be a *random variable*, only depending on the layer Z and *without any interlayer correlations* This model was first considered by Matheron and de Marsily [Mat80], to study diffusion in porous rocks exhibiting large-scale permeability – and thus velocity – fluctuations

This is the first example of a disordered system encountered in this article: two statistical ensembles are involved, namely the distribution of $\{V(Z)\}$ and the different walks (or thermal histories) for a given $\{V(Z)\}$ The position of a walker at time t depends on both, and one should pay attention to this fact when discussing its statistical properties In a real experiment, one is in general interested in the properties of walks for a given ("quenched") $\{V(Z)\}$ In this section, we shall not discuss in detail the question of ensemble averaging, we only want to point out the statistical mechanism leading to an anomalous diffusion law, and to predict this law by a simple argument [Bou87c] Appendix C is devoted to a more detailed study of the diffusion behaviour and diffusion front of this model, for which several analytical results can be obtained

1 3 2 1 The diffusion behaviour The *tranverse* motion (with respect to the flow) is a usual Brownian walk, characterized by $Z^2 = 2D_\perp t$ Each layer is thus visited a large number of times, the probability for

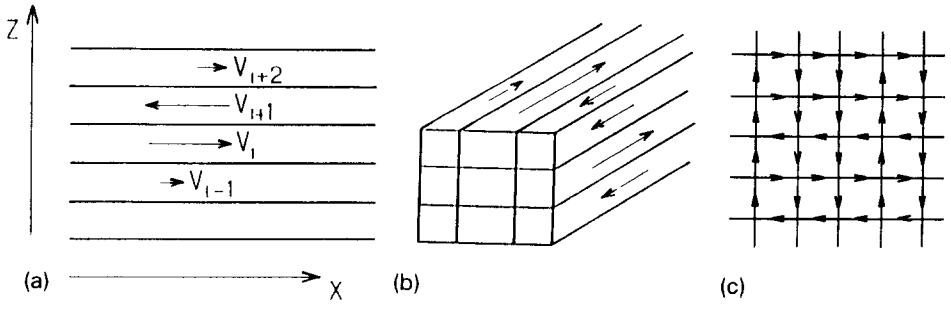


Fig 1 12 (a) Model of a two-dimensional stratified porous medium, permeability fluctuations induce a random distribution of local flow velocity (b) Three-dimensional analogue with channels of constant velocity (c) "Random Manhattan" lattice in two dimensions

the particle to be in the initial layer after time t is simply

$$P_0(t) \sim (D_\perp t)^{-1/2} .$$

This means that of the order of $\sqrt{D_\perp t}$ different layers have been probed, each of them $N_{id}(t) \sim t/\sqrt{D_\perp t}$ times Said differently, the one-dimensional motion along Z induces a temporal correlation of the particle velocity decaying as

$$\langle X_{t=0} X_t \rangle \sim \sigma_V / \sqrt{D_\perp t} , \quad \sigma_V \equiv \int_{-\infty}^{+\infty} V^2 \rho(V)\, dV \tag{1 69}$$

(we consider the problem in the reference frame in which $\langle V \rangle = 0$).

The total displacement along X is simply the sum of the velocities seen by the walker According to the previous section, one thus has

$$\sqrt{\overline{X_t^2}} \sim \sqrt{\sigma_V / D_\perp^{1/2}}\, t^{3/4} , \tag{1.70}$$

which is (up to a numerical factor, see appendix C) the result obtained in [Mat80] for $\sqrt{\langle \overline{X_t^2} \rangle}$: this model exhibits *correlation-induced hyperdiffusion.*

1 3 2 2 Five remarks

(i) In the laboratory frame, we must obviously superimpose the convective motion $X_t = \langle V \rangle t$ on the previous hyperdiffusion If, however, the average velocity $\langle V \rangle$ is not strictly parallel to the layers, diffusion is asymptotically normal, since the number of visited layers becomes of order t Diffusion is also normal if the sample is finite in the z direction In both cases, however, the above behaviour holds for intermediate times

(ii) If the velocity field is so strongly disordered that $\langle V^2 \rangle - \langle V \rangle^2 = \infty$ (if, for example, $\rho(V) \sim V^{-(1+\mu)}$, $V \to \infty$), then diffusion is even more rapid,

$$X_t \sim \sqrt{\frac{t}{D_\perp}} \sum_{i=1}^{\sqrt{D_\perp t}} V_i \sim D_\perp^{(1/\mu - 1)/2} t^{(1/\mu + 1)/2} \quad (1 < \mu < 2)$$

(iii) If the flow profile is not random, but a Poiseuille flow, $V(Z) \sim Z^2$, or a shear flow, $V(Z) \sim Z$, then one obtains

$$X_t \sim \sqrt{\frac{t}{D_\perp}} \int^{\sqrt{D_\perp t}} dZ\, V(Z) \sim D_\perp^{\beta/2} t^{1+\beta/2} , \tag{1 71}$$

with $\beta = 1$ (shear flow) or $\beta = 2$ (Poiseuille flow)

(iv) One can generalize the problem to higher dimensionalities and consider the geometry of figure 1 12b In this case the number of different "cells" probed by the particle is $t/\ln t$ (see appendix A) and the resulting diffusion behaviour is thus

$$\overline{X_t^2} \sim t \ln t \tag{1 72}$$

(v) One may also consider, following Redner [Red89, Bou89f], an isotropic version of the above model which, in two dimensions, is defined through

$$V_x(x, z) \equiv V_x(z) \text{ random} , \qquad V_z(x, z) \equiv V_z(x) \text{ random} \tag{1 73a}$$

("random Manhattan" velocity field, see fig 1 12c) Assuming $R \sim t^\nu$, one has

$$\langle V_x(0) V_x(t) \rangle = \langle V_z(0) V_z(t) \rangle \sim t^{-1} ,$$

and hence, using (1 63),

$$R \sim t^\nu \sim t^{1-1/2} \quad \text{or} \quad \nu = \tfrac{2}{3} \tag{1 73b}$$

1.3 3 Memory effects and long-range correlations. new insights into the Flory approach to polymers

Self-avoiding walks (SAWs) faithfully model linear polymers in a good solvent [G16–G18], that is when the attractive part of the interaction is screened out and only the strongly repulsive hard core must be taken into account In three dimensions, for example, the self-avoiding constraint makes the chain "swell" the end to end distance (or the gyration radius) follows, as a function of the number of monomers N, a "hyperdiffusive" law,

$$R \sim N^{\nu}, \quad \nu \geq \tfrac{1}{2} \tag{1 74}$$

Long before the identification of this problem with the critical behaviour of a n-component spin model in the limit $n \to 0$ [dGe72], Flory did propose [G15, Fis69] an *approximate* formula for ν, which turns out to be remarkably accurate Flory's approach has been extended to a host of other problems, and is very often successful Nevertheless, the very method of Flory has often been described as ill founded, and its success fortuitous The aim of this subsection is to show how Flory's method can in fact be understood in statistical terms, as a self-consistent approach to a problem in which long-range correlations play a dominant role

1 3.3 1 Flory's approximation: the conventional picture The usual way to establish Flory's formula is to estimate the different contributions to the "free energy" of a chain of N monomers and size R, and then to find the optimal size minimizing this free energy

– One thus considers the repulsive energy coming from the hard core repulsion, and thus proportional to the number of contacts within the chain A mean field estimate of the latter is simply the product of the number of monomers times the average density,

$$F_{\text{rep}} \simeq kTa^{d}N^{2}R^{-d} \tag{1 75}$$

(a is a typical monomer size) This term obviously favours large values of R and thus swelling of the chain

– This swelling is limited by an entropic factor which expresses the fact that there are much fewer *stretched* configurations than typical ones This entropy reads

$$S = k \ln \mathcal{N}(R, N), \tag{1 76}$$

where $\mathcal{N}(R, N)$ is taken to be the total number of conformations *of the free chain* $\mathcal{N}(R, N) = z^{N}P_{0}(R, N)$, z being the lattice coordination number, and $P_{0}(R, N)$ the Gaussian distribution associated with the free walk

The total free energy $F = F_{\text{rep}} - TS$ thus reads

$$\frac{F}{kT} = a^{d}\frac{N^{2}}{R^{d}} + \frac{R^{2}}{a^{2}}N - N \ln z, \tag{1 77}$$

which, upon minimization, yields

$$R \simeq aN^{\nu}, \quad \nu = 3/(d+2) \tag{1 78}$$

This value of ν is the Flory approximant. It is exact in $d = 1$ ($R = N$ trivially), and in $d = 2$ ($\nu = 3/4$

Table 1 1
Exponent ν for linear polymers

	"$d = 1 + \varepsilon$" [Sta82]	$d = 2$	$d = 3$	$d = 4 - \varepsilon$
"exact"	$1 - \varepsilon/2$	3/4	0 588	$1/2 + \varepsilon/16$
Flory ($\beta = 1$, upper bound?)	$1 - \varepsilon/3$	3/4	3/5	$1/2 + \varepsilon/12$
lower bound? ($\beta = 1/2$)	$4/5 - 4\varepsilon/25$	2/3	4/7	$1/2 + \varepsilon/16$
$\beta = \nu$	$1 - \varepsilon/2$	0 707	0 571	$1/2 + \varepsilon/16$

from conformal invariance [Nie82]); it predicts $d_c = 4$ as the upper critical dimension above which the self-avoiding constraint is irrelevant ($\nu = 1/2$) It is remarkably precise in $d = 3$ $\nu_{\text{Flory}} = 0.6$, which is 2% off the theoretical [LeG85] and experimental [Cot80] values $\nu = 0\ 588\ldots$ Its $\varepsilon = 4 - d$ expansion does, however, not reproduce the correct result [dGe72] (table 1.1)

Numerous authors have nevertheless underlined some weak points in the above argument.

(a) Both terms (N^2/R^d and R^2/N) are grossly *overestimated* compared to their *actual* value for the self-avoiding walk (for example, the estimate N^2/R^d assumes no correlations between monomers—but obviously the interactions correlate them in such a way as to reduce the number of intersections). De Gennes [G16] argues that the two errors fortuitously cancel to yield a reasonable value for ν (see fig. 1 13), making improvement very delicate since the balance between the two terms should not be ruined.

(b) Flory's equilibrium free energy is found to behave as $-N \ln z + N^{2\nu-1}$, whereas the true free energy can be written as $-N \ln \bar{z} + (\gamma - 1) \ln N$. (See below for the interpretation of the exponent γ.) Hence, the minimization procedure concerns subdominant terms (which are not predicted correctly at

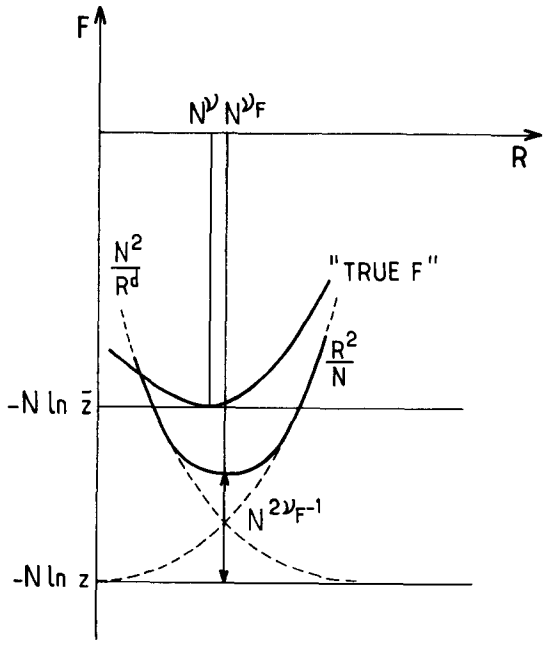


Fig 1 13 The success of Flory's approximation is usually understood as illustrated both parts of the free energy (1 77) are overestimated, but their intersection happens to be close to the exact value

the end), although Flory's approach is not able to account for the dominant term [dC170]. Furthermore, the subdominant term $N^{2\nu-1}$ has not the correct sign [dC170]

Many other works (which we shall not discuss here though some are of great interest) try to explain the reasons for this apparently undue success [G17, Edw65, Dek87] The interpretation we propose below is satisfying because it nowhere involves a "free energy" concept, but is entirely based on a statistical reasoning

1 3 3.2 Flory approximation: a statistical picture [Bou89c] The physical image that we promote is to consider a polymer as a random walk with elementary steps exhibiting long-range correlations, induced by the self-avoiding constraint and thus by the past "track" of the chain itself. This suggests a natural way to construct those correlations· first, one should notice that the self-avoiding constraint changes the large-scale structure of the walk, but that at small length scales the walk locally behaves as a random walk· local constraints are "irrelevant" (This is well illustrated by the fact that K-tolerant walks – walks such that Kth order contacts are allowed – have the same large-scale structure as truly self-avoiding walks; see also [Dek87]) We shall thus define two types of self-intersections: local self-intersections and long-range self-intersections. only the latter are responsible for the long-range "transmission" of the correlations and thus of the large-scale non-Brownian character of the walk.

We thus view the nth displacement as completely random except when this nth monomer makes a "long-range self-intersection" The integral of the correlation function between displacements,

$$N_{\mathrm{id}} = \int^{N} C(n)\,\mathrm{d}n = \overline{\sum_{n}^{N} \Delta \boldsymbol{x}_0\, \Delta \boldsymbol{x}_n} \tag{1 79}$$

(the bar stands for an average over starting points, or configurations) can be written as

$$\sum_{\substack{\text{long-range}\\ \text{contacts}}} \Delta \boldsymbol{x}_0\, \Delta \boldsymbol{x}_n + \sum_{\text{free}} \Delta \boldsymbol{x}_0\, \Delta \boldsymbol{x}_n \tag{1 80}$$

The "free" part averages to zero since the displacements in this sum are supposed to be uncorrelated (A weaker statement might be more correct: in this picture, the "free" part contributes in a subdominant way to N_{id}) If the number of long-range contacts is N_{c}, the first part in the sum contains N_{c} terms and is thus expected to scale as

$$N_{\mathrm{id}} \sim \sum_{\substack{\text{long-range}\\ \text{contacts}}} \Delta \boldsymbol{x}_0\, \Delta \boldsymbol{x}_n \sim N_{\mathrm{c}}^{\beta}\,, \tag{1.81}$$

where the exponent β measures "how correlated" these displacements are· $\beta = 1$ corresponds to perfect transmission of the correlations at the contact point More generally, one expects (1 81) to be bounded from below by Brownian behaviour and from above by the ballistic one,

$$\tfrac{1}{2} \le \beta \le 1\,. \tag{1 82}$$

Now, the end to end radius of the polymer is a sum of correlated variables, which thus scales as

$$R \sim N_{\mathrm{id}} \sqrt{N/N_{\mathrm{id}}} \tag{1 83}$$

Hence $R = N^{\nu} = \sqrt{N_c^{\beta} N}$ It is reasonable to estimate the number of *long-range* contacts in the mean field way, as

$$N_c \sim N^2 / N^{\nu d} . \tag{1 84}$$

This is much smaller than N if $\nu > 1/d$, i e., if the chain is not collapsed; the real total number of contacts is, however, known to be N, but most of them are *local* contacts. Therefore, one obtains

$$N^{\nu} \simeq N^{(2-\nu d)\beta/2+1/2} ,$$

or

$$\nu = (1 + 2\beta)/(2 + d\beta) \tag{1.85}$$

Without specifying the choice for β, one observes that the correlation function $C(n)$ decays as n^{-y} with $y = 1 + \beta(\nu d - 2)$. Correlations are relevant only if $y \leq 1$ or $\nu \leq 2/d$, and in any case $\nu \geq 1/2$. We thus find that the critical dimension above which correlations do not change the Brownian character of the walk is $d_c = 4$ independently of β. Right at $d = 4$, correlations decay as n^{-1} and in this case $N_{id} \sim \ln N$; logarithmic corrections appear naturally within this approach at the critical dimension while this is out of reach in the usual Flory argument

Now, what is the value to be given to β? A "strong" correlation hypothesis corresponds to $\beta = 1$ Equation (1.85) then gives $\nu = 3/(d + 2)$, which is precisely Flory's formula (1.78). This suggests that ν_{Flory} might in fact be *an upper bound to* ν. This indeed seems to be the case (see table 1.1). $\beta = 1/2$, on the other hand, would yield a lower bound (which is also obeyed by known results),

$$\nu \geq 4/(4 + d) . \tag{1 86}$$

A "self-consistent" choice is $\beta = \nu$, since it states that the sum in eq. (1.81) statistically behaves as the full sum of individual displacements. For $\beta = \nu$, eq (1.85) reads

$$\nu = 1/\sqrt{d} . \tag{1.87}$$

Note that, quite amazingly, this formula is exact for $d = 4 - \varepsilon$ and $d = 1 + \varepsilon$!

Now, the points we want to make appear clearly:

(a) The Flory formula combines an exact result on sums of correlated random variables (eq. 1.63) with an approximate way of estimating the correlations, namely a "mean field" approximation of the number of *long-range* contacts (and not of the *total* number of them), and "strong" transmission of the correlations hypothesis. The Flory formula more naturally appears as one member of a whole family parametrized by the exponent β, which in fact suggest bounds on ν

(b) The generalization to a large number of other problems is straightforward, without any need of constructing an elastic "free energy". The "recipe" is the following: One has to know the behaviour of (i) the sum of *perfectly correlated* variables of the type considered (for example, on a tortuous fractal, a minimal path along the structure of length s leads to a displacement $R \sim s^{\alpha}$ with $\alpha \leq 1$, see section 6.1) and (ii) the sum of *independent* random variables of the type considered, that is the value of the exponent ν above the critical dimension, which we shall denote below by ν_0.

Equation (1 85) will then be replaced by

$$R \simeq N_{id}^{\alpha}(N/N_{id})^{\nu_0} \tag{1.88}$$

Application of this method yields "Flory formulae" (with the same way of estimating N_c and with $\beta = 1$) to, e.g., self-avoiding Levy flights, self-avoiding surfaces, polymers on fractals, etc. Table 1 2 summarizes the quantities of interest for these various problems

One problem deserves special consideration: that of branched polymers for which $\alpha = 1$ and $\nu_0 = 1/4$ We obtain for the exponent governing the size of a branched polymer

$$\nu = 7/(4 + 3d)\,, \tag{1 89}$$

instead of the value proposed long ago by Isaacson and Lubensky [Isa80] (based on an uncontrolled "elastic" free energy),

$$\nu = 5/(4 + 2d)$$

Our formula is also expected to be an upper bound (which is satisfied by all known results) Obviously it reproduces $\nu = 1/4$ for $d_c = 8$ but contrarily to the previous one yields the exact result in $d = 1$ The corresponding conjectured lower bound ($\beta = \frac{1}{2}$) reads

$$\nu \geq 8/(8 + 3d)\,, \tag{1 90}$$

which again is satisfied by all known results (see table 1 2)

Table 1 2
Some Flory-type approximations and lower bounds

Self-avoiding Levy flights [dGe86]
Distribution of elementary step length $\psi(l) \sim l^{-(1+\mu)}$
Value of ν_0 $\nu_0 = 1/\mu$
Value of α $\alpha = 1$ if $\mu > 1$ and $\alpha = 1/\mu$ for $\alpha < 1$
Generalized Flory approximation ($\beta = 1$) $\nu = \nu_0$ for $\mu < 1$, $\nu = (2\mu - 1)/(d\mu - d + \mu)$ for $\mu > 1$

Self-avoiding walks on fractals
(d_f fractal dimension, d_s spectral dimension, $\hat{d}$ spreading dimension, see section 6 1)
Value of ν_0 $\nu_0 = d_s/2d_f$
Value of α $\alpha = \hat{d}/d_f$
Generalized Flory approximation ($\beta = 1$) $\nu d_f = (4\hat{d} - d_s)/(2 + 2\hat{d} - d_s)$ [G12, Bou89c, Aha89]

Self-avoiding manifolds [Kan86],
(internal dimension of the manifold D)
Value of ν_0 $\nu_0 = 1 - D/2$ (logarithm for $D = 2$)
Value of α $\alpha = 1$
Generalized Flory approximation $\nu = [2 + D(2\beta - 1)]/(2 + d\beta)$

Branched polymers

	$d = 1$	$d = 2$	$d = 3$	$d = 8 - \varepsilon$
"exact"	1	0 6408 [Der82a]	1/2 [Par81]	$1/4 + 3\varepsilon/118$ [Lub79]
Flory ($\beta = 1$)	1	0 70	7/13	$1/4 + 3\varepsilon/112$
lower bound? ($\beta = 1/2$)	8/11	4/7	8/17	$1/4 + 3\varepsilon/132$
Lubensky–Isaacson [Isa80]	5/6	5/8	1/2	$1/4 + \varepsilon/80$

We feel that more work should be done to establish rigorously the bounds suggested here, which naturally emerge from the physical picture and methods that we propose This would at last settle the precise status of the "Flory approximation"

1 3 3 3 Limiting distribution for the end to end distance Once the "diffusion law" relating the average radius of gyration to the number of monomers is established, one would like to go further and obtain the full probability distribution $P(R, N)$ For large R and N, one expects that *in the scaling region* $R \sim N^{\nu}$, $P(R, N)$ takes a scaled, universal shape, of the form

$$P(R, N) = N^{-\nu d} f(R/N^{\nu}) , \tag{1 91}$$

independently of the microscopic details of the model (generalized CLT) Using the equivalence with the $n = 0$ ϕ^4 field theory, the asymptotic behaviour of $f(u)$ (both for $u \to 0$ and for $u \gg 1$) can be found One has [Fis66, dC180, G16]

$$f(u) \simeq u^{(\gamma-1)/\nu} \quad \text{for } u \to 0 , \tag{1 92a}$$

$$f(u) \simeq u^{\sigma} \, e^{-u^{\delta}} , \quad \delta = 1/(1-\nu) , \quad \sigma = \delta(1 - \gamma + \nu d - d/2) \tag{1 92b}$$

The shape of f is sketched in fig 1 14 Equation (1 92a) shows that the probability for the last point of the polymer to lie very close to the initial point is *reduced* by a factor $N^{1-\gamma}$ as compared to the naive "mean field" prediction $P_0(R = a, N) \simeq (a/R)^d \sim N^{-\nu d}$ In other words, there are $N^{1-\gamma}$ fewer self-avoiding polygons than self-avoiding walks with free ends (see the detailed discussion in ref [G16]) Note that it has been suggested in ref [G17] that in the spirit of a Flory approach, one should take $\gamma = 2\nu$ Equation (1 92b) has already been encountered in section 1 2 3 1 (eq 1 42) and is closely related to the response of a SAW to an external bias, as will be extensively discussed in chapter 5

1.3 4 Spin models at their critical point

As a last illustration of the physics of long-range correlations, we wish to present some features of spin models at criticality using the statistical language and tools introduced above (for early related ideas, see [Cas78]).

As is well known, the magnetization

$$\boldsymbol{M} = \sum_{i=1}^{L^d} \boldsymbol{S}_i$$

of a spin model on a d-dimensional lattice of size L^d is the result of a competition between entropy (inducing disorder) and energy (acting to correlate the spins on large distances). At high temperature, the former dominates and correlations between spins are short ranged. One thus has (say for Ising spins

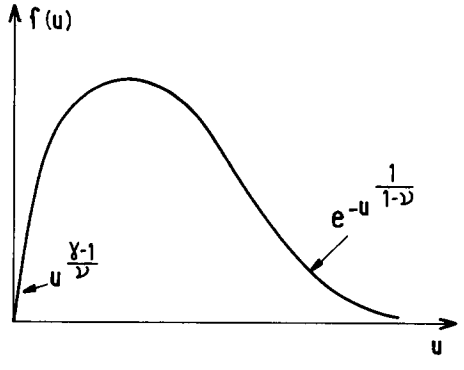


Fig 1 14 Scaling function characterizing the limit distribution of the position $P(R, N)$ for a polymer

$S_i = \pm 1$) in the whole high-temperature region

$$M = \sum_{i=1}^{L^d} S_i \sim \sqrt{L^d} \tag{1 93}$$

More precisely, the CLT applies in the usual Gaussian form,

$$P(M, L^d) \sim \frac{1}{\sqrt{2\pi\chi L^d}} \exp(-M^2/2\chi L^d) ,$$

with a variance equal to the magnetic susceptibility On the contrary, in the low-temperature (pure) phase, perfect correlations are favoured and lead to a non-zero value of the correlation function at large separations, equal to the square of the remnant magnetization m,

$$M = \sum_{i=1}^{L^d} S_i \simeq L^d m \pm \sqrt{L^d} \tag{1 94}$$

In this phase, the ratio M/L^d has a non-vanishing limit for large L The distribution $P(M, L)$ for large L is essentially a Gaussian distribution centered at $+mL^d$ (or $-mL^d$)

Precisely at the critical temperature T_c, the spin correlations decay algebraically,

$$\langle S(0)S(r)\rangle \sim r^{-(d-2+\eta)} ,$$

and, according to section 1 3 1, the system can be pictured as N_{eff} "effectively independent" families of N_{id} perfectly correlated spins, with

$$N_{\text{id}}(L^d) = \int^{L} \frac{r^{d-1}\,\mathrm{d}r}{r^{d-2+\eta}} \simeq L^{2-\eta}$$

The fluctuations of the total magnetization are thus of order

$$M \simeq \sqrt{L^d N_{\text{id}}} \sim (L^d)^{\nu} , \tag{1 95}$$

$$\nu = (d + 2 - \eta)/2d \neq \tfrac{1}{2} \tag{1 96}$$

Thus, at the critical point, the total magnetization follows an anomalous power law as a function of the number of sites L^d with an "anomalous diffusion exponent" ν More precisely, it is the variable $M/(L^d)^{\nu}$ which has a limit distribution for $L \to \infty$ at the critical temperature:

$$P(M, L^d) = L^{-d\nu} f(M/L^{d\nu}) \tag{1 97}$$

The following information is available about the scaling function f for the Ising model (sometimes called the "block-spin" distribution in the framework of real-space renormalization group studies or of rigorous approaches)

– In one dimension, an analytical expression due to Bruce [Bru81] can be found from the solution of the

model in a field, yielding

$$f(x) = \tfrac{1}{2}\delta(x-1) + \tfrac{1}{2}\delta(x+1)\,.$$

– Numerical simulations by Binder [Bin81] show that f depends on the boundary conditions For free boundary conditions, its shape evolves from a double-peaked to a single-peaked structure when the dimension is increased from $d=2$ to $d=3$. Some approximate analytical treatments have been performed in [Bru79, Bru81] using Wilson's approximate recursion relations. Finally, some *moments* of f are known in two dimensions, using conformal invariance [Bur85] (in particular $\langle M^4\rangle / \langle M^2\rangle^2 = 1.33$ instead of 3 for a Gaussian).
– As we shall argue in section 5.3.3, the shape of the *tails* of $f(u)$ only depends on the dimensionality and on the critical exponent at criticality. This follows from the analysis of the response to a weak magnetic field, which can be shown on a general basis (section 5 3.3) to follow a non-linear law,

$$M \sim L^d H^{\nu/(1-\nu)}\,, \tag{1 98}$$

from which it follows that ($u \gg 1$)

$$f(u) \simeq \mathrm{e}^{-u^{\hat\delta}}\,, \quad \hat\delta = 1/(1-\nu) \tag{1.99}$$

(i e., $\hat\delta = 4$ for $d=4$, $\hat\delta \simeq 6$ for $d=3$ and $\hat\delta = 16$ for $d=2$), and that one has the well-known relation between the critical exponents δ and η ($M \sim H^{1/\delta}$),

$$\delta = \nu/(1-\nu) = (d+2-\eta)/(d-2+\eta) \tag{1.100}$$

These probabilistic remarks concerning the critical state provide some insights on the neighbourhood of T_c, and in particular allow one to emphasize the statistical content of the scaling relations between critical exponents, defined conventionally by

$$\begin{aligned} &F \simeq L^d |T-T_c|^{2-\alpha}\,, \qquad \chi \simeq |T-T_c|^{-\gamma}\,, \\ &\xi \simeq |T-T_c|^{-\nu_{\mathrm{th}}}\,, \qquad m \simeq |T-T_c|^{\beta}\,. \end{aligned} \tag{1 101}$$

For T close to T_c, one can picture the system as made up of regions of size ξ in the critical state There are $(L/\xi)^d$ such regions, which each contribute to the total free energy by an amount of order kT,

$$F/kT \simeq L^d/\xi^d \tag{1 102a}$$

Hence the Josephson relation

$$2-\alpha = \nu_{\mathrm{th}} d\,. \tag{1 102b}$$

The typical size of the fluctuations of the total magnetization can be estimated as ($T < T_c$)

$$M = (\xi^d)^{\nu} L^d \xi^{-d} \;\Rightarrow\; m \sim \xi^{d(\nu-1)}\,; \tag{1 103a}$$

this leads to the relation

$$\beta = d\nu_{\rm th}(1-\nu) \tag{1.103b}$$

Accordingly, the susceptibility to an external magnetic field is enhanced by correlations In the high-temperature phase, for example, the fluctuations of the total magnetization read

$$\delta M \sim \sqrt{(L/\xi)^d}\ \xi^{d\nu} \tag{1 104a}$$

Hence, from $\chi T_c = \delta M^2/L^d$, one obtains $\gamma = \nu_{\rm th} d(2\nu - 1)$, leading to

$$\gamma \equiv \nu_{\rm th}(2-\eta) \tag{1 104b}$$

Using (1 104) and (1 100), eq (1 103a) can be written as

$$\beta(\delta - 1) = \gamma$$

Making use of (1 102), this also reads

$$\alpha + 2\beta + \gamma = 2$$

Equations (1.102)–(1 104) form the well-known set of relations between critical exponents, which have been recovered here within the framework of sums of random variables

1 4 Conclusion and main points of chapter 1

In this chapter, we have tried to underline two major ideas.

(a) The breakdown of the canonical CLT and of the "mean field" value of the diffusion exponent can be due either to very large fluctuations (broad distributions) or to long-range correlations The most interesting situation occurs when those statistical pathologies are *induced by the dynamics of the problem* itself, and not introduced ab initio One should therefore look for possible mechanisms generating large fluctuations and/or strong correlations Once those mechanisms are identified, their consequences for the transport properties of the problem studied can be discussed using the simple tools developed in sections 1 2 1 and 1 3 1 It can happen that the identification of the mechanism suggests a self-consistent solution to the problem this has been encountered for polymers and should perhaps be dwelled upon further for spin models.

More precise statements can be made using more sophisticated techniques, In the next chapters we shall present and develop some of them, trying as much as possible to discuss the results in the language introduced in this chapter

(b) The concepts of universality classes, fixed points, scaling, which have emerged in the last twenty years within the framework of critical phenomena and the renormalization group have a very deep statistical origin [Cas78], the theory of stable distributions, and in particular the central limit theorem, shows very pedagogically how the microscopic information is grinded and processed to extract very few "relevant" quantities, the remaining part of it being irremediably lost in the tails (recall the H-theorem, which relies on the same mechanism) The robustness of this mechanism is in fact what makes possible at all the description of large complex systems in terms of very few "macroscopic" parameters

While the theory of stable laws for uncorrelated variables is very well established, the case of correlated sums does not stand on equally systematic ground In a sense field theory and renormalization group have handled this problem efficiently in the case of correlations generated by a Gibbs measure; those are perhaps the natural tools to explore this path further

2. Diffusion on lattices with random hopping rates: introduction, related problems and simple results

2 1 *Introduction and basic quantities*

2.1.1 *Random walks in quenched disordered media*

The most natural model for a heterogeneous material in which the local transport coefficients and driving fields are highly complex and irregular is to consider a given sample as a particular statistical realization of an ensemble, constructed by randomly choosing those local quantities according to a certain probability distribution "Quenched" disorder means that those random quantities *do not evolve with time*: this is usually the model considered to describe the dynamical properties of materials containing impurities, defects, or intrinsic randomness (amorphous systems), provided that the time scale of the observed dynamical property (e.g. the diffusion of a tracer or of a carrier) is much shorter than that of, e g , diffusion of the impurities, or more generally the "turnover" time of the disorder Many examples of such physical situations will be encountered in the following. conductivity of amorphous materials or of quasi-one-dimensional ionic conductors, dynamics of domain wall or dislocations, transport of a dye in porous materials (with or without flow), etc

More generally, one can consider the dynamics of a complex system as the diffusion of its representative point in its tortuous phase space (chaotic dynamical system, configuration space of a spin glass, etc)

The models studied below are random walks on euclidean lattices with *inhomogeneous* transition rates W_{nm} from site m to site n (for diffusion on fractal lattices see chapter 6). If on the time scale of the experiment disorder evolves slowly, the physical problem amounts to considering a given configuration of hopping rates (W_{nm}) chosen once and for all with probability $\psi[\{W_{nm}\}]$. One is then interested in the properties of random walks in this quenched environment It is crucial to realize that one must pay special attention to the definition of the various quantities of interest. It often happens that two equivalent ways of defining a physical quantity in an ordered medium no longer lead to the same result in the presence of disorder. In particular, one should carefully distinguish the following two averages·
– Averages over the different "thermal histories" (each one being defined by the succession of hops performed by the particle), the environment being fixed. Such averages will be denoted in the following of this article by an overbar· $\overline{(\cdot)}$
– Averages over the possible environments, according to the distribution $\psi(W_{nm})$. Such averages will be denoted by brackets: $\langle(\cdot)\rangle$.

The position X_t of a single walker is a random variable depending both on the thermal history (which we shall sometimes denote symbolically by η) and on the environment $\omega = \{W_{nm}\}$ The latter being fixed, one can then obtain information on the statistical properties of $X_t(\eta; \omega)$ in the following ways·

(1) One can study a packet of particles, all starting at $t = 0$ at the *same initial site* n_0, but undergoing *different thermal histories* The probability of presence on site n at time t, $P_n(t|n_0, 0)$ is defined by the thermal average (a denotes the lattice spacing),

$$P_n(t|n_0, 0) = \overline{\delta[n - a^{-1}X_t(\eta; \omega)]},$$

and still depends on the environment It describes the shape of the packet at time t and obeys the following master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P_n(t) = \sum_m W_{nm} P_m(t) - \Big(\sum_m W_{mn} P_n(t) \Big) , \tag{2 1}$$

with initial condition and normalization

$$P_n(t=0|n_0, 0) = \delta_{n\, n_0} , \qquad \sum_n P_n(t|n_0, 0) = 1$$

Introducing the local currents $J_{nm} = W_{mn} P_n(t) - W_{nm} P_m(t)$, eq (2 1) can be written in the form of a conservation equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_n(t) + \sum_m J_{nm} = 0 \tag{2 2}$$

(ii) One can also study the time evolution of a packet of particles having some distribution of initial positions $\rho_0(n_0)$. Equation (2 1) being a linear equation, the shape of the packet at time t is given by

$$\rho_n(t) = \sum_{n_0} \rho_0(n_0) P_n(t|n_0, 0) \tag{2.3}$$

An extreme case is the uniform distribution $\rho_0 = 1/N$ (N is the total number of sites) In this case one averages overall initial positions and it is clear that, provided the distribution $\psi(W_{nm})$ is translationally invariant, this coincides with an average over environments,

$$\frac{1}{N} \sum_{n_0} P_{n_0+n}(t|n_0, 0) \xrightarrow[N\to\infty]{} \langle P_{n_0+n}(t|n_0, 0) \rangle , \tag{2 4}$$

where the r h s no longer depends on n_0 (2.4) is the distribution of the random variable $X_t(\eta, \omega)$ over *both* thermal histories and environments.

One could also study other quantities such as the distribution of $X_t(\eta, \omega)$ over environments for a single thermal history, etc.

2 1 2. Characterization of the diffusion process

2 1 2 1 Behaviour of the moments, fluctuations The shape of the packets $P_n(t|n_0, 0)$ and $\rho_n(t)$ are conveniently characterized by their moments, in particular the first one measures the position of its "centre of mass" and the variance gives an estimate of its spreading

From $P_n(t|n_0, 0)$, one defines the (asymptotic) *velocity* and *diffusion tensor* as

$$\begin{aligned} V &= \lim_{t\to\infty} (\mathrm{d}/\mathrm{d}t) \overline{X_t(\eta, \omega)} , \\ D_{\alpha\beta} &= \tfrac{1}{2} \lim_{t\to\infty} (\mathrm{d}/\mathrm{d}t) [\overline{X_t^\alpha(\eta, \omega) X_t^\beta(\eta, \omega)} - \overline{X_t^\alpha(\eta, \omega)}\, \overline{X_t^\beta(\eta, \omega)}] \end{aligned} \tag{2 5}$$

These quantities a priori still depend on the environment. However, in all the cases studied in the following in which diffusion is normal, it will turn out that they take, with probability one, a value independent of the specific environment chosen [and which is a function only of a few parameters

characterizing the distribution $\psi(W_{nm})$] This is the so-called "self-averaging" property, which physically relies on the fact that at large times the whole configuration $\{(W_{nm})\}$ has been properly sampled Despite its clear physical origin, up to now this property has only been "rigorously" proven in a limited number of cases (see section 3.1 and [Asl89a]). However, as will be explained below, one should *not* conclude, even when this property holds, that at large time the full diffusion process no longer exhibits fluctuations with respect to the environment.

When the effect of disorder is not dramatic, one expects a finite (i e. non-zero and non-infinite) diffusion constant, together with a finite velocity if the process is biased; this will be called "*normal diffusion*" in the following All other situations will be called "*anomalous*", this encompasses non-linear time dependence of $\overline{X_t}$ in the presence of a bias (global or local) and non-linear dependence of the spreading $\overline{X^\alpha X^\beta} - \overline{X^\alpha}\,\overline{X^\beta}$ We shall encounter situations in which these quantities follow non-Brownian power laws as well as a more complicated time dependence than power laws

The centre and width of an arbitrary packet $\rho_n(t)$ also define a velocity and diffusion constant,

$$V[\rho_0] = a \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\boldsymbol{n}} \boldsymbol{n}\rho_n(t)\,,$$
$$D_{\alpha\beta}[\rho_0] = \frac{a^2}{2} \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{\boldsymbol{n}} n_\alpha n_\beta \rho_n(t) - \sum_{\boldsymbol{n}} n_\alpha \rho_n(t) \sum_{\boldsymbol{n}} n_\beta \rho_n(t) \right) \tag{2 6a}$$

For the same physical reason as above, these quantities no longer depend on the environment in general Furthermore, the self-averaging property for V in (2 5) implies that $V[\rho_0]$ in fact does not depend on the initial distribution: $V[\rho_0] = V$,

$$V[\rho_0]t + \cdots \sim \sum_n \boldsymbol{n} \sum_{n_0} \rho_0(n_0) P_n(t|n_0, 0) = \sum_{n_0} \rho_0(n_0) \sum_n \boldsymbol{n} P_n(t|n_0, 0) \sim Vt\,. \tag{2.6b}$$

This is, however, *not always the case* [Asl89d, Dou89a] for the diffusion constant $D[\rho_0]$, which can be put in the form,

$$D_{\alpha\beta}[\rho_0] = D_{\alpha\beta} + \frac{a^2}{2} \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{n_0} \rho_0(n_0) \overline{n_\alpha}\,\overline{n_\beta} - \sum_{n_0} \rho_0(n_0) \overline{n_\alpha} \sum_{n_0} \rho_0(n_0) \overline{n_\beta} \right),$$
$$\overline{n_\alpha} \equiv \sum_n n_\alpha P_n(t|n_0, 0)\,. \tag{2 6c}$$

The last term is the fluctuation of the centre of mass position, which in general does not vanish at large time, and depends a priori on the shape of the initial distribution $\rho_0(n_0)$. In particular, for a uniform initial distribution $\rho_0(n_0) = 1/N$, $D[\rho_0]$ measures the spreading of the full distribution $\langle P_n(t|n_0, 0)\rangle$, and will for this reason be called D_{av} ("av" standing for "average", D_{av} has been called "annealed diffusion constant" in [Dou89a], but it should not be confused with the diffusion constant of the annealed model):

$$D^{\mathrm{av}}_{\alpha\beta} = \frac{a^2}{2} \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left(\langle \overline{X^\alpha_t X^\beta_t} \rangle - \langle \overline{X^\alpha_t} \rangle \langle \overline{X^\beta_t} \rangle \right).$$

Four remarks

– As is clear from (2.6c), the reason which allows D_{av} to be different from D is the sample to sample fluctuation of the corrections to the leading behaviour of the thermal average $\overline{X_t}$ When diffusion is

normal, this fluctuation turns out to exist only in one dimension and when a global bias exists [Dou89a] $D_{av} = D$ in more than one dimension, when diffusion is normal Note that the fact that $\langle \bar{X}_t^2 \rangle / \langle \bar{X}_t \rangle^2 - 1$ behaves at most as $1/t$ precisely guarantees the self-averaging property of the velocity.

- One can wonder whether $D[\rho_0]$ can have some continuous dependence on ρ_0 As pointed out in [Dou89a], for "reasonable ρ_0", D and D_{av} are the only generic values. For a superposition $p\delta_{n\,n_0} + (1-p)/N$, one easily sees that $D[\rho_0] = p^2 D + (1-p^2) D_{av}$ [Dou89a]

- D_{av} could have some relevance to experiments, despite the fact that it involves an average over environments Suppose that one performs a diffusion experiment in a system made up of one-dimensional channels, following in each one the diffusion of a concentrated packet (fig 2.1) Each channel being a different environment, one realizes an average over samples if one observes the packet with some low resolution This picture (fig 2 1) clearly illustrates why one has $D < D_{av}$, as is clear from (2.6c) Furthermore, *ergodicity* suggests that the histogram in time of the positions of a single particle for fixed η and ω should coincide with $\langle P(X,t) \rangle$, and thus be characterized by D_{av}.

- When diffusion is anomalous, one should realize that the *typical* $\overline{X_t^\alpha X_t^\beta} - \overline{X_t^\alpha} \overline{X_t^\beta}$ and *average* $\langle \overline{X_t^\alpha X_t^\beta} \rangle - \langle \overline{X_t^\alpha} \rangle \langle \overline{X_t^\beta} \rangle$ spreading can have a very different behaviour in the presence of a bias (even only a *local bias*, zero on average) This is particularly dramatic in the case of Sinai's diffusion discussed in section 3 3, as first pointed out by Golosov [Gol84] A clear-cut example is provided by the layered medium of section 1 3 2, which, despite the absence of a global bias, is such that $\overline{X_t^2} - \bar{X}_t^2 \sim Ct^{3/2}$ while $\langle \overline{X_t^2} \rangle \sim C_{av} t^{3/2}$ with $C_{av} > C$, as pointed out in [Dou89a] The values of C and C_{av} are computed in appendix C, where these fluctuation properties are studied in detail

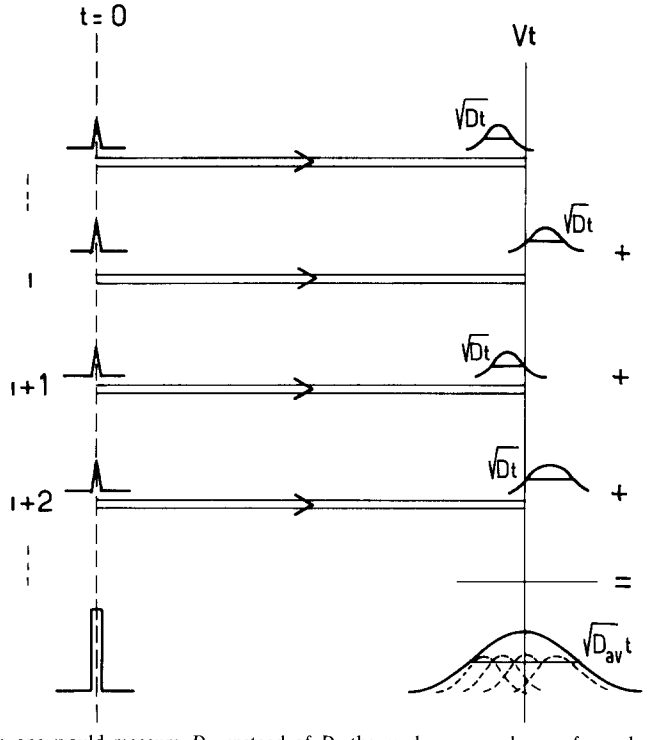


Fig 2 1 Physical situation where one would measure D_{av} instead of D the medium is made up of one-dimensional "fibres" along which the particles may diffuse The diffusion front for each channel is characterized by D, but the position of the centre of mass depends upon the channel If one cannot "resolve" the different channels, one will observe the envelope of the whole diffusion front, which has a width D_{av}, obviously larger than D

2 1 2.2. On the existence of a generalized CLT for the distribution of the position As discussed at length in chapter 1, for an ordered lattice $P(X, t)$ obeys the central limit theorem, which provides a precise characterization of the diffusion process in the large-time limit. One would like to extend such a characterization of the probability distribution of $X_t(\eta, \omega)$ in the presence of disorder in the cases where diffusion is normal as well as when it is anomalous One can consider several distributions, e g ,

(i) the distribution over thermal histories $P(X, t)$, for a given environment,

(ii) any of the distributions $\rho(X, t)$, and in particular the average one $\langle P(X, t)\rangle$, with respect to both thermal histories and environments,

(iii) one can also construct the histogram of the positions of a *single* particle for a given thermal history and environment

For each of these distributions, the question is whether it is possible to find $V(t, \quad)$ and $\xi(t, \quad)$ such that it reaches at large time a scaling form ($d = 1$),

$$\frac{1}{\xi(t)} f([X - V(t)]/\xi(t)) \tag{2 7}$$

(in the usual sense of a CLT, that is, a scaling region being understood) In addition, one can wonder whether it is possible to choose $V(t)$ and $\xi(t)$ to be functions of t only, devoid of any fluctuations with respect to the environment In all the cases considered in the following, the answer will turn out to be in the affirmative provided no local or global bias exists ($W_{nm} = W_{mn}$) In the opposite case, it can happen in low dimension, or in the presence of long-range correlations that $\langle P(X, t)\rangle$ but *not* $P(X, t)$ satisfies a generalized CLT with non-fluctuating $V(t)$ and $\xi(t)$. This is obviously the case, for normal diffusion, whenever $D \neq D_{\mathrm{av}}$ Examples with only local bias and anomalous diffusion are provided by the layered medium of section 1.3.2, for which this question is studied in appendix C, and by Sinai's random walk (section 3 3) Finally, general considerations about ergodicity suggest that the histogram should in general satisfy the same CLT as $\langle P(X, t)\rangle$ However, it could be that in some cases ergodicity is broken (see section 3 3)

A quantity we shall often be interested in in the following is $P(X = 0, t|0, 0)$, that is, the probability of finding the particle on its initial site after time t It characterizes the *relaxation properties of the model*. Even at large time, this quantity may contain different information than the one provided by the CLT, since X is kept fixed and t arbitrary: $X = 0$ can be inside or outside the scaling region and may thus be unrelated to the scaling function f (if any). As illustrated, e g , by section 3 3, $P(X = 0, t)$ *is not expected to be self-averaging* in general [Del89, Bou89b] (see, however, [Der83c] for a discussion of the symmetric case $W_{nm} = W_{mn}$)

2 1 3 The steady state

Starting from any initial distribution function, an equilibrium state can be reached – the boundary conditions being chosen such that no "macroscopic" current exists – if *all local currents* can vanish simultaneously, that is, if the "detailed balance" condition is satisfied,

$$P_n^{\mathrm{eq}}/P_m^{\mathrm{eq}} = W_{nm}/W_{mn} \quad \Rightarrow \quad J_{nm}^{\mathrm{eq}} = 0 \tag{2 8}$$

For this condition to be satisfied, the product

$$\prod_{\mathscr{C}_{12}} W_{nm}/W_{mn}$$

must be independent of the path $\mathscr{C}_{12}$ linking any two sites n_1 and n_2 If this is not the case, no true equilibrium distribution exists but rather the system reaches a *dynamical steady state* with uncompensated local currents An example of such a situation is presented in the next section.

2 1 4 Three classes of models

We shall mainly be concerned with the three following classes of models

(A) *Traps* Each lattice site is considered as a trap with mean desorption time τ_n (fig 2 2),

$$W_{nm} = 1/z\tau_m \,, \qquad W_{mn} = 1/z\tau_n \,, \tag{2 9}$$

where z is the lattice coordination number The master equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t} P_n(t) = \sum_m \frac{P_m(t)}{z\tau_m} - \frac{P_n(t)}{z\tau_n} \,, \tag{2 10}$$

and the equilibrium density satisfying (2 8) is

$$P_n^{\mathrm{eq}} \propto \tau_n$$

(B) *Symmetrical barriers* Each *link* of the lattice acts as a symmetrical barrier (fig 2.3),

$$W_{nm} = W_{mn} \,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} P_n(t) = \sum_m W_{nm}(P_m(t) - P_n(t)) \tag{2 11}$$

In this case the equilibrium density is uniform· $P_n^{\mathrm{eq}} = \text{constant}$

(C) *Random forces* On each link of the lattice lives a *force* F_{nm} $(= -F_{mn})$ (fig 2 4) and the transition rates are *activated*,

$$W_{nm} = W_0 \exp(-F_{nm}a/2kT) \,, \qquad W_{mn} = W_0 \exp(+F_{nm}a/2kT) \,, \tag{2 12}$$

where kT is the temperature This model does not in general*) satisfy detailed balance and local

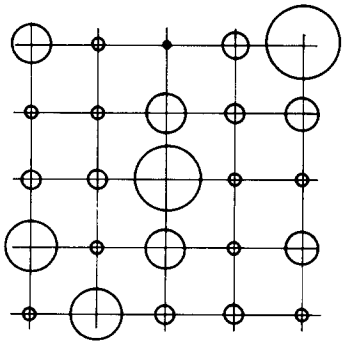


Fig 2 2 Schematic view of model A diffusion among static random traps

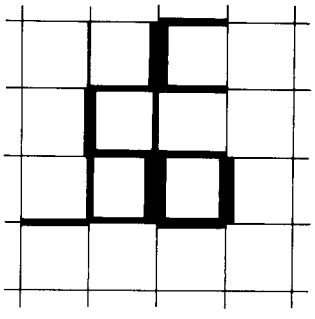


Fig 2 3 Schematic view of model B random barriers controlling the local current

*) Condition (2 8) can be satisfied only if the discretized curl of $\boldsymbol{F}$ vanishes, i e , if $\boldsymbol{F}$ is the gradient of a potential

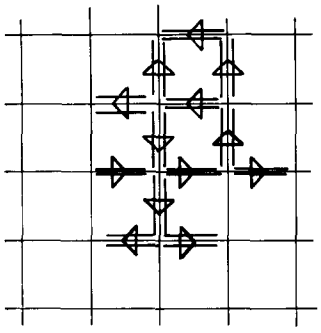


Fig 2 4 Schematic view of model C random forces, creating a local random bias

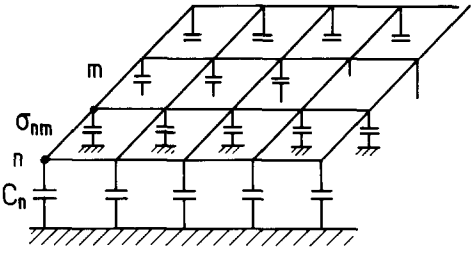


Fig 2 5 Random resistor network, with each node connected to the ground by random capacitors

currents can survive in the stationary state Only if the forces F_{nm} are the "gradient" of a potential U, $F_{nm} = (U_n - U_m)/a$, will an equilibrium state exist. It then takes the Boltzmann form, $P_n^{eq} \sim \exp(-U_n/kT)$. (Note that this is always the case in one dimension.) The opposite case, where $\boldsymbol{F}$ is divergenceless (div $\boldsymbol{F} = 0$), corresponding, e g., to an incompressible hydrodynamic flow, clearly leads to a stationary state with closed current loops, indeed, the stationary state reads in this case $P_n = \text{const.} = P_0$, and hence

$$J_{mn} = (W_{mn} - W_{nm})P_0 = 2W_0P_0 \sinh(aF_{nm}/2kT) .$$

The diffusion behaviour for models A and B will be given in this chapter, in all dimensions. Type C models, far more complex and richer, will be dealt with in chapters 3 ($d = 1$) and 4 ($d > 1$).

2.2. *Equivalence with electrical and mechanical problems*

The diffusion problems presented above have electrical and mechanical analogues of great practical importance (For a general reference on this topic, see [Doy84])

2.2.1. *Random resistor and capacitance network*

The evolution equation of the *charges* of the capacitances C_n located at the nodes of the electrical network drawn in fig 2.5 reads

$$dq_n/dt = \sum_m \sigma_{nm}(q_m/C_m - q_n/C_n) . \tag{2 13}$$

This is identical to the master equation (2 1) upon the identification

$$P_n(t) \leftrightarrow q_n(t) , \qquad W_{nm} \leftrightarrow \sigma_{nm}/C_m . \tag{2 14}$$

The above three types of models correspond to the following choices:

* C_n random and $\sigma_{nm} \equiv \sigma$ for the trapping model (A) with local trapping times defined by $z\tau_n = C_n/\sigma$ (which is the local time constant of the capacitance C_n).
* $C_n = C$ and σ_{nm} random for the symmetrical barrier model (B), with $W_{nm} = \sigma_{nm}/C$ In this case the potential $V_n = q_n/C$ follows the same equation. The "equilibrium" situation is obviously $V_n = \text{constant}$.

* C_n random and $\sigma_{nm} \sim \sqrt{C_n C_m}$ corresponds to model C (in the potential case), with $C_n = \exp(U_n/kT)$

2 2 1 1 The "Einstein relation" for the random resistor network. If *only the resistances are random* (case B), the average conductivity of the lattice σ and the diffusion constant D of the equivalent symmetrical barrier model are related by

$$\sigma/C = D/a^2 , \tag{2 15}$$

whenever they exist (a is the lattice spacing) A general derivation of this property, due to Derrida [G14], is given in appendix D Three very important remarks must be made, in order to clarify somewhat points which are often obscure in the literature (see, however, [Gef87])

(a) We have improperly called eq (2.15) an "Einstein relation"; in fact, it is *not* a statement about the response of a diffusive particle to an external electric field, but only a "translation" between the electric language and the random walk language. In particular, a random network subjected to an external potential drop is *not* related to a biased random walk problem (see chapter 5 for a discussion of the response of random walks to a bias). We shall no longer use the term "Einstein relation" in the present context in the following

(b) The derivation of ref. [G14] in fact holds for periodic lattices obtained by "tiling" boxes of size L^d, all containing the *same* random configuration of W_{nm} It is, however, expected to be true in all situations where there exists a length scale above which the medium can be considered as homogeneous (for example, the percolation structure above the correlation length ξ)

(c) This theorem does *not* in general extend to the case of random capacitors (an example will be given below in which the charges q_n diffuse anomalously with $D = 0$, while the conductivity is finite) nor to the finite-frequency case If diffusion is anomalously slow (e g on the percolation cluster, cf section 6.3.2), one can argue that the finite size d c conductivity of the sample will scale as (ν is the diffusion exponent)

$$\sigma(L) \sim D(L) \sim L^{2-1/\nu} \xrightarrow[L\to\infty]{} 0 \tag{2 16}$$

At finite frequency ω, a length scale appears (the penetration depth of the charges), connected to ω by $L_p \sim \omega^{-\nu}$. One very often argues [Gef83] that for $a \ll L_p \ll L$, σ becomes independent of L and has the scaling form

$$\sigma(L, \omega) = \sigma(L, 0) f(i\omega L^{1/\nu}) , \tag{2 17}$$

leading to

$$\sigma(L; \omega) \sim (i\omega)^{1-2\nu} \tag{2 18}$$

This, however, cannot be true in general (see also sections 2 2 1 3 and 6 3 2). the complex impedance (defined, e g , by U_B/I_A with $U_A = 0$, A and B being the end points) of a one-dimensional *regular* chain is of the form $[f(\omega)]^L$ This is also true for the fractal Koch curve with a different $f(\omega)$ [Gef87]. Scaling relations of the form (2.17) are nevertheless useful when one deals with the *input impedance* of a random network; see section 2 2 1.3 below (and section 6 3) for a detailed example

2 2 1 2 Diffusion and conductivity a second theorem [Doy84, Gef87] Another theorem relating the conductance between two points of a network (say A and B) and properties of the random walkers on this network may be obtained This theorem makes the following intuitive relation precise

The conductance Σ_{AB} between any two points is proportional to the total probability that a walker starting on A *reaches B* (while never returning to A) *in an arbitrary number of steps*, the weight of a path $\mathscr{C}_{AB}$ simply being

$$W[\mathscr{C}_{AB}] = \prod_{\mathscr{C}_{AB}} W_{ij}/\Sigma_k W_{ik} \tag{2 19}$$

A trivial example on which one sees this theorem at work is presented schematically in fig 2 6 One readily sees that

$$\sigma = \sigma_1\left(\frac{\sigma_2}{(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)} + \frac{\sigma_2^3}{(\sigma_1+\sigma_2)^2(\sigma_2+\sigma_3)^2} + \frac{\sigma_2^5}{(\sigma_1+\sigma_2)^3(\sigma_2+\sigma_3)^3} + \right)\sigma_3 \equiv \frac{\sigma_1\sigma_2\sigma_3}{\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3},$$

which reproduces the rule for resistances in series

A proof in the general case can be found in appendix D

2 2 1 3 Illustration chain with a broad distribution of capacitances [Hul89] Consider the electrical network drawn in fig 2 7 The conductances σ_{nm} are constant, equal to σ, while the capacitances are chosen according to a "broad" distribution law,

$$\psi(C) \sim C^{-(1+\mu)}, \quad C \to \infty$$

According to the above general discussion, this corresponds to the problem of diffusion among random traps of release time $\tau_n \sim C_n$, which we have considered in section 1 2 3 1 It is obvious that the *end to end conductance*, Σ_{AB}, is equal to σ *whatever the distribution* ψ, even in the case $\mu < 1$, in which the diffusion constant *vanishes* in the associated diffusion problem This illustrates the remarks made above concerning the non-existence of an "Einstein relation"

The diffusion properties are nevertheless of interest for the electrical problem, in particular if one considers the *input admittance* $A(\omega)$ of the chain [Rig88] Indeed, in a period ω^{-1} of the applied voltage, the charges diffuse over a penetration depth $L_p(\omega)$ given by (2 40) below,

$$L_p(\omega) \sim \omega^{-1/2}, \quad \mu > 1, \qquad L_p(\omega) \sim \omega^{-\mu/(1+\mu)}, \quad \mu < 1 \tag{2 20}$$

The *resistive* (real) part of the admittance is thus given by that of $L_p(\omega)$ conductances in series,

$$G(\omega) \equiv \mathrm{Re}\, A(\omega) \sim \sigma / L_p(\omega), \tag{2 21a}$$

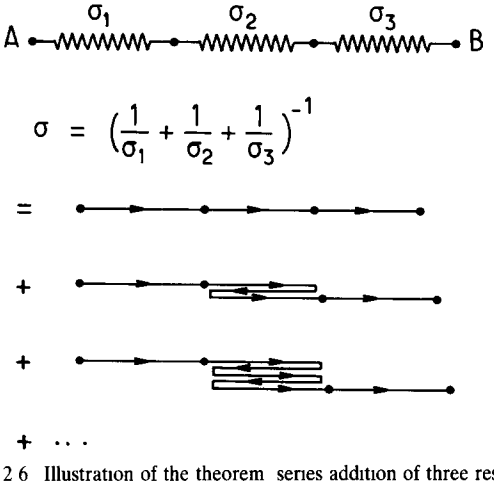


Fig 2 6 Illustration of the theorem series addition of three resistors (section 2 2 1 2)

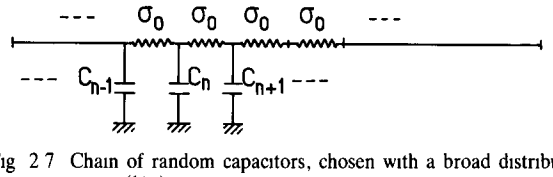


Fig 2 7 Chain of random capacitors, chosen with a broad distribution $\psi(C) \sim C^{-(1+\mu)}$ for $C \to \infty$

while the *capacitive* (imaginary) part is that of $L_p(\omega)$ random capacitances in *parallel*,

$$C(\omega) \equiv -\frac{1}{\omega} \operatorname{Im} A(\omega) \sim \sum_{n=1}^{L_p(\omega)} C_n \tag{2 21b}$$

The last sum, as usual, behaves as

$$C(\omega) \sim L_p(\omega), \quad \mu > 1, \qquad C(\omega) \sim [L_p(\omega)]^{1-\mu}, \quad \mu < 1$$

The ratio of the real to imaginary part of the admittance is thus

$$\operatorname{Re} A(\omega) / \operatorname{Im} A(\omega) \sim 1, \quad \forall \mu, \tag{2 21c}$$

both being of order

$$\operatorname{Re} A(\omega) \sim \operatorname{Im} A(\omega) \sim L_p^{-1}(\omega)$$

As recently pointed out by Mitescu et al [Mit89], Kramers–Kronig relations show that if (2 20) and (2 21) are true then the phase of the admittance $A(\omega)$ is *exactly* given by

$$\text{phase } A(\omega) = \frac{\pi}{4}, \quad \mu > 1, \qquad \text{phase } A(\omega) = \frac{\pi}{2} \frac{\mu}{1+\mu}, \quad \mu < 1, \tag{2 22}$$

independently of the frequency (See also [Cle84, 90])

The same results may be obtained using "scaling" arguments For a chain of length L, one expects $A(L, \omega) \sim A(L, 0) f(L_p(\omega)/L)$ Obviously, $A(L, 0) = \sigma L^{-1}$, demanding that for $L \gg L_p(\omega)$ the admittance ceases to depend on L, one recovers at once $A(L, \omega) \sim L_p(\omega)^{-1}$ Note, however, that the frequency dependent *end to end* conductivity has the following "localized" behaviour

$$\sigma(L, \omega) \sim (\sigma/L) \exp[-L/L_p(\omega)]$$

2 2 2. The random masses and springs model

Consider the network of masses M_n and springs drawn in fig. 2.8 The displacement $\boldsymbol{X}_n$ of the nth atom from its equilibrium position follows the equation of motion

$$M_n \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{X}_n = \sum_m K_{nm} (\boldsymbol{X}_m - \boldsymbol{X}_n) \tag{2 23a}$$

in the zero-friction limit, or

$$\gamma \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{X}_n = \sum_m K_{nm} (\boldsymbol{X}_m - \boldsymbol{X}_n) \tag{2 23b}$$

in the large-friction limit The problem is often called *scalar* elasticity, since eqs. (2.23) can be written for each component of X independently For work on "vectorial" elasticity, see, e.g., [Fen84, Rou87a,b, 88]

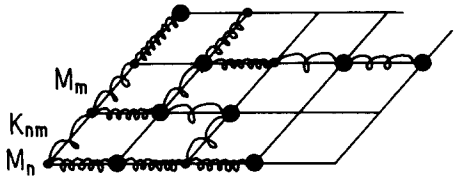


Fig 2 8 Random masses and springs network

Let us specialize to the case $M_n = M$ The matrix Λ defined by

$$\Lambda_{nm} = K_{nm}\,, \quad n \neq m\,, \qquad \Lambda_{nn} = -\sum_l K_{ln}\,,$$

has only *negative* eigenvalues λ_α, corresponding to the frequencies of the purely propagating modes (phonons) of eq. (2.23a), $-M\omega_\alpha^2 = \lambda_\alpha$, and to the inverse relaxation terms of the purely damped modes of eq. (2.23b), $\tau_\alpha^{-1} = -\lambda_\alpha/\gamma$. The density of those eigenvalues $\rho(\lambda)$ is simply related to the probability of finding the particle on its initial site in the *symmetric barrier model* defined by $W_{nm} = K_{nm}$.

Indeed, the vector $\boldsymbol{P} = \{P_i(t),\ 0 \le i \le N\}$ evolves according to $(\mathrm{d}/\mathrm{d}t)\boldsymbol{P} = \boldsymbol{\Lambda P}$, from which one formally deduces

$$\boldsymbol{P}(t) = \mathrm{e}^{\boldsymbol{\Lambda} t}\boldsymbol{P}(0)\,.$$

One thus has

$$P_n(t|n,0) = [\mathrm{e}^{\boldsymbol{\Lambda} t}]_{nn}\,. \tag{2.24}$$

The average over all starting points of the probability of being at the initial site is then simply

$$\begin{aligned} P_0(t) &\equiv N^{-1}\,\mathrm{Tr}\,\mathrm{e}^{\boldsymbol{\Lambda} t} = \int \rho(\lambda)\,\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda\,, \\ \rho(\lambda) &= N^{-1}\sum_\alpha \delta(\lambda - \lambda_\alpha)\,. \end{aligned} \tag{2.25}$$

If $P_0(t)$ decays asymptotically as $P_0(t) \sim t^{-\nu d}$, the density of phonon modes in the lattice is, for low frequencies,

$$\rho(\omega) \sim \omega^{2\nu d - 1}\,, \quad \omega \to 0 \tag{2.26}$$

(we have used $\rho(\omega)\,\mathrm{d}\omega = \rho(\lambda)\,\mathrm{d}\lambda$, $\lambda = -\omega^2$)

For a regular lattice, $\rho(\omega)$ behaves as ω^{d-1} when $\omega \to 0$; subdiffusive behaviour – generated by low values of K_{nm} – corresponds to an enhancement of the low-frequency density of phonons; in other words, the lattice is anomalously "soft". Let us finally notice that the diffusion constant in the diffusion problem corresponds to the square of the (long-wavelength) sound velocity associated with eq. (2 23a)

2.3. *Continuous space formulation: Fokker–Planck and Langevin equations*

Very often the diffusion of a point particle is described by a continuous space *Langevin equation*,

$$\gamma\,\mathrm{d}\boldsymbol{X}/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{\eta}(t)\,, \tag{2.27}$$

which corresponds to the equation of motion of a strongly damped (or massless) point particle subjected to*)

*) The possibility of separating F and η comes, as usual, from the assumption of "slow" and "fast" degrees of freedom involved in the interaction with the medium

- a force $\boldsymbol{F}(\boldsymbol{X})$, modelling the interaction with the medium,
- a thermal noise $\boldsymbol{\eta}(t)$, randomly evolving with time, representing the interaction with a thermal bath

Usually, one takes white noise,

$$\overline{\eta_\alpha(t)\eta_\beta(t')} = \gamma kT\delta_{\alpha\beta}\delta(t-t')\,, \tag{2 28}$$

where T is the temperature

From the Langevin equation one canonically obtains a Fokker–Planck equation for the probability distribution $P(\boldsymbol{X}, t)$ (see, e g., [G5]),

$$\frac{\partial P(\boldsymbol{X}, t)}{\partial t} = -\mathrm{div}\,\boldsymbol{J}\,, \qquad \boldsymbol{J}(\boldsymbol{X}, t) = \frac{\boldsymbol{F}}{\gamma}\,P - D_0\boldsymbol{\nabla}P\,, \quad D_0 \equiv \frac{kT}{\gamma} \tag{2 29}$$

We wish to show here how eq (2 29) can be obtained as a *continuum limit* of the master equation (2 1) Indeed, consider the following form for the transition rates $W_{n\,n+1}$, $W_{n+1\,n}$ (we consider for simplicity the one-dimensional case, but the generalization to higher dimensions is straightforward),

$$\begin{aligned} W_{n,n+1} &= \frac{D_{n,n+1}}{a^2}\exp\!\left(-\frac{aF_{n\,n+1}}{2\gamma D_{n\,n+1}}\right), \\ W_{n+1\,n} &= \frac{D_{n,n+1}}{a^2}\exp\!\left(+\frac{aF_{n\,n+1}}{2\gamma D_{n\,n+1}}\right), \end{aligned} \tag{2 30}$$

where a is the lattice spacing. $D_{n\,n+1}$ corresponds to the local temperature, with the associated "trial frequency" $D_{n\,n+1}/a^2$ *) $W_{n,n+1}$ and $W_{n+1\,n}$ are thus taken in activated form, with a local potential difference $\Delta U_n = aF_{n\,n+1}$ Expanding the W's in powers of a, one thus obtains

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}P_n = {} & \frac{1}{a}\left(D_{n\,n+1}\,\frac{P_{n+1}-P_n}{a} - D_{n-1,n}\,\frac{P_n-P_{n-1}}{a}\right) + \frac{1}{a}\left(\frac{F_{n-1,n}}{\gamma}\,\frac{P_n+P_{n-1}}{2} - \frac{F_{n\,n+1}}{\gamma}\,\frac{P_{n+1}+P_n}{2}\right) \\ & + \frac{1}{8}\left(\frac{F^2_{n\,n+1}}{\gamma^2 D_{n\,n+1}}\,(P_{n+1}-P_n) - \frac{F^2_{n-1\,n}}{\gamma^2 D_{n\,n-1}}\,(P_n-P_{n-1})\right) + \mathrm{O}(a) \end{aligned} \tag{2 31}$$

Defining

$$\lim_{a\to 0}\frac{1}{a}\,P_{n=X/a}(t) \equiv P(X, t)\,, \tag{2 32}$$

one recovers, in the continuum limit, the Fokker–Planck equation,

$$\frac{\partial}{\partial t}P(X, t) = \frac{\partial}{\partial X}\left(-\frac{F(X)}{\gamma}\,P + D(X)\,\frac{\partial}{\partial X}\,P\right)$$

In d dimensions, it reads

$$\frac{\partial P}{\partial t} = \boldsymbol{\nabla}\left(-\frac{\boldsymbol{F}}{\gamma}\,P + D(\boldsymbol{X})\boldsymbol{\nabla}P\right) \tag{2 33}$$

*) In the heavily damped limit the trial frequency corresponds to the diffusion time in the potential wells

In the above example, the local temperature has been defined on the *links*: $\gamma D_{n,n+1}$. One could very well have chosen to define the temperature on the *sites*,

$$W_{n\,n+1} = \frac{D_{n+1}}{a^2} \exp\left(-\frac{aF_{n,n+1}}{2\gamma D_{n+1}}\right), \qquad W_{n+1,n} = \frac{D_n}{a^2} \exp\left(\frac{aF_{n,n+1}}{2\gamma D_n}\right) \tag{2 34}$$

In this case, the continuum limit corresponds to a *different* Fokker–Planck equation,

$$\frac{\partial}{\partial t} P(X, t) = \frac{\partial}{\partial X}\left(-\frac{F(X)}{\gamma} P + \frac{\partial}{\partial X} D(X)P\right). \tag{2 35}$$

The difference between eqs (2.33) and (2 35) corresponds to two different order prescriptions in stochastic calculus (see ref [G5, G8] for a thorough discussion) It is seen that they reflect two different physical situations More generally, depending on the discrete model underlying the Fokker–Planck equation, the diffusion term reads

$$\frac{\partial}{\partial X}\left(D(X)^p \frac{\partial}{\partial X} D(X)^q P\right), \quad p + q = 1 .$$

Let us finally mention that the use of a fluctuation–dissipation theorem, implicit in the above presentation, imposes that

$$|(\partial/\partial X)\ln D(X)| \ll 1/\sqrt{M\gamma D(X)}$$

(M is the mass of the particle). This condition expresses the fact that the particle must have time to become thermalized in the non-uniform environment For the corrections to (2 33) arising from a small but non-zero mass in eq (2 27), see ref. [G8]

2.4. Random traps and random barriers· qualitative analysis

In this section we analyse the diffusion behaviour of random traps (A) and random barriers (B) models, for a lattice of arbitrary dimension Once the important statistical mechanisms are identified, this can be achieved in a very simple manner using the tools of chapter 1 We thus postpone more sophisticated techniques to chapters 3 and 4

2.4.1 Random traps

We proceed in a way very similar to section 1 2 3 1, where the "annealed" version of this model (CTRW) was dealt with. The difference here lies in the fact that the trapping time at a given site is the same for each visit of this site ("quenched disorder"), thus inducing a *correlation* between the successive trapping times encountered. Whenever this correlation is relevant, it will induce a diffusion law different from that of the corresponding "annealed" problem (CTRW).

Let N be the number of jumps, and $\{\tau_i\}$ the trapping times encountered The time t and average position are still related by eqs (1 25)–(1.27),

$$\overline{X_\alpha^2} = l_\alpha N , \qquad t = \sum_{i=1}^{N} \tau_i ,$$

the last sum no longer being a sum of independent random variables. One is nevertheless led to distinguish the same two cases as in section 1 2 3

(i) If $\langle\tau\rangle$ is finite, t behaves as $N\langle\tau\rangle$ for large N (whatever the correlation of the τ_i is) and the diffusion behaviour is thus the same as for a CTRW,

$$\overline{X_\alpha^2} \sim 2D_{\alpha\alpha}t\,, \quad D_{\alpha\alpha} = l_\alpha^2/2\langle\tau\rangle \tag{2 36}$$

(on a hypercubic lattice $D = (a^2/2d)\langle\tau\rangle^{-1} = a^2\langle 1/W\rangle^{-1}$)

(ii) If the τ_i are distributed according to a broad distribution [$\psi(\tau) \sim \tau^{-(1+\mu)}$, $\tau \to \infty$] with $0 < \mu < 1$, $\langle\tau\rangle$ is infinite, and one has to treat properly the correlations in the sum (1.27) Following section 1.3 1, one has to estimate the number of "effectively independent" trapping times in (1 27), that is, the number S_N of *different* sites visited by the Brownian motion between traps. One has (cf appendix A)

$$S_N \sim \begin{cases} N^{d/2}\,, & d<2\,, \\ N/\ln N\,, & d=2\,, \\ N\,, & d>2\,, \end{cases} \quad N\to\infty \tag{2 37}$$

The results of section 1.2 1 on broad distributions can now be applied to the sum $t \sim (N/S_N)\sum_{i=1}^{S_N}\tau_i$, leading to

$$t \sim NS_N^{1/\mu-1}\,, \quad 0<\mu<1\,, \qquad t \sim N\ln S_N\,, \quad \mu = 1 \tag{2 38}$$

Combining (2 37) and (2 38), one has the following results·

– If $d > 2$, each trap is visited only a finite number of times as $N \to \infty$; correlations are not strong enough to induce a different diffusion behaviour from the annealed case,

$$\overline{X_\alpha^2} \sim t^\mu\,, \quad 0<\mu<1\,, \qquad \overline{X_\alpha^2} \sim t/\ln t\,, \quad \mu = 1 \tag{2 39}$$

– If $d < 2$, each site is visited infinitely often when $N \to \infty$; one has $t \sim N^{1-d/2+d/2\mu}$, and a new diffusion behaviour arises,

$$\overline{X_\alpha^2} \sim \begin{cases} t^{2\nu}\,, & 1/\nu = 2-d+d/\mu\,, \quad 0<\mu<1\,, \\ t/\ln t\,, & \mu = 1 \end{cases} \tag{2.40}$$

In the marginal dimension $d = 2$, (2 40) is modified by logarithmic terms if $0 < \mu < 1$,

$$\overline{X_\alpha^2} \sim t^\mu(\ln t)^{1-\mu} \tag{2 41}$$

The derivation given here follows [Bou87c], but the results (2 39) and (2 40) were first obtained by Machta using a real-space renormalization method [Mac85]. It is worth noticing that the "fixed point" (equivalent model at large scales) obtained by this method is simply a CTRW for $d > 2$, while correlations are not eliminated at large scales for $d < 2$ (the quenched nature of the problem remains) We refer to [Mac81, 85, Bro89a,b] for details on such real-space renormalization methods.

The diffusion exponent ν is plotted as a function of the dimension in fig 2 9 One should note that, in contrast with naive intuition, diffusion becomes *slower* as the dimension increases (for $d < 2$) As noted in [Ale81], this is because the probability to visit a trap with a long trapping time increases with d (since the larger d, the faster S_N grows) These "deep" traps control the

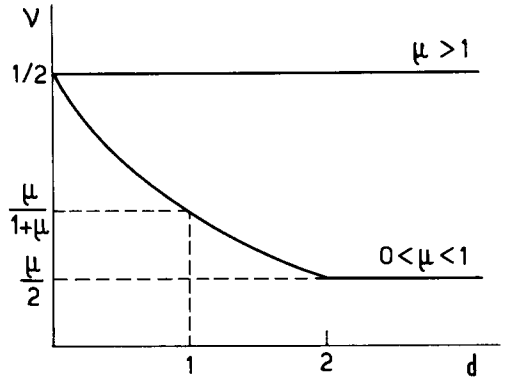


Fig 2 9 The diffusion exponent $\nu(\mu)$ for model A as a function of the space dimension d

diffusion process, the largest trapping time encountered is of order

$$\tau_{\max} \sim S_N^{1/\mu} \sim \begin{cases} t^{\nu d/\mu}, & d<2, \\ t^{1/\mu}, & d>2, \end{cases}$$

and thus increases more slowly with time for small dimensions

Whether the full distribution $P(X, t)$ satisfies a "generalized CLT" for this model has not been rigorously established. When diffusion is normal, it is clear that this is indeed the case, with a Gaussian limit distribution. When diffusion is anomalous and $d > 2$ the renormalization procedure of [Mac85] and the above derivation strongly suggest that $P(X, t)$ does reach a scaling form for a given sample analogous to the one of a CTRW (see section 1.2.3.1)

2.4.2. Random symmetric barriers

Random barriers (type B) models are most easily analyzed – at least at a qualitative level – using the equivalence with a random resistor network (section 2 2 1). They turn out to have very different diffusion properties in one dimension [G10] and in more than one dimension [Ale81]

2.4.2 1 One dimension In $d = 1$, the problem simply amounts to a series addition of random resistances, $\sigma_i^{-1} = (CW_{i,i+1})^{-1}$ The total conductivity of a chain of N units (N jumps for the diffusion process) is thus given by

$$\frac{N}{\sigma(N)} = \sum_{i=1}^{N} \frac{1}{\sigma_i} . \tag{2 42}$$

Again, two cases must be distinguished:

(1) *If* $\langle 1/W \rangle$ *is finite*, the total conductivity has a finite limit for large N,

$$\frac{1}{\sigma(N)} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{-1} \xrightarrow[N\to\infty]{} \langle 1/\sigma \rangle \tag{2.43}$$

Making use of eq. (2 15), which relates the conductivity to the diffusion constant, one finds that the latter is finite,

$$D = a^2\sigma/C = a^2\langle 1/W \rangle^{-1} . \tag{2.44}$$

Analytical methods allow one to obtain finite-time corrections, together with systematic short-time expansions (see ref [G13] and references therein)

(ii) *If* $\langle 1/W \rangle = +\infty$, for example, if $1/W_{i\,i+1}$ has a "broad" distribution,

$$\psi(W) \sim W^{\mu-1}, \quad 0 < \mu \leq 1, \quad W \to 0,$$

then the total conductivity *vanishes for large sizes*, as

$$\sigma(N) \sim N^{1-1/\mu}, \quad 0 < \mu < 1, \qquad \sigma(N) \sim (\ln N)^{-1}, \quad \mu = 1 \tag{2 45}$$

The diffusion behaviour can be guessed from this result by assuming that the "effective diffusion constant" at scale X, $D(X) = X^2/t$, has the same scale dependence as $\sigma(N)$ [cf. eq (2.16)]. This leads to a subdiffusive law [$D(X) \sim X^{1-1/\mu}$]

$$\overline{X^2} \sim \begin{cases} t^{2\nu}, & \nu = \mu/(1+\mu), \quad 0 < \mu < 1, \\ t/\ln t, & \mu = 1 \end{cases} \tag{2 46}$$

This can of course only be considered as a rough argument, since it makes extended use of relation (2 15) between σ and D in some transient regime. The result (2 46) is nevertheless the correct one [G10], as will be shown by analytical methods in section 3 2

One thus observes that, in one dimension, the asymptotic diffusion behaviour of random barrier models is identical to that of random traps provided one identifies $W_{n\,n+1} = W_{n+1\,n}$ with $1/2\tau_n$ This is true in the normal case, eq (2 44), as well as in the anomalous one, eq (2 46) This was to be expected on physical grounds Indeed, for a given link, it is always possible to find an energy barrier approximately equal to that of this link, lying at a finite distance (fig 2 10) (the probability that this can be achieved in a distance smaller than L reads $1 - \exp[-L\,\Delta W\,\psi(W)]$ for a given precision ΔW) In one dimension, the region delimited by these two approximately equal barriers can be viewed as an effective trap in a coarse-grained picture of the lattice (fig 2 10) Barriers and random traps models are thus expected to behave similarly at large scales (long time)

Note the existence of a *duality* between site and bond disorder in one dimension Indeed, the time evolution of the current $J_{n\,n+1} = W_{n\,n+1}(P_n - P_{n+1})$ for symmetric barriers obeys

$$\frac{1}{W_{n\,n+1}} \frac{\mathrm{d}}{\mathrm{d}t} J_{n\,n+1} = J_{n+1\,n+2} + J_{n-1\,n} - 2J_{n\,n+1} \tag{2 47}$$

This coincides with the equation satisfied by $2\tau_n P_n$ in a trapping model with $2\tau_n = W_{n\,n+1}^{-1}$

2 4 2 2 Higher dimensions In more than one dimension, this duality no longer holds The local currents of a symmetric model $W_{nm} = W_{mn}$ obey the following equation.

$$\frac{1}{W_{nm}} \frac{\mathrm{d}}{\mathrm{d}t} J_{nm} = \sum_k J_{mk} - \sum_l J_{nl} \tag{2 48}$$

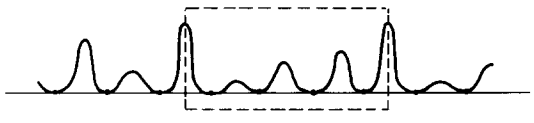


Fig 2 10 Duality between models A and B in one dimension effective trap limited by two comparable barriers

(where the two sums are over neighbours of m and n, respectively). This defines a lattice structure which is no longer identical to the original one. It follows that random barrier models do not have the same diffusion behaviour as random traps in more than one dimension [Ale81]

Indeed, we now show that very high energy barriers can always be avoided thanks to the existence of several paths, and as a result that diffusion is always normal, even if the (W_{nm}^{-1}) have a "broad" distribution. This is most easily seen following a percolation type of argument [Ale81] for the equivalent resistor network Assuming that all resistances $\rho > \rho_m$ (i.e. $W < W_m$) have been removed from the lattice leads to a lower bound for the conductivity σ (since the removed bonds are in fact infinite resistors). This lower bound is certainly non-zero provided ρ_m can be chosen to be a finite cut off such that $p(\rho_m) > p_c$, where p_c is the bond percolation threshold and $p(\rho_m)$ is the fraction of removed bonds, given by

$$1 - p(\rho_m) = \int_{\rho_m}^{\infty} \psi(\rho)\, d\rho .$$

Thus, as long as $p_c < 1$ (i.e., provided $d > 1$) the conductivity (and accordingly the diffusion constant) is non-zero

Calculating this diffusion constant (or the conductivity of the associated random resistor network) is, however, a difficult task (it contains percolation as a limiting case!), no exact expression is known in arbitrary dimension, in contrast to the random traps model. This is equivalent to finding the permeability of a disordered porous medium [Mag89], the magnetic susceptibility of a random mixture of magnets, the permittivity of a mixture of dielectrics [Ber88], etc. For references see [Eto77, 88].

Two classes of methods can be used systematic weak disorder or cumulant expansions [Zwa82, Der83b, Den84, Kar84, Nie85], and effective medium types of approximation The former will be discussed in section 4 2.3 for a general lattice hopping model, while the latter is briefly presented in the next section A field theoretic formulation of the random barrier model can also be given, using, e g , replicas [Ste78] or interacting Bose and Fermi fields [Car83a] This can be used as a starting point for various expansions [Kar84] or for devising alternative effective medium approximations [Car83a]

In addition, one should mention that rigorous upper and lower bounds can be obtained on variational grounds [Has62] Finally, an exact result is known in the two-dimensional square-lattice or continuum case, for binary distributions (two-phase mixtures),

$$\psi(W) = p\delta(W - W_1) + (1 - p)\delta(W - W_2) .$$

Then, a duality type of analysis [Men75] allows one to show that

$$D(W_1, W_2, p)D(W_1, W_2, 1 - p) = \sqrt{D_1 D_2} \tag{2.49}$$

(D_1 and D_2 are the diffusion coefficients of a homogeneous sample with $W = W_1$ and $W = W_2$, respectively). In particular, if 1 and 2 play a symmetrical role (i.e., $p = 1/2$), then

$$D = \sqrt{D_1 D_2} \tag{2 50}$$

(A generalization of this result to a tensorial local "diffusion coefficient", which arises in the study of the a.c susceptibility of magnetic materials, can be found in [Bou89d].)

2.4.2.3. Effective medium approximation. The effective medium approximation (see, e g , [Lan77, Gub77, Kir73, Eto88] for reviews), is in a sense a kind of mean field theory. The idea is the following: one assumes that the diffusion coefficient D (or conductivity, etc.) is known. The random medium is then replaced everywhere but in a small sphere by the equivalent "effective" medium. One then successively fills the small sphere with the different constituents i of the initial random medium, each with its own weight ψ_i, and computes the new diffusion constant,

$$D_i = D + \delta D(D, D_i) \tag{2 51}$$

D is then self-consistently determined by the condition

$$\sum_i \psi_i \, \delta D(D, D_i) = 0 , \tag{2 52}$$

stating that, on average, the correction induced by the "impurity" vanishes The method is thus clearly in the family of "molecular field" approximations, but is optimal in the sense that it reproduces *all* the previously listed known results it is exact in one dimension, and for weak disorder satisfies the rigorous bounds, and yields the special result (2 49)' In view of its interest and usefulness, we quote here the self-consistent equation in more explicit form (for a hypercubic lattice in d dimensions),

$$D_{\mathrm{EMA}} = \frac{a^2}{W_{\mathrm{EMA}}} , \qquad \int \mathrm{d}W \, \psi(W) \, \frac{W_{\mathrm{EMA}} - W}{W + (d-1)W_{\mathrm{EMA}}} = 0 \tag{2.53}$$

Generalization to non-zero frequency can be found in [Web81, Der83b]

3. One-dimensional models

This chapter is devoted to the study of one-dimensional hopping models General analytical techniques are presented in section 3 1, the case of random symmetric barriers $W_{n,n+1} = W_{n+1,n}$ is considered in section 3 2, and the rich variety of anomalous diffusion properties of the asymmetric case with bond disorder is discussed at length in section 3 3

3 1 Green function methods, calculation of the velocity and diffusion constant, nature of the fluctuations

In this section we shall present Green function techniques for a general one-dimensional hopping model, allowing one to obtain *explicit* expressions for the velocity and diffusion constant. That such a calculation is possible in one dimension relies on the fact that the constant current solutions of the master equation (2 1) can then be obtained in explicit form. This general analysis was first performed by Derrida [Der83a], making use of a periodization of the medium. Further developments have been achieved in [Asl89a,b]

In order not to deal from the beginning with the more complicated discrete case, we shall illustrate the techniques on the continuous model described by the Fokker–Planck equation (2 29) The basic quantity used in the following is the Laplace transform,

$$P(x, x_0, E) = \int_0^\infty \mathrm{e}^{-Et} P(x, t|x_0, 0) \, \mathrm{d}t , \tag{3 1}$$

where $P(x, t|x_0, 0)$ denotes the solution of (2.29) with initial condition

$$P(x, t=0|x_0, 0) = \delta(x - x_0) \tag{3 2}$$

In the discrete case, we shall define in the same way $P_n(E)$ from the probability $P_n(t)$ with initial condition $P_n(t=0) = \delta_{nn_0}$. These quantities are in fact *Green functions* of the Fokker–Planck (or master) equations (2 29) [or (2 1)] Indeed, they satisfy

$$(E - H_{\mathrm{FP}})P(x, x_0; E) = \delta(x - x_0)\,, \quad H_{\mathrm{FP}} \equiv \frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\cdot\right) - \frac{\partial}{\partial x}[F(x)\cdot] \tag{3 3}$$

(in the following we set $\gamma \equiv 1$), and similarly

$$EP_n(E) - [W_{n,n+1}P_{n+1}(E) + W_{n,n-1}P_{n-1}(E) - (W_{n+1,n} + W_{n-1,n})P_n(E)] = \delta_{nn_0} \tag{3 4}$$

3 1 1. The Green function at $E = 0$

3.1.1.1. Constructing the steady state. For $E = 0$, these Green functions have a simple physical meaning As is clear from eqs (3.1)–(3.3), they correspond to the steady-state probability generated by a source of particles located at the initial site, and emitting one particle per unit time Accordingly, the current associated with this steady state reads (for a particular choice of boundary conditions)

$$J(x) = 1\,, \quad x \geq x_0\,, \qquad J(x) = 0\,, \quad x < x_0 \tag{3 5}$$

(and similarly $J_{n,n+1} = W_{n+1\,n}P_n - W_{n,n+1}P_{n+1} = 1$ for $n > n_0$ and 0 for $n < n_0$)

This remark allows one to construct this steady state in explicit form, for an arbitrary configuration of the hopping rates, from the knowledge of the constant-current solutions of the master equation. Let us begin with the continuous case, for which the general form of a solution with constant current equal to J reads

$$P_J(x) = C\int^{x} \frac{F(y)}{D(y)}\,\mathrm{d}y - J\int_{b}^{x} \frac{\mathrm{d}y}{D(y)}\exp\left(-\int_{x}^{y}\frac{F(z)}{D(z)}\,\mathrm{d}z\right), \tag{3 6}$$

in which b and C are arbitrary constants The first term is a zero-current solution and the second term is obtained from the first one by "varying the constant C" These two constants are fixed by boundary conditions We shall consider first the problem on the whole line $]-\infty, +\infty[$. To fix orientation let us assume that $F(z)$ is positive for large z (i e , in the case where $F(z)$ and $D(z)$ are random, we shall assume $\langle F/D\rangle > 0$). Then imposing the decrease of $P(x, x_0; E = 0)$ for $x \to -\infty$ and continuity at $x = x_0$ leads to

$$P(x, x_0; E=0) = \begin{cases} \displaystyle\int_{x}^{+\infty} \frac{\mathrm{d}y}{D(y)}\exp\left(-\int_{x}^{y}\frac{F(z)}{D(z)}\,\mathrm{d}z\right), & x \geq x_0\,, \\ \displaystyle\exp\left(-\int_{x}^{x_0}\frac{F(y)}{D(y)}\,\mathrm{d}y\right) P(x_0, x_0; E=0)\,, & x \leq x_0\,. \end{cases} \tag{3 7}$$

When specialized to the non-random case with $D(x) = D_0$, $F(x) = F_0 > 0$, this formula reads

$$P_{\text{ordered}}(x, x_0, E=0) = \begin{cases} F_0^{-1}, & x \geq x_0, \\ F_0^{-1} \, e^{-(F_0/D_0)(x_0 - x)}, & x \leq x_0, \end{cases} \tag{3 8}$$

which of course can also be obtained by direct Laplace transform of the Gaussian distribution. The exponential decrease of the steady state for $x < x_0$ stems from the bias towards $x > 0$, and the velocity $V_0 (= F_0)$ is related to the flux at infinity,

$$1/V_0 = P_{\text{ord}}(+\infty, x_0, E=0) - P_{\text{ord}}(-\infty, x_0, E=0) \tag{3.9}$$

This whole construction can be repeated for a general discrete hopping model with only slightly more involved calculations For $n > n_0$, iteration of the constant-current condition,

$$W_{n+1\,n} P_n - W_{n\,n+1} P_{n+1} = \begin{cases} 1 & \text{for } n \geq n_0, \\ 0 & \text{for } n < n_0, \end{cases}$$

allows one to relate $P_n(E=0)$ and $P_{n+N}(E=0)$,

$$\begin{aligned} P_n(E=0) &= \frac{1}{W_{n+1\,n}} + \sum_{i=n+1}^{n+N-1} \frac{1}{W_{i+1\,i}} \prod_{j=n}^{i-1} \frac{W_{j\,j+1}}{W_{j+1\,j}} + P_{n+N}(E=0) \prod_{j=n}^{n+N-1} \frac{W_{j\,j+1}}{W_{j+1,j}}, \quad n \geq n_0, \\ P_n(E=0) &= P_{n_0}(E=0) \prod_{j=n}^{n_0-1} \frac{W_{j\,j+1}}{W_{j+1\,j}}, \quad n < n_0 \end{aligned} \tag{3 10}$$

Boundary conditions remain to be imposed Let us again consider the problem on the whole line and assume that the product

$$\prod_{j=n}^{n+N-1} \frac{W_{j\,j+1}}{W_{j+1\,j}} = \exp\left(\sum_{j=n}^{n+N-1} \ln(W_{j\,j+1}/W_{j+1\,j}) \right) \tag{3 11}$$

vanishes for $N \to \infty$ In the random case with bond disorder, this is insured with probability one provided

$$\langle \ln(W_{n\,n+1}/W_{n+1\,n}) \rangle < 0, \tag{3.12}$$

which amounts to saying that the mean bias is towards $n > 0$ [see eq (2.12)]. One then comes up with the discrete generalization of (3 7),

$$P_n(E=0) = \begin{cases} \dfrac{1}{W_{n+1\,n}} + \displaystyle\sum_{i=n+1}^{\infty} \frac{1}{W_{i+1\,i}} \prod_{j=n}^{i-1} \frac{W_{j\,j+1}}{W_{j+1\,j}}, & n \geq n_0, \\ P_{n_0}(E=0) \displaystyle\prod_{j=n}^{n_0-1} \frac{W_{j\,j+1}}{W_{j+1\,j}}, & n < n_0 \end{cases} \tag{3 13}$$

For a given set of W_{ij}, the series on the first line converges with probability one under the same condition (3 12)

Alternatively, we could have chosen, following [Der83a], to consider a periodic medium of period N, $W_{ij} = W_{i+N\,j+N}$. Solving (3 10) with $P_{n+N} = P_n$, one then obtains (after some reshuffling of indices)

$$P_n^{\mathrm{per}}(E=0) = \frac{C_N}{W_{n+1,n}}\left(1+\sum_{i=1}^{N-1}\prod_{j=1}^{i}\frac{W_{n+j-1\,n+j}}{W_{n+j+1\,n+j}}\right),$$
$$C_N = \left(1-\prod_{n=1}^{N}\frac{W_{n,n+1}}{W_{n+1,n}}\right)^{-1} \tag{3.14}$$

In contrast with (3.13), this only involves finite sums and no such condition as (3 12) must be assumed.

3 1 2 A digression on Kramers' problem Having established the general forms (3 6) and (3 7) of constant-current solutions for an arbitrary configuration $(F(x), D(x))$, it is natural to show how they allow one to deal with a classical problem the thermally activated escape from a metastable state (Kramers' problem, for a review see [Han86] and other papers in the same volume) Consider the potential $U(x)$ of fig 3 1, one asks at which rate Γ particles initially located in the first well will jump over the barrier to reach the second one In the strongly overdamped limit, in which neither the potential nor the force $F(x) = -U'(x)$ vary appreciably on the length scale $(MkT/\gamma^2)^{1/2}$ (where M is the mass of the particle), the answer can be constructed from (3 6) Indeed imagine that particles are injected at $x = x_1$ and removed at $x = x_2$, thus creating a constant current J The steady state with boundary conditions $P(x \simeq x_2) = 0$ reads

$$P(x) = -Jp_{\mathrm{eq}}(x)\int_{x_2}^{x}\frac{\mathrm{d}y}{D(y)p_{\mathrm{eq}}(y)}, \tag{3 15}$$

where $p_{\mathrm{eq}}(y) \propto \exp[-U(y)/\gamma D(y)]$ is the Boltzmann equilibrium distribution The escape rate is then obtained as the ratio of the current to the population in the initial well,

$$\Gamma = J\left(\int_{x_1}^{x_b} P(x)\,\mathrm{d}x\right)^{-1} = \left(\int_{x_1}^{x_b}\mathrm{d}x\, p_{\mathrm{eq}}(x)\int_{x}^{x_2}\frac{\mathrm{d}y}{D(y)p_{\mathrm{eq}}(y)}\right)^{-1} \tag{3 16}$$

For the configuration of fig 3 1, a steepest descent approximation would lead to

$$\Gamma = \frac{M\omega_0\omega_b}{2\pi\gamma}\,\mathrm{e}^{-E_b/kT}, \quad \omega_x^2 \equiv U''(x)/M, \tag{3 17}$$

which is the Arrhenius–Kramers result [Kra40] in the overdamped limit

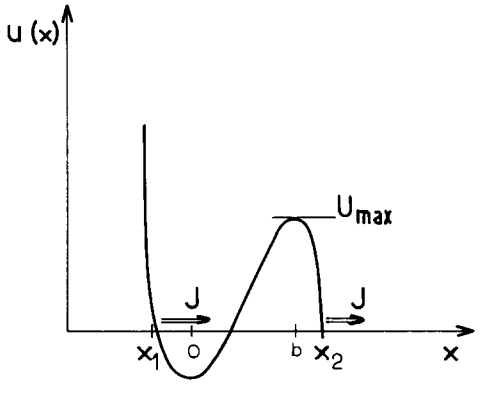


Fig 3 1 Kramers' problem exit time of a potential well for an overdamped particle A current J is injected at $x = x_1$ and retrieved at $x = x_2$

More generally, the steady-state distribution (3 7) allows one to compute for an arbitrary potential $U(x)$ the mean exit time (over thermal histories) of a given interval I, given an initial position x_0 inside this interval Indeed, the probability of being still inside I after a time t reads

$$\int_I P(y, t|x_0, 0)\,dy \tag{3 18}$$

Hence the distribution of exit times is

$$-\frac{\partial}{\partial t}\int_I P(y, t|x_0, 0)\,dy \tag{3 19}$$

Its average value thus involves $P(x, x_0, E=0)$ only,

$$\bar{T}(x_0) = \int_0^{\infty} dt \int_I P(y, t|x_0, 0) \equiv \int_I P(y, x_0, E=0)\,dy \tag{3 20}$$

Given the situation and boundary conditions of fig 3 1, (3 16) is recovered from (3 20), (3 7)

3 1 1 3 Deducing the asymptotic velocity from the steady state Before going further, we want to show that the asymptotic velocity for a general one-dimensional hopping model can be directly obtained from the steady state $P_n(E=0)$

This can be achieved by two different methods Following Derrida [Der83a], one can consider a given *periodic* sample W_{ij} and, making use of (3 14), obtain the asymptotic velocity for a *finite* period N Then, taking the limit $N\to\infty$, the result is found to converge with probability one towards a limit independent of the particular configuration chosen Alternatively [Geo88], one can relate the velocity to the flux at infinity associated with the *average* steady state on the whole line Both methods give the same result but both have their loopholes In the latter one takes explicitly averages over disorder and one would like to prove that sample to sample fluctuations do vanish in the long-time limit No averages are taken in the former, but one should prove that the limits $t\to\infty$ and $N\to\infty$ do commute (that is, that the result on a periodic lattice of large period does coincide with the result on the open line) In order to settle these difficulties, one needs information on the approach to the infinite-time limit, which requires the knowledge of the Green function at non-zero E This will be tackled in the next section, where it will be shown that *the asymptotic velocity is indeed a self-averaging quantity*, the value of which coincides with the one deduced from the present steady-state analysis

Let us first follow the second method and take the average over disorder of the open-line steady state (3 13), for a model in which the pairs $(W_{n\,n+1}, W_{n+1\,n})$ are independently distributed on each link Then

$$\langle P_n(E=0)\rangle = \begin{cases} \left\langle \dfrac{1}{W_{j+1\,j}} \right\rangle \left(1 + \displaystyle\sum_{i=n+1}^{\infty} \left\langle \frac{W_{j,j+1}}{W_{j+1\,j}} \right\rangle^{i-n}\right), & n \geq n_0\,, \\ \left\langle \dfrac{W_{j,j+1}}{W_{j+1\,j}} \right\rangle^{n_0-n} \langle P_{n_0}(E=0)\rangle\,, & n < n_0 \end{cases} \tag{3 21}$$

The geometrical series in (3.21) converges only provided $\langle W_{j\,j+1}/W_{j+1\,j}\rangle < 1$, therefore

$$\langle P_n(E=0)\rangle = +\infty\,, \quad \text{if } \langle W_{j\,j+1}/W_{j+1\,j}\rangle \geq 1\,, \tag{3.22a}$$

$$\langle P_n(E=0)\rangle = \left\langle \frac{1}{W_{\rightarrow}} \right\rangle \left(1 - \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle\right)^{-1} \times \begin{cases} 1, & n \geq n_0, \\ \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle^{-|n-n_0|}, & n < n_0, \end{cases}$$

$$\text{if } \langle W_{j\,j+1}/W_{j+1,j}\rangle < 1 \tag{3.22b}$$

[where in the last expression and in the following, obvious notations have been used for $W_{j,j+1}$ $(W_{\leftarrow})$ and $W_{j+1,j}$ $(W_{\rightarrow})$]

The dependence of the average steady state is thus very similar to that of the pure system (3 8); quite intuitively, the velocity is deduced from the flux at infinity,

$$1/V = (1/a) \lim_{n\to\infty} [\langle P_n(E=0)\rangle - \langle P_{-n}(E=0)\rangle],$$

a being the lattice spacing. One thus obtains (see [Tem72] for an early derivation)

$$V = 0, \quad \text{if } \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle \geq 1, \tag{3.23a}$$

$$\frac{V}{a} = \left\langle \frac{1}{W_{\rightarrow}} \right\rangle^{-1} \left(1 - \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle\right), \quad \text{if } \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle < 1 \tag{3 23b}$$

For a model with *site* disorder, in which the pairs $(W_{n-1,n}, W_{n+1,n})$ are independent random variables (e.g the random trap model A), one gets the same expressions (3.23), provided $(W_{\leftarrow}, W_{\rightarrow})$ denote $(W_{n-1,n}, W_{n+1,n})$ [Asl89c]

The main outcome of this calculation is that asymmetric models can display zero asymptotic velocity when the "slow" bonds (opposed to the average bias – here towards $n > 0$) have a large enough weight or strength This is most likely the signal of a phase with anomalous drift, as will be discussed at length in section 3.3

The advantage of considering a periodic sample [Der83a] is that at large times a stationary state is established with a *non-vanishing* probability on each site, related to $P_n^{\mathrm{per}}(E=0)$ (corresponding to a constant non-zero current and periodic boundary conditions) Defining

$$R_n(t) = \sum_{k=-\infty}^{+\infty} P_{n+k\,N}(t),$$

one has

$$R_n(t) \xrightarrow[t\to\infty]{} \frac{P_n^{\mathrm{per}}(E=0)}{\sum_{n=1}^{N} P_n^{\mathrm{per}}(E=0)} \tag{3 24}$$

The velocity is then computed as [Der83a]

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{x(t)} = a \sum_{n=-\infty}^{+\infty} n \frac{\mathrm{d}P_n}{\mathrm{d}t} = a \sum_{n=1}^{N} (W_{n+1,n} - W_{n-1,n}) R_n(t), \tag{3.25}$$

where the master equation (2 1) has been used, together with the periodicity of the medium Making

use of (3 24) and observing that

$$\sum_{n=1}^{N} (W_{n+1,n} - W_{n-1\,n}) P_n^{\mathrm{per}}(E=0) = N ,$$

one gets in the limit $t \to \infty$

$$V_N = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \overline{x(t)} = a \frac{N}{\Sigma_{n=1}^{N} P_n^{\mathrm{per}}(E=0)} \tag{3 26}$$

One has thus obtained the expression of the asymptotic velocity for a *given* periodic sample of arbitrary period Taking the limit $N \to \infty$, one finds that V_N converges with probability one towards the previously obtained result (3.23)

3 1.2. The Green function at $E \neq 0$

3 1 2 1 Recursive construction Obtaining the full Green function for $E \neq 0$ is of course a formidable task, which cannot be achieved in closed form However, a recursive construction can be given following an ingenious scheme elaborated in [Ber78, G10, Ber85] This scheme makes use of the auxiliary quantities $G_n^+(E)$, $G_n^-(E)$ $(n>0)$ defined by (in the following we set $n_0 = 0$ for the sake of simplicity)

$$\begin{aligned} G_n^+(E) &= J_{n\,n+1}(E)/P_n(E) = W_{n+1\,n} - W_{n,n+1} P_{n+1}(E)/P_n(E) , \\ G_n^-(E) &= J_{-n-1\,-n}(E)/P_{-n}(E) = W_{-n-1\,-n} - W_{-n\,-n-1} P_{-n-1}(E)/P_{-n}(E) \end{aligned} \tag{3 27}$$

Being the ratio of a current to a probability at the "frequency" E, $G_n^+(E)$ $[G_n^-(E)]$ can be thought of as an effective admittance [G10] for the part of the chain to the right of site n (to the left of site $-n$) The Laplace transform of the master equation (2.1) can then be rewritten, using these auxiliary variables, as $(n>0)$

$$EP_n(E) = -G_n^+(E) P_n(E) + G_{n-1}^+(E) P_{n-1}(E) , \tag{3 28a}$$

$$EP_0(E) - 1 = -[G_0^+(E) + G_0^-(E)] P_0(E) , \tag{3.28b}$$

$$EP_{-n}(E) = -G_n^-(E) P_{-n}(E) + G_{n-1}^-(E) P_{-n+1}(E) , \tag{3 28c}$$

which are of first order in space, and as such can be easily iterated to yield

$$P_0(E) = [E + G_0^+(E) + G_0^-(E)]^{-1} , \tag{3.29a}$$

$$P_n(E) = P_0(E) \prod_{m=1}^{n} \frac{G_{m-1}^+(E)}{E + G_m^+(E)} , \tag{3 29b}$$

$$P_{-n}(E) = P_0(E) \prod_{m=1}^{n} \frac{G_{m-1}^-(E)}{E + G_m^-(E)} \tag{3 29c}$$

The auxiliary variables G_n are easily seen to be infinite continued fractions, recursively defined for a given sample (W_{ij}) by

$$G_n^+ = \frac{W_{n+1,n}}{1+\dfrac{W_{n\,n+1}}{E+G_{n+1}^+}}, \qquad G_n^- = \frac{W_{-n-1,-n}}{1+\dfrac{W_{-n,-n-1}}{E+G_{n+1}^-}} \tag{3.30}$$

We shall assume throughout this section that condition (3 12) defining the direction of the mean bias is satisfied and we shall work on the infinite line Then it is easily seen that the construction of the steady state given above is recovered for $E=0$. Indeed, the iteration of the recursion relation for $1/G_n^+(E)$ is *convergent* for $E\to 0$, while the recursion relation for $1/G_n^-(E)$ *diverges* in this limit (this is because $W_{n,n+1}/W_{n+1,n}$ is typically smaller than 1 because of (3 12), while $W_{-n,-n-1}/W_{-n-1,-n}$ is typically larger than 1). More precisely, one notices that for $E\to 0$

$$G_n^+(E)\to G_n^+(0)\,, \qquad G_n^-(E)\sim Eg_n^- + \cdots\,,$$

where $G_n^+(0)$ is defined by the same relation as (3.13),

$$\frac{1}{G_n^+(0)} = \frac{1}{W_{n+1\,n}} + \sum_{i=n+1}^{\infty} \frac{1}{W_{i+1,i}} \prod_{j=n}^{i-1} \frac{W_{j,j+1}}{W_{j+1,j}} \tag{3 31}$$

[and thus $P_n(E=0) = 1/G_n^+(0)$ for $n>0$], while the g_n^- are defined by

$$g_n^-/(1+g_{n+1}^-) = W_{-n-1,-n}/W_{-n\,-n-1}\,, \tag{3.32}$$

leading back to expression (3.13) for $P_{-n}(E=0)$. The quantities $1/G_n^+(0)$, g_n^- are *random variables*, whose average values are easily computed in the case where the pairs $(W_{j,j+1},\ W_{j+1\,j})$ are random independent variables,

$$\left\langle \frac{1}{G_n^+(0)} \right\rangle = \langle g_n^- \rangle = 0\,, \quad \text{if } \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle \geq 1\,, \tag{3.33a}$$

$$\left\langle \frac{1}{G_n^+(0)} \right\rangle = \left\langle \frac{1}{W_{\rightarrow}} \right\rangle \left(1-\left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle\right)^{-1}, \qquad \langle g_n^- \rangle = \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle \left(1-\left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle\right)^{-1}, \quad \text{if } \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle < 1\,, \tag{3.33b}$$

with similar expressions for site disorder [Asl89c]

3.1.2.2 The velocity and diffusion constant for a given sample: self-averaging property. As first shown in [Asl89a], the above construction of the Green function can be used to study the long-time behaviour of the thermally averaged position $x(t)$ *for a given sample*. Its Laplace transform reads

$$x_1(E) \equiv \int_0^{\infty} \mathrm{e}^{-Et}\overline{x(t)}\,\mathrm{d}t = a\sum_{n=0}^{\infty} n[P_n(E) - P_{-n}(E)]\,. \tag{3.34}$$

Using eq (3 29), this can be rewritten as

$$x_1(E) = aP_0(E)[G_0^+(E)S^+(E;\,\xi=1) - G_0^-(E)S^-(E,\,\xi=1)]\,, \tag{3 35}$$

where $S^{\pm}(E;\,\xi)$ are infinite series defined by

$$S^{\pm}(E,\,\xi) = 1 + \sum_{n=1}^{\infty} \xi^n \prod_{i=1}^{n} \frac{G_i^{\pm}(E)}{E + G_i^{\pm}(E)} \tag{3 36}$$

(the parameter ξ has been introduced for further use). Studying the long-time behaviour of $\overline{x(t)}$ amounts to studying the singularities of these series for $E \to 0$. This is easy for $S^-(E,\,\xi)$, which obviously has a finite limit as $E \to 0$, thus the term involving S^- contributes to $\overline{x(t)}$ only through corrections of order $1/t$. Analysing the divergence of $S^+(E,\,\xi)$ without taking averages is much more difficult It requires the use of a resummation procedure [Asl89a], which we now briefly sketch Introducing the centred random variables

$$\gamma_i(E) = \frac{1}{G_i^+(E)} - m_1(E) \quad \text{with } m_1(E) = \left\langle \frac{1}{G_i^+(E)} \right\rangle, \tag{3 37}$$

one writes

$$\frac{G_i^+(E)}{E + G_i^+(E)} = \frac{1}{1 + E[m_1(E) + \gamma_i(E)]} = \frac{1}{1 + Em_1} \sum_{n=0}^{\infty} \left(\frac{-E\gamma_i}{1 + Em_1} \right)^n . \tag{3 38}$$

Using this form in (3 36) and collecting terms involving products of a given number of γ_i, one obtains the following convenient expansion of $S^+(E,\,\xi)$

$$\begin{aligned} S^+(E,\,\xi) = {} & \frac{1 + Em_1(E)}{1 - \xi + Em_1(E)} - \frac{E\xi}{1 - \xi + Em_1} \sum_{n=1}^{\infty} \frac{\xi^{n-1}}{(1 + Em_1)^n} \gamma_n \\ & + \frac{E^2\xi}{1 - \xi + Em_1} \sum_{n=1}^{\infty} \frac{\xi^{n-1}}{(1 + Em_1)^{n+1}} \sum_{m=1}^{n} \gamma_n \gamma_m + \cdots \end{aligned} \tag{3 39}$$

Thus, one sees that the most divergent term of $S^+(E,\,\xi=1)$ for $E \to 0$ is $1/[Em_1(E=0)]$ independently of the specific sample considered. One thus gets from (3.35)

$$x_1(E) \sim \frac{1}{m_1(E=0)} \frac{1}{E^2} + \cdots, \quad E \to 0\,,$$

which means that the large-time behaviour of $\overline{x(t)}$ reads for a given sample

$$\overline{x(t)} = Vt + \cdots, \quad t \to \infty\,, \quad V = \langle 1/G_n^+(E=0) \rangle^{-1}$$

Using (3 33) this provides a rigorous derivation of the result derived from the steady state in the previous section and shows that the velocity is indeed a *self-averaging quantity*. The *non-leading* contributions to the thermal average of the position, $\overline{x(t)}$, do, however, display fluctuations from one environment to the other, which will be studied in section 3.1.3

The same techniques can be used in principle to calculate the diffusion constant for a given sample, when it exists,

$$D = \frac{1}{2} \lim_{t \to \infty} \frac{\mathrm{d}}{\mathrm{d}t} [\overline{x(t)^2} - \overline{x(t)}^2]$$

Indeed, the Laplace transform $x_2(E)$ of the mean square $\overline{x(t)^2}$ can be written in terms of $S^+(\xi, E)$ and of its derivative as

$$x_2(E) = a^2 \sum_{n=0}^{\infty} n^2 [P_n(E) + P_{-n}(E)]$$

$$= \frac{a^2}{E} P_0(E) \left[G_0^+(E) \left(S^+(E;1) + 2 \frac{\partial S^+}{\partial \xi} \bigg|_{\xi=1} \right) + G_0^-(E) \left(S^-(E;1) + 2 \frac{\partial S^-}{\partial \xi} \bigg|_{\xi=1} \right) \right], \qquad (3\,40)$$

where again the dominant contributions up to order $1/E^2$ come only from the term involving S^+ However, obtaining these contributions in explicit form for a given sample is fairly tedious; furthermore, the second term $\overline{x(t)}^2$ in D requires the computation of the convolution $x_1 * x_1$ of the Laplace transforms For these reasons, this direct calculation has not yet been performed in the literature for the general case Instead, the following results have been published:

– In [Der83a, Asl89b], the diffusion constant of a given *periodic* sample has been obtained through steady-state methods generalizing the calculation of the velocity presented above

– In [Asl89a], the calculation of the behaviour at large time of the *average over disorder* $\langle \overline{x^2(t)} \rangle - \langle \overline{x(t)}^2 \rangle$ for an infinite sample has been performed, starting from eq. (3.40). The resulting expression of the diffusion constant coincides with the one of the periodic case in the limit of an infinite period It is found to be finite (and non-zero) provided

$$\langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle < 1 ,$$

in which case it reads

$$\frac{D}{a^2} = \frac{1 - \langle W_{\leftarrow}/W_{\rightarrow} \rangle}{1 - \langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle} \left\langle \frac{1}{W_{\rightarrow}} \right\rangle^{-3} \left[\left\langle \frac{1}{W_{\rightarrow}} \right\rangle \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}^2} \right\rangle + \frac{1}{2} \left\langle \frac{1}{W_{\rightarrow}^2} \right\rangle \left(1 - \left\langle \frac{W_{\leftarrow}}{W_{\rightarrow}} \right\rangle \right) \right] ; \qquad (3.41)$$

$D = \infty$ for $\langle W_{\leftarrow}/W_{\rightarrow} \rangle < 1 < \langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle$ The calculation of D in the last regime, $\langle W_{\leftarrow}/W_{\rightarrow} \rangle > 1$, is more complicated, because the velocity vanishes simultaneously, this will be clarified in a particular case in section 3 3

Because the direct calculation for a given infinite sample has not been performed in the general case, the self-averaging (i.e. sample independent) nature of D remains to be proven (though it is most likely to hold). It has been shown in [Asl89a] that it holds true in the limiting case of a strictly directed walk (in which the $W_{j,j+1}$ are all zero)

A remark on the Green function at coinciding points. It is to be noticed that for a general hopping model, the velocity and diffusion constant can in fact be computed from the knowledge of the expansion at small E of the average Green function at coinciding points [Geo88, Asl89b],

$$\langle P(x, x; E) \rangle = 1/V - 2(D/V^3)E + \cdots . \qquad (3.42)$$

3.1 3 Transients and sample to sample fluctuations

As is clear from (3 35) and (3.36), the thermal average of the position, $\overline{x(t)}$, is a sum of (correlated) random variables, and has thus a *distribution over environments* Because the system is one dimensional, the relative standard deviation of this distribution only decays as $1/t^{1/2}$, thus leading to sample to sample fluctuations resulting in $D_{\rm av} \neq D$ in general [Dou89a, Asl89d]

The above techniques can be used to obtain information on this distribution (and thus also on the *transient regimes*) Let us first consider its mean $\langle \overline{x(t)} \rangle$ It was first shown in [Asl89a] that one can use (3 35) and (3 37) to compute the first correction to $\langle \overline{x(t)} \rangle$, which turns out to be a constant,

$$\langle \overline{x(t)} \rangle = V(t + t_0) + \qquad (3\ 43)$$

In order to obtain the expression of t_0, one has to calculate correlations between the γ_n at $E = 0$, together with the subdominant contributions to $G_n^+(E)$ close to $E = 0$. The former are easily calculated from (3.37),

$$\langle \gamma_n(E=0)\gamma_m(E=0) \rangle = \langle 1/G_0^+(0)^2 \rangle \langle W_{\leftarrow}/W_{\rightarrow} \rangle^{|n-m|} ,$$

with ($a = 1$)

$$\langle 1/G_0^+(0)^2 \rangle = \begin{cases} \infty , & \text{if } \langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle \geq 1 , \\ \left(1 - \left\langle \left(\dfrac{W_{\leftarrow}}{W_{\rightarrow}}\right)^2 \right\rangle\right)^{-1} \left(\left\langle \dfrac{1}{W_{\rightarrow}^2} \right\rangle + \dfrac{2}{V} \left\langle \dfrac{W_{\rightarrow}}{W_{\leftarrow}^2} \right\rangle \right) , & \text{if } \langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle < 1 \end{cases} \qquad (3\ 44)$$

Expanding the recursion relation (3 30), one observes that for small E,

$$G_n^+(E) \sim G_n^+(0) + E g_n^+ + \quad ,$$

where the g_n^+ are given by

$$\frac{g_n^+}{G_n^+(0)^2} = \frac{1}{G_n^+(0)^2} + \sum_{i=n+1}^{\infty} \frac{1}{G_i^+(0)^2} \prod_{j=n}^{i-1} \frac{W_{j\,j+1}}{W_{j+1\,j}} \qquad (3\ 45)$$

This quantity has a finite average provided $\langle W_{j\,j+1}/W_{j+1\,j} \rangle < 1$, given by

$$\left\langle \frac{g_n^+}{G_n^+(0)^2} \right\rangle = \left\langle \frac{1}{G_n^+(0)^2} \right\rangle \frac{\langle W_{\leftarrow}/W_{\rightarrow} \rangle}{1 - \langle W_{\leftarrow}/W_{\rightarrow} \rangle} \qquad (3\ 46)$$

The constant correction to the average $\langle \overline{x(t)} \rangle$ is then calculated to be [Asl89a] ($a = 1$)

$$Vt_0 = \frac{1 + \langle W_{\leftarrow}/W_{\rightarrow} \rangle}{1 - \langle W_{\leftarrow}/W_{\rightarrow} \rangle} \left(V^2 \left\langle \frac{1}{G^+(0)^2} \right\rangle - 1 \right) , \qquad (3\ 47)$$

provided $\langle (W_{j\,j+1}/W_{j+1\,j})^2 \rangle < 1$. It diverges when $\langle (W_{j,j+1}/W_{j+1\,j})^2 \rangle$ reaches 1; thus, if the distribution of the W_{ij} is such that $\langle W_{j,j+1}/W_{j+1,j} \rangle < 1 < \langle (W_{j,j+1}/W_{j+1\,j})^2 \rangle$, a finite velocity will be reached for a given sample but the *sample-averaged* correction to this behaviour *diverges*. t_0 must be interpreted as the time scale after which the regime $\overline{x(t)} = Vt$ is reached on average

The same techniques can be used to study the variance of the distribution of $\overline{x(t)}$ over samples. As announced above, one finds that it is of order t and can be expressed in terms of V and the time scale t_0,

$$\langle \overline{x(t)}^2 \rangle - \langle \overline{x(t)} \rangle^2 = V^2 t_0 t + \cdots \tag{3.48}$$

From this result, one concludes that the diffusion constant D_{av}, defined in section 2 1 and associated with the *average* diffusion front $\langle P(x, t) \rangle$, *does not coincide with D whenever the velocity is non-zero*, and reads [Asl89d, Dou89a] ($a = 1$)

$$D_{\mathrm{av}} = D + V^2 t_0 / 2 = \frac{V}{2} \left(\frac{1 + \langle W_{\leftarrow} / W_{\rightarrow} \rangle}{1 - \langle W_{\leftarrow} / W_{\rightarrow} \rangle} + 2Vt_0 \right) = 2D - \frac{1}{2} \frac{1 + \langle W_{\leftarrow} / W_{\rightarrow} \rangle}{\langle 1 / W_{\rightarrow} \rangle} \tag{3.49}$$

Since the variance of $\overline{x(t)}$ is of order t as soon as $V \neq 0$, it is clear that the distribution (over thermal histories) of the scaled variable $(x(t) - Vt)/t^{1/2}$ cannot satisfy a "generalised" CLT (i.e. reach a limit form) for a *given environment* The basic mechanism underlying these sample to sample fluctuations can be most simply illustrated on the following example [Dou89a]. Consider a sequence of random variables $(t_1, t_2, \ldots)$, which are independent but distributed according to *different* distributions $p_1(t)$, $p_2(t)$, etc. One asks whether the CLT can be generalized to the sum

$$T_n = \sum_{i=1}^{n} t_i \tag{3.50}$$

This is the generic situation one encounters in a disordered medium. T_n describes, for example, the first passage time at site n of a random walker with $W_{i,i+1} = 0$ and $W_{i+1,i} = W_i$ (directed walk), $p_i(t)$ being the waiting time distribution, $p_i(t) = W_i \exp(-W_i t)$ Denoting by τ_i and σ_i the mean and variance of $p_i(t)$, one has

$$\bar{T}_n = \sum_{i=1}^{n} \tau_i , \qquad \overline{T_n^2} - \bar{T}_n^2 = \sum_{i=1}^{n} \sigma_i . \tag{3 51a,b}$$

The proof of the CLT (in its usual Gaussian form) given in section 1.1 can be readily extended (cf. e g , ref. [G2]) to such an inhomogeneous situation when one considers the rescaled variable

$$(T_n - \overline{T_n}) \Big/ \left(\sum_{i=1}^{n} \sigma_i \right)^{1/2} , \tag{3.52}$$

provided that the series (3 51b) diverges. Let us assume furthermore that the series $(1/N) \Sigma_{i=1}^{N} \tau_i$ converges (as would be the case for the above walk when $\langle 1/W_{\rightarrow} \rangle < \infty$); it is easily seen that *no CLT holds* if one rescales T_n as $T_n - n \lim_{N \to \infty} N^{-1} \Sigma_{i=1}^{N} \tau_i$ instead of $T_n - \overline{T_n}$. The basic mechanism encountered in this section is indeed that one cannot forget the fluctuations of $\overline{x(t)}$ by replacing it by its leading term, Vt However, this suggests that for the above hopping models, the distribution of the variable $[x(t) - \overline{x(t)}]/t^{1/2}$ – when diffusion is normal – could have a limit form for a given environment

3 2 Analytical results for one-dimensional symmetric barriers

The above Green function techniques, and in particular the recursive construction of section 3 1 2, were first applied (see ref [G10] for a review) to the case of random symmetric barriers $W_{n\,n+1} = W_{n+1\,n}$ ($= W_n$), independently distributed according to

$\psi(W)$ In section 2 4 2 we have performed a qualitative analysis of such models, based on their electrical analogue and on general statistical mechanisms Analytical results and techniques will be briefly summarized in this section, referring for further details to the review article of Alexander et al [G10]

The starting point is the recursive relation (3 30), which, in the symmetric case, becomes

$$\frac{1}{G_n^+} = \frac{1}{W_n} + \frac{1}{E + G_{n+1}^+} \tag{3 53}$$

(with a similar relation for G_n^-) The G_n are random (correlated) variables depending on the sample (W_n) Their distribution $Q_E(G)$ satisfies an integral equation which follows from the above recursion relation,

$$\begin{aligned} Q_E(G) &= \int \mathrm{d}W\, \psi(W) \int \mathrm{d}G'\, Q_E(G')\delta\left[G - \left(\frac{1}{W} + \frac{1}{E+G'}\right)^{-1}\right] \\ &= \int_{G-F}^{\infty} \mathrm{d}G'\, Q_F(G')\psi\left(\frac{G(G'+E)}{G'+E-G}\right)\left(\frac{G'+E}{G'+E-G}\right)^2 \end{aligned} \tag{3 54}$$

(The use of such an integral equation, initiated by the works of Dyson [Dys53] and Schmidt [Sch57] (see also [Lie66]) on random harmonic chains, is a powerful tool to study one-dimensional systems) Because the G_n are correlated, the distribution $Q_E(G)$ does not contain all the desired information on the diffusion behaviour, however, it allows one, for example, to compute the average over disorder of the probability of presence at the initial site (autocorrelation function), whose Laplace transform reads [using (3 29)]

$$\langle P_0(E)\rangle = \int \mathrm{d}G \int \mathrm{d}G'\, \frac{Q_E(G)Q_E(G')}{E+G+G'} \tag{3 55}$$

Detailed studies of the solution of the integral equation (3 54) have been performed, in both limits $E \to +\infty$ (short times) and $E \to 0$ (large times) (see ref [G10]) Note that, for the pure system $\psi(W) = \delta(W - W_0)$, $Q_E(G)$ is a delta function $\delta(G - G_0)$ with

$$2G_0 = \sqrt{E^2 + 4W_0E} - E \tag{3 56}$$

In the limit of infinite E, one immediately sees from (3 54) that $Q_\infty(G) = \psi(G)$ Provided the successive moments $\langle W^k\rangle$ of W_n exist, a systematic expansion of $Q_E(G)$ in powers of $1/E$ can be performed [G10] This can also be done for the integral equation satisfied by the full average Green function $\langle P_n(E)\rangle$, yielding a *short-time* expansion of the diffusion law ($a = 1$),

$$\overline{\langle x^2(t)\rangle} = 2\langle W\rangle t - 2[\langle W^2\rangle - \langle W\rangle^2]t^2 + \tfrac{2}{3}(2\langle W^3\rangle - 3\langle W^2\rangle\langle W\rangle + \langle W\rangle^3)t^3 + \tag{3 57}$$

However, the relevant information on the *large-time* behaviour is contained in the *small-E* regime, where (3 54) is much more difficult to analyse For $E = 0$, it is clear that $Q_0(G) = \delta(G)$ satisfies (3 54) Since in the infinite-time limit a generalized central limit theorem is expected to apply [e g to $P_n(t)$], one can suspect that $Q_E(G)$ will depend only on a scaled variable $G/\varepsilon(E)$ (in a limit in which G and E are both small) Solutions of (3 54) have thus been looked for under the form

$$Q_E(G) = \frac{1}{\varepsilon(E)}\, h(G/\varepsilon(E)) \tag{3 58}$$

The scaling factor $\varepsilon(E)$ and scaling function $h(x)$ were found to have a qualitatively different behaviour according to whether the first moment $\langle 1/W_n\rangle$ is finite or not [Ber80]

(i) if $\langle 1/W_n\rangle$ is finite, the problem is found to be very similar to the *non-disordered* case, with an effective hopping rate $W_{\mathrm{eff}} = \langle 1/W_n\rangle^{-1}$, indeed, one finds that

$$\varepsilon(E) = \sqrt{E}\langle 1/W\rangle^{-1/2}, \qquad h(x) = \delta(x-1) \tag{3 59}$$

This leads to the familiar large-time decay of the density on the initial site,

$$\langle P_0(t)\rangle \underset{t\to\infty}{\sim} \frac{1}{\sqrt{4\pi Dt}} \tag{3 60}$$

(Recall that in this case diffusion is normal with a diffusion constant $D = \langle 1/W_n\rangle^{-1}$)

(ii) If, on the contrary, $\langle 1/W_n \rangle = \infty$, for example, if

$$\psi(W) \sim W^{\mu-1}, \quad 0<\mu<1, \quad W \to 0,$$

then $h(x)$ is found [Ber80] to be a non-trivial function continuously depending on μ and very different from the result for the pure system $h(x)$ satisfies the following integral equation

$$h(x) = x^{\mu} \int_x^{\infty} \mathrm{d}y\, h(y) \left(\frac{y}{y-x}\right)^{\mu} \tag{3 61}$$

(whose solution can be expressed in terms of a generalized hypergeometric function [Ber80]) The scaling factor $\varepsilon(E)$ reads

$$\varepsilon(E) \propto E^{1/(1+\mu)}, \tag{3 62}$$

leading to an anomalous decay at large time of $\langle P_0(t) \rangle$,

$$\langle P_0(t) \rangle \sim C t^{-\mu/(1+\mu)}, \quad t \to \infty \tag{3 63}$$

In section 2 4 2 1 diffusion was shown to be anomalous in this case, with a diffusion exponent $\nu = \mu/(1+\mu)$, which precisely corresponds to (3 63) if one takes the scaling form (2 7) of $P_n(t)$ into account Note that, in this unbiased situation, a CLT is expected to hold for $P_n(t)$ for a *given* environment, however, the scaling function $f(x)$ associated with $P_n(t)$ is not known analytically for this problem, as opposed to $h(x)$, a purely exponential form $f(x) = \mathrm{e}^{-C|x|}$ has been advocated on a numerical basis [Ber80]

3.3. *Anomalous diffusion behaviour of the asymmetric hopping models with bond disorder*

The general analysis of section 3.1 reveals that *asymmetric* hopping models with bond disorder can display anomalous diffusion behaviour without assuming a priori broad distributions of hopping rates This is at variance with the symmetric case studied in the previous section and with the random traps models of section 2 4. The aim of this section is to discuss this anomalous behaviour through (i) a physical explanation of the underlying statistical mechanism and (ii) some analytical results. In so doing, the continuous-space formulation of the random force model (section 2 1 4, 2.3) is most helpful, since it retains the essential features of all asymmetric hopping models while allowing the exact calculation of some quantities [Bou87a, 89a]. Section 3 3 1 is devoted to a presentation of this model.

3.3.1. *The continuous random-force model*

3 3.1.1 The model. The continuous random-force model is defined through*)

$$W_{n,n+1} = \frac{T}{\gamma a^2} \mathrm{e}^{-aF_n/2T}, \qquad W_{n+1,n} = \frac{T}{\gamma a^2} \mathrm{e}^{+aF_n/2T},$$

with F Gaussian distributed ($\langle F \rangle = F_0$, $\langle F_n F_m \rangle_c = (\sigma/a)\delta_{n,m}$). The continuous limit ($a \to 0$) of the associated master equation corresponds to the following Langevin equation:

$$\gamma\, \mathrm{d}x/\mathrm{d}t = F(x) + \eta(t), \tag{3 64}$$

where $F(x)$ is Gaussian white noise (in space) and $\eta(t)$ Gaussian white noise (in time) (see section 2 3) Equation (3.64) is thus the simplest Langevin equation describing the motion of a particle in a random medium one can think of. As we are in one dimension, $F(x)$ can be written as $F(x) = -\mathrm{d}U/\mathrm{d}x$; $U(x)$ is

*) In the following, we set the Boltzmann constant equal to unity ($k = 1$)

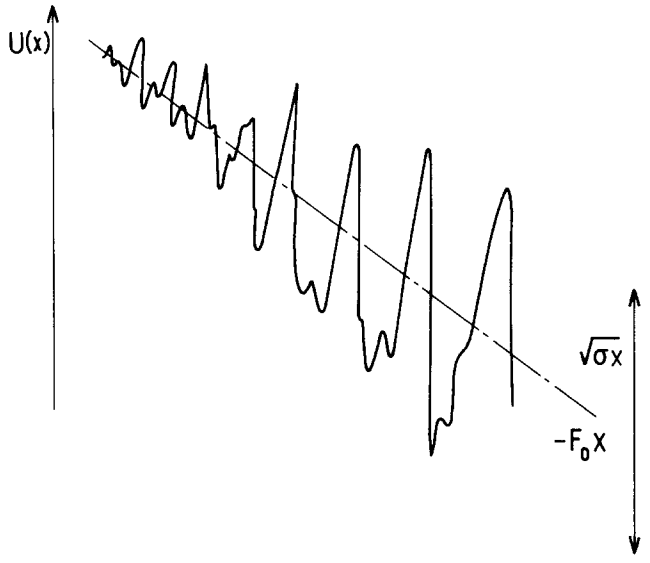


Fig 3 2 Typical configuration of the potential of which the random force $F(x)$ is the derivative The fluctuations of $U(x)$ around the average slope $-F_0 x$ typically grow as $\sqrt{\sigma x}$

thus a "Brownian motion" in x, a typical configuration of which is depicted in fig 3 2. One immediately sees that the motion of the particle under the external bias is strongly impeded by the fluctuations of the potential

3.3.1 2 Velocity and diffusion constant The results of section 3 1 allow one to calculate the asymptotic velocity and diffusion constant for model (3 64). This shows at once that non-trivial properties appear for some values of the parameters defining the problem. For a Gaussian F, one has (in the following we shall take $F_0 > 0$)

$$\langle (W_{n,n+1}/W_{n+1,n})^k \rangle = \langle e^{-(aF_n/T)k} \rangle = e^{-(a\sigma/2T^2)k(\mu - k)}, \tag{3.65}$$

where μ is a dimensionless parameter,

$$\mu = 2F_0 T/\sigma \tag{3 66}$$

μ is thus the ratio of the mean energy gain (over a lattice site) $F_0 a$ times the thermal energy to the mean square of the energy fluctuation σa (again over a lattice site). Applying eqs (3 23) and (3 41) to this particular case, one finds

* $0 < \mu < 1$: $\langle W_{\leftarrow}/W_{\rightarrow} \rangle \geq 1$, and hence $V = 0$,
* $1 < \mu < 2$: $\langle W_{\leftarrow}/W_{\rightarrow} \rangle < 1 < \langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle$ The velocity is now finite and reads

$$V = V_0(1 - 1/\mu) \tag{3.67}$$

($F_0/\gamma = V_0$ is the velocity the particle would acquire in the absence of disorder.) The diffusion coefficient, in this phase, is infinite

* $\mu > 2$ or $\langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle < 1$. V and D are both finite; V retains the same expression as above and D

reads

$$D = D_0(\mu - 1)/(\mu - 2)\,, \tag{3 68}$$

where T/γ is the "bare" diffusion constant D_0. The width D_{av} of the average front can also be obtained from (3.49) and reads

$$D_{av} = D_0\mu/(\mu - 2)\,.$$

Note that in the limit of an ordered system ($\sigma \to 0$, $\mu \to \infty$), one recovers $D = D_{av} = D_0$.

These results are summarized in fig 3 3, where V/V_0 and D/D_0 are plotted versus μ One should note that these two quantities are singular for *different* values of the control parameter, this is often encountered in the physics of disordered systems (see, e.g., [Der84a]).

The only parameter controlling the transport properties is thus the dimensionless parameter μ, which is independent of the lattice spacing a and of the friction coefficient γ. μ increases as the mean bias or the temperature grows, and decreases if the disorder gets stronger

The appearance of a phase where the velocity vanishes simply means that the relation between position and time is sublinear [Der82b, Sol75, Kes75], $\langle \bar{x} \rangle \sim t^{\alpha}$ with $\alpha < 1$. The divergence of the diffusion constant signals the fact that the spreading of the probability distribution grows faster than $t^{1/2}$ ($x - \bar{x} \sim t^{\nu}$)

The aim of the next sections is to show how these laws can be understood on physical grounds [Bou87c, 89a, Vin86, Fei88] and the diffusion exponents calculated.

For a general asymmetric model, these two anomalous diffusion phases read (section 3 1)

$$\langle W_{\leftarrow}/W_{\rightarrow} \rangle \geq 1\,, \qquad \langle W_{\leftarrow}/W_{\rightarrow} \rangle < 1 \leq \langle (W_{\leftarrow}/W_{\rightarrow})^2 \rangle\,,$$

which are easily shown to yield $0 < \mu \leq 1$ and $1 < \mu \leq 2$ in the continuous limit (3.64).

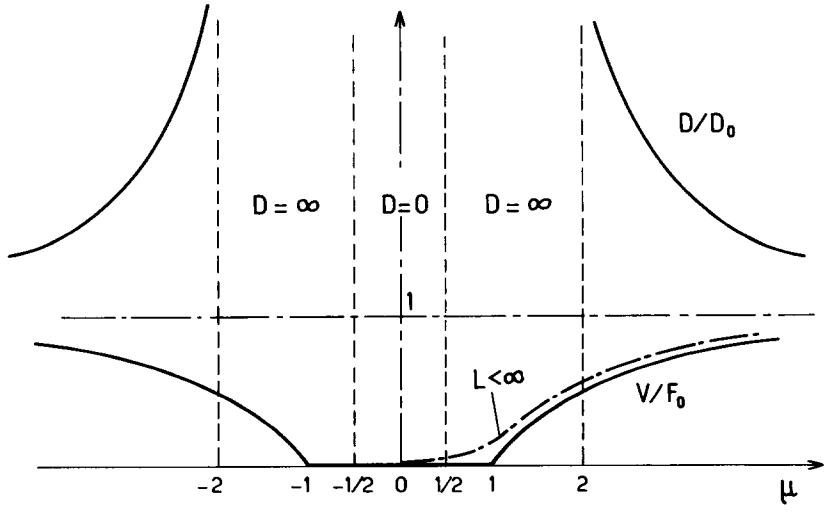


Fig 3 3 Velocity and diffusion constant (divided by their "pure" values) as a function of the dimensionless ratio μ The dash–dotted line shows the behaviour of the current going through a finite-size sample

3 3 1 3. Length and time scales The problem at hand is thus the thermally activated motion of a particle within a potential, schematically depicted in fig 3 2 One is led at once to single out two length scales, defined by comparing the fluctuational energy $\sqrt{\sigma x}$ (a) to the mean energy gain due to the external force, $-F_0 x$, or (b) to the thermal energy T Then for $x \ll x_0 = \sigma/F_0^2$, the particle's motion is mainly determined by the energy fluctuations and insensitive to the presence of the mean bias F_0 The thermal energy T allows the particle to diffuse without being too much affected by those fluctuations until $\sqrt{\sigma x} \sim 2T$ (or $x \sim x_1 = 4T^2\sigma^{-1}$); this distance is reached after a time corresponding to free diffusion,

$$\tau_1 = x_1^2/2D_0 = \gamma \ 8T^3/\sigma^2$$

Now, two very different physical situations occur when $x_0 \gg x_1$ or when $x_1 \gg x_0$ The former case indicates that, well before being able to feel the external force, the particle is "pinned" by strong fluctuations at scales x_1 and must then wait for a large energy fluctuation $\Delta E \gg T$ from the thermal bath to overcome the potential barrier and continue on its way

On the other hand, if $x_1 \gg x_0$, thermal energy is able to bring the particle far enough so that the mean bias F_0 exceeds the force fluctuations; the subsequent motion is then only slowed down by rare, very large, force fluctuations occurring on a scale x_1. The relevant control parameter thus naturally appears as

$$x_1/x_0 = (4T^2/\sigma)F_0^2/\sigma \equiv \mu^2$$

[see eq. (3 66)] or equivalently, as the ratio of the energy barrier associated with the scale x_0 to the thermal energy,

$$\Delta U(x_0)/2T = \sigma/2TF_0 = 1/\mu \ .$$

As is clear from formulae (3.67) and (3.68), $\mu = 1$ does not merely correspond to a cross-over region but to a true *phase transition* separating a region of zero velocity ($\mu < 1$) from a region of finite velocity

This physical analysis suggests that the motion of the particle can be seen at large scales as a *succession of trappings* within regions of size x_1 (fig. 3.2), characterized by a release time distribution corresponding to the different times needed for the particle to receive the right amount of thermal energy. As will be reviewed below, this analogy can be made more precise the distributions of relaxation times and of the thermal average of the waiting time can be obtained in closed form for this model [Bou87a, 89a] and indeed turn out to be broad.

3 3 2 Sinai's diffusion for zero global bias

3 3 2 1 Ultra-slow diffusion When the external bias vanishes ($F_0 = 0$, or more generally when $\langle \ln(W_{\leftarrow}/W_{\rightarrow}) \rangle = 0$), the previously defined length scale x_0 diverges and the particle can only rely on the thermal bath to progress and overcome ever increasing potential barriers: in order to span a distance x, the particle must be given an energy typically of the order of $\sqrt{\sigma x}$; this takes a time governed by an Arrhenius law,

$$t \simeq \tau_1 \exp(\sqrt{\sigma x}/2T) \qquad (3\ 69)$$

(τ_1 is obviously the order of magnitude of the "trial" frequency) This suggests

$$\langle \overline{x(t)} \rangle = 0 , \qquad \langle \overline{x^2(t)} \rangle = C x_1^2 [\ln(t/\tau_1)]^4 \tag{3 70}$$

This remarkable slowing down of the diffusion process has been discovered by Sinai [Sin81, 82], who gave a rigorous proof of the above law for a discrete time version of the hopping models considered here

While the *typical* time needed to reach x increases as (3 70), one may, however, ask what the *average* (over different starting points) of this time is. As first discussed by de Gennes [dGe75] and recently shown by Noskowitz and Goldhirsch [Nos88] (see also [Dou89b]), this is obtained as

$$\langle \overline{t(x)} \rangle = \left\langle \exp\left(-\frac{1}{2T} \int_0^x F(z)\, dz \right) \right\rangle = e^{\sigma x / 8T^2} , \tag{3 71}$$

which is much greater than the *typical* time. (As will be discussed in chapter 4, long-range correlations $\langle F(x)F(y) \rangle \sim |x-y|^{-a}$ with $a<2$ lead to a modified Sinai law $\overline{x^2(t)} \sim (\ln t)^{4/(2-a)}$ [Bou87b, Hav88b])

3.3.2.2. Diffusion front and the Golosov phenomenon More precise results about Sinai's diffusion have been obtained recently by Golosov [Gol84] Here again the point is that the thermal average $\overline{x}_t$ fluctuates from sample to sample (while, since no global bias is present, $\langle \overline{x}_t \rangle = 0$). In the case at hand, these fluctuations turn out to be of the same order as $\langle \overline{x}_t^2 \rangle^{1/2}$ For a *fixed* configuration of the potential, the mean position of a packet of particles initially at $x = 0$ evolves as

$$\overline{x(t)} = \xi_t(\omega)(\ln t)^2 , \tag{3 72}$$

where $\xi_t(\omega)$ is noise of order 1. The width of the packet, however, quite surprisingly *does not grow with time*· one has

$$\overline{x^2(t)} - \overline{x(t)}^2 \simeq x_1^2 , \quad t \to \infty . \tag{3 73}$$

Two initially close particles, undergoing different thermal histories, stay close to each other *for all times*, this means that, at any given time, one deep valley (at a distance $\ln^2 t$ from the origin) dominates all others and gathers all the particles (see fig. 3 4a) It is thus clear that $P(x, t)$ *does not* evolve towards a limit distribution for large t, and remains concentrated around its centre of gravity

For disorder-averaged quantities, one has

$$\langle \overline{x(t)} \rangle = 0 , \qquad \langle \overline{x^2(t)} \rangle \sim \ln^4 t \tag{3 74}$$

An intriguing remark [Dou87] must be made concerning the full *averaged* limit distribution of the scaled variable $x/\ln^2 t$: as the potential $U(x)$ is a "random walk" as a function of x, the probability that $U(x)$ remains confined between $-U/2$ and $+U/2$ in the segment $[0, x]$ is a well-known quantity [G1], which reads

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2 \pi^2}{2U^2 \sigma} x \right) . \tag{3.75}$$

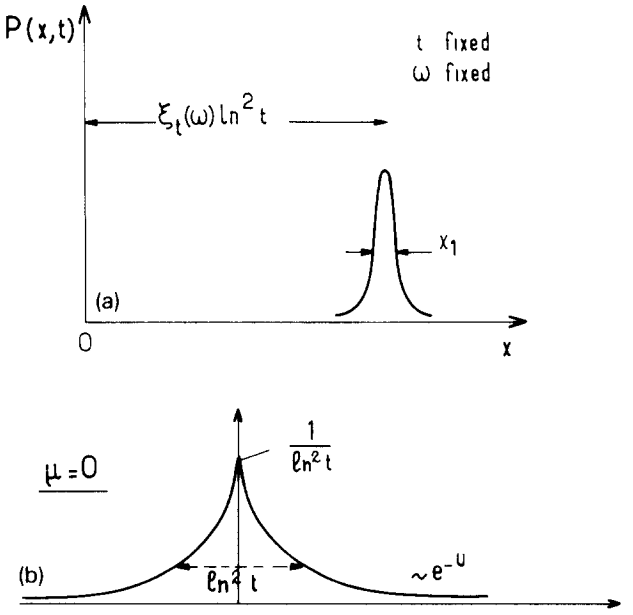


Fig 3 4 (a) The Golosov phenomenon in a given environment, the probability distribution is peaked, its centre of mass is at a distance $\xi_t \ln^2 t$ from the origin, while its width is finite, $\sim x_1 = (kT)^2/\sigma$ (b) Scaling function for the average diffusion front in Sinai's ($F_0 = 0$) case

Now, the highest potential barrier being of the order U, the associated time is $t \sim \exp(U/2T)$ and one can think of replacing U by $2T \ln t$ in the above formula to get $\langle P(x, t)\rangle$ for large t Remarkably this leads essentially to the exact result for $\langle P(x, t)\rangle$, which takes the scaled form

$$x_1 \langle P(x, t)\rangle \to \ln^{-2}(t/\tau_1) f_0\left[\frac{x}{x_1 \ln^2(t/\tau_1)}\right], \tag{3 76}$$

$$f_0(u) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp[-\tfrac{1}{2}\pi^2(2k+1)^2|u|] \tag{3 77}$$

This exact expression for f_0 (see fig. 3 4b) has recently been derived by Kesten [Kes86] for a related discrete time model, and is in very good agreement with the numerical results of [Nau85] (see, however, [Bun88]). It has also been partially recovered in [Bou89a] using the replica trick and a WKB method

One can also wonder whether the histogram of the positions of a single walker for a single thermal history and a given environment reaches a limiting form when $x(t)$ is rescaled by $(\ln t)^2$ While ergodicity would suggest that it coincides with $\langle P(x, t)\rangle$, this property has, to our knowledge, not been proven

3 3 3 Non-zero bias induced broad distribution of trapping times

As suggested by the analysis of characteristic scales of section 3 3 1 3, trapping regions are induced at large scales in the models considered here We show in this section that the corresponding trapping time distribution is "broad", i e , has a slow power law decay, which is identified precisely

3 3 3.1 Physical origin: rare events The appearance of a broad trapping time distribution has a very general origin in thermally activated systems it comes from the fact that exponentially rare energy

barriers take an exponentially long time to be crossed. If $p(\Delta U) \sim \exp(-\Delta U/2T_0)$ and $\tau \propto \exp(\Delta U/2T)$, one has

$$p(\tau)\,\mathrm{d}\tau = p(\Delta U)\,\mathrm{d}\Delta U \;\to\; p(\tau) \underset{\tau\to\infty}{\sim} \tau^{-(1+T/T_0)} \tag{3.78}$$

The origin of the (Poisson) exponential distribution of energy barriers can be understood on the binary model for which the local force is equal to F_0 with probability $1-p$ and $-F_0$ with probability $p \ll 1$

The probability of encountering a sequence of "unfavourable" drifts of length Na is obviously $P(N) \simeq p^N$ The corresponding barrier height is $\Delta U = NaF_0$, and hence

$$p(\Delta U) \simeq \mathrm{e}^{-\Delta U/2T_0} \quad \text{with } 2T_0 = aF_0/\ln p\,. \tag{3.79}$$

Let us first describe a heuristic way of obtaining the large-time behaviour [Fei88] of the corresponding distribution for the continuous model (3 64), which is easily generalizable to, e.g , *correlated* random local forces. Define $p_L(\Delta U)$ as the probability that the potential energy is equal to $U=0$ for $x=0$ and $U=\Delta U$ for $x=L$,

$$\begin{aligned} p_L(\Delta U) &\propto \int \mathscr{D}F\,\delta\Big(\int_0^L F(x)\,\mathrm{d}x + \Delta U\Big)\exp\Big(-\int_0^L \mathrm{d}x\,\frac{[F(x)-F_0]^2}{2\sigma}\Big) \\ &\propto \int \mathscr{D}F \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\exp\Big[\int_0^L \mathrm{d}x\,\Big(\frac{\mathrm{i}k\,\Delta U}{L} + \mathrm{i}kF(x) - \frac{[F(x)-F_0]^2}{2\sigma}\Big)\Big], \end{aligned} \tag{3.80}$$

which yields, for L large enough,

$$p_L(\Delta U) \propto \mathrm{e}^{-F_0\,\Delta U/\sigma}\,, \quad L\to\infty\,, \tag{3 81}$$

and hence, using $\tau \sim \exp(\Delta U/2T)$, $p(\tau) \sim \tau^{-(1+\mu)}$ for $\tau\to\infty$ with μ given by (3.66)

The saddle point of integral (3.80) is at $F(x) = \Delta U/L$, which shows that indeed ΔU can be interpreted as the highest energy barrier between 0 and L. Note also that the case of a Gaussian *correlated* force $[\langle F(x)F(y)\rangle_c = G(x-y)]$ can be easily treated with this method and leads to the same probability distribution for trapping times upon the replacement

$$\sigma \to \int_0^\infty G(x)\,\mathrm{d}x\,. \tag{3.82}$$

If the correlations are long ranged [i.e., $G(x) \sim x^{-a}$ with $a<1$], then $p_L(\Delta U)$ will never cease to depend on L, and one should work with an "effective parameter" $\mu(L)$ going to zero for large L as L^{a-1}

The above argument shows that indeed a broad distribution of trapping times appears in the random force model. This is a generic feature of asymmetric hopping models, the exponent governing the decay of this trapping time distribution being in general given by the equation [Der82b, 83a, Kes75]

$$\langle (W_{\leftarrow}/W_{\rightarrow})^{\mu}\rangle = 1 \tag{3 83}$$

(a justification of which will be given below) The solution of this equation for the continuous model considered here indeed coincides with $2F_0T/\sigma$

Moreover, one may for this model calculate *exactly* the distribution of two particularly interesting quantities [Bou87a, 89a]·
- the distribution over environments of the thermal average of the local "sojourn time", denoted as $\bar{\tau}$;
- the *density of (inverse) relaxation time* [Bou87a] defined through

$$\langle P(x_0, t|x_0, 0)\rangle = \int_0^\infty \mathrm{d}\tau\, \rho(\tau)\, \mathrm{e}^{-t/\tau}$$

These quantities will be calculated in the next section and in section 3.5, respectively Both indeed show the expected algebraic tail for large time

3 3 3 2 Exact calculation of the distribution of the mean local trapping time Let us isolate a region I of size x_1 and put a particle at $t = 0$ on site x, taken as the "entry point" of the trap, $I = [x, x + x_1]$ The probability that the particle is still inside I after time t is obviously (cf section 3 1 1 2)

$$\rho_I(t) = \int_I \mathrm{d}y\, P(y, t|x, 0) \tag{3 84}$$

The probability of leaving I between t and $t + \mathrm{d}t$ is simply

$$-(\partial/\partial t)\rho_I(t)\,\mathrm{d}t \tag{3 85}$$

Note that it is very small for $t \leq \tau_1$ The mean exit time [mean is here over different particles or thermal histories, but for a given configuration of $F(x)$] is then

$$\bar{\tau}(x) = -\int_0^\infty \mathrm{d}t\, t\, \frac{\mathrm{d}\rho_I}{\partial t} \equiv \int_I \mathrm{d}y\, P(y, x, E = 0) \tag{3 86}$$

Taking $I = [x, x + \mathrm{d}x]$, one may thus define the local "sojourn time" $\bar{\tau}(x)$ to be proportional to $P(x, x, E = 0)$ For physical reasons the proportionality constant is chosen to be x_1

From (3 7), it is easy to see that $\bar{\tau}(x)$ satisfies the following equation

$$\frac{\mathrm{d}\hat{\tau}}{\mathrm{d}\hat{x}} = -2 + 2\mu\, \frac{F(x)}{F_0}\, \hat{\tau}\,, \quad \hat{\tau} = \frac{\bar{\tau}}{\tau_1}\,, \quad \hat{x} = \frac{x}{x_1}\,, \tag{3 87}$$

which is a "Langevin equation" for $\hat{\tau}$ to which one naturally associates a Fokker–Planck equation*) for the probability $\psi(\hat{\tau}, \hat{x})$,

$$\frac{\partial}{\partial \hat{x}}\, \psi(\hat{\tau}, \hat{x}) = 2\, \frac{\partial}{\partial \hat{\tau}} \left(\hat{\tau}\, \frac{\partial}{\partial \hat{\tau}}\, \hat{\tau}\psi + (1 - \mu\hat{\tau})\psi \right) \tag{3 88}$$

The normalizable stationary distribution $\psi(\hat{\tau}, \hat{x} = -\infty)$ reads

$$\psi(\hat{\tau}) = \frac{1}{\Gamma(\mu)\hat{\tau}^{1+\mu}}\, \mathrm{e}^{-1/\hat{\tau}}\,, \tag{3 89}$$

which exhibits the expected power law decay for large $\hat{\tau}$ The moments of $\bar{\tau}$ then read

$$\langle \bar{\tau}^n \rangle = \tau_1^n\, \frac{\Gamma(\mu - n)}{\Gamma(\mu)}\,, \quad n < \mu\,, \qquad \langle \bar{\tau}^n \rangle = +\infty\,, \quad n > \mu \tag{3 90}$$

Note that this expression can also be directly computed from (3 7), using the fact that $F(x)$ is Gaussian distributed

*) Note that the correct prescription to be used (Stratonovich's) is fixed by working on a lattice

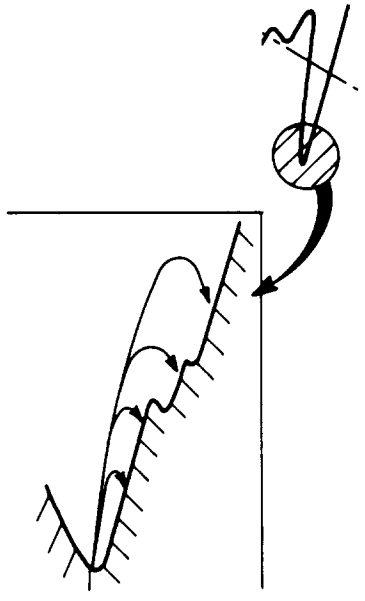


Fig 3 5 Detail of a trapping region and typical succession of trials leading to escape, which generate the "Kesten" variable $z_1 + z_1 z_2 +$

Remarks

(a) Expression (3 7) for $\bar{\tau}(x)$ has a simple physical interpretation if one writes

$$\bar{\tau}(x) = \tau_1\left[1 + \exp\left(\frac{U(x) - U(x+a)}{xT}\right) + \exp\left(\frac{U(x) - U(x+2a)}{xT}\right) + \quad\right], \tag{3 91}$$

which expresses the fact that – in a region of large fluctuations – the particle will first make a hop to $x + a$ but then will roll downward until it has sufficient thermal energy to reach $x + 2a$, etc , until it finally reaches a crest beyond which the mean bias takes over (see fig 3 5) The typical time needed to exit the trap is the sum of the times corresponding to the intermediate steps $\bar{\tau}(x)$ is thus of the form $\bar{\tau} = z_1 + z_1 z_2 + z_1 z_2 z_3 +$, with $z_i \propto e^{-aF_i/xT}$ Such random variables have been extensively studied in [Kes73, dCa85, Der83d], for large $\bar{\tau}$, one can show that the resulting distribution of $\bar{\tau}$ decays as $\bar{\tau}^{-(1+\mu)}$ with μ defined by the equation $\langle z^{\mu} \rangle = 1$, which provides a justification of (3 83) [Indeed, ψ satisfies the following integral equation

$$\psi(\bar{\tau}) = \int d\bar{\tau}'\, \psi(\bar{\tau}') \int dz\, \rho(z)\delta[\tau - z(1 + \bar{\tau}')] = \int dz\, \frac{\rho(z)}{z}\, \psi(\bar{\tau}/z - 1)$$

Assuming an algebraic $\psi(\bar{\tau})$ for large $\bar{\tau}$ immediately leads to $\langle z^{\mu} \rangle = 1$] The case studied here (ln z Gaussian) is remarkable since it leads to an explicit solution (in the limit $a \to 0$) of the above integral equation, (while the general case is quite complex [dCa85])

(b) An interesting issue of the above analysis is a more quantitative description of the transient regimes, already discussed on physical grounds in section 3 3 1 3 This amounts to studying how the distribution $\psi(\hat{\tau}, \hat{x})$ approaches its large-scale limit $\psi(\hat{\tau})$, and requires the computation of the spectrum of eigenvalues of the Fokker–Planck operator (3 88) This has been done in [Bou89a] and the result of this study is that for $\mu < 1$ the spectrum of (3 88) only has a continuous part, and that

$$\langle \bar{\tau}(0)\bar{\tau}(x) \rangle \sim x^{-3/2}\, e^{-\mu^2 x/2x_1}$$

As foreseen in sections 3 3 1 1 and 3 3 1 3, it is thus the characteristic length $x_0 = x_1/\mu^2$ which governs the approach to the asymptotic regime in this phase The corresponding time to reach this scale is indeed $\tau_1\, e^{1/\mu}$, since for $x < x_0$ the diffusion follows $x^2 \sim x_1 (\ln t)^4$ For $\mu > 2$, the correlation function of the τ's is easily obtained from (3 7) and (3 86) The result has in fact already been stated in (3 44) and reads, when specialized to the model at hand,

$$\langle \bar{\tau}(0)\bar{\tau}(x) \rangle - \langle \bar{\tau}(0) \rangle \langle \bar{\tau}(x) \rangle = \frac{\tau_1^2}{(\mu - 1)(\mu - 2)}\, e^{-2(\mu-1)x/x_1}$$

Thus it is seen that the effective traps are correlated, with a correlation length $x_1/2(\mu - 1)$

For a more general asymmetric model, these conclusions are qualitatively unchanged, with $W_{i,i+1}/W_{i+1,i}$ playing the role of the variable $\exp(-aF_i/T)$ on each link

3.3 4 Phase diagram and asymptotic probability distribution for non-zero bias

The previous section justifies the idea that, when $F_0 \neq 0$, the physics of the random force model is well captured, in the long-time limit, by a *directed walk among traps with a broad release time distribution* [Bou87c, Geo88, Bou89a] This simpler model can be solved exactly [Dou87] (section 3.3 4 1) This equivalence can only be approximate since it neglects correlations between trapping times (no steps backward), and its status will be discussed in section 3 3.4.2 It leads, however, to the correct diffusion behaviour and average limit distributions of the position, the form of which is independent of the details of the correlations at short distance Such an idea was also expressed in [Ber86] on the basis of a decimation approach

3 3 4.1. The directed walk with a broad $\psi(W)$ Let us consider the directed walk on a lattice of spacing ξ, described by the master equation

$$dP_n/dt = W_n P_{n-1} - W_{n+1} P_n \,, \tag{3 92}$$

where the W_n are independently distributed according to a broad distribution $\psi(W)$ which, for small W, behaves as

$$\psi(W) \sim A\tau_\xi^\mu W^{\mu-1} \tag{3 93}$$

The waiting time distribution on each site is simply $W_n \exp(-W_n t)$, and its mean, equal to $\tau_n = 1/W_n$, has a broad distribution behaving as $\tau_n^{-(1+\mu)}$ for large τ_n. The thermal average of the first passage time at site x thus reads

$$\overline{t(x)} = \sum_{i=1}^{x/\xi} W_i^{-1} \tag{3 94}$$

When the W_i^{-1} are broadly distributed, the results of chapter 1 (section 1 2) thus allow one to guess the diffusion behaviour

– If $\mu < 1$, the sum $\overline{t(x)}$ grows as $x^{1/\mu}$, and hence the position will behave typically as t^μ, i.e., *more slowly* than t ("creep phase")

– If $1 < \mu < 2$, the sum $\overline{t(x)}$ grows as x but the fluctuations are of the order of $x^{1/\mu}$, thus the position will be *typically* of order $Vt \pm t^{1/\mu}$ ("anomalous dispersion").

– If $\mu > 2$, the fluctuations recover a "normal" character and x is typically of order $Vt \pm \sqrt{Dt}$, with V and D finite

The interest of the directed walk is that much more precise statements can be made, in particular about the diffusion fronts. Indeed, the general formula (3.29) boils down, in the directed case, to

$$\xi P(x = n\xi; E) = \frac{1}{E + W_n} \prod_{i=0}^{n-1} \frac{W_i}{E + W_i} \tag{3 95}$$

Hence, the asymptotic form of the average diffusion front is obtained as the inverse Laplace transform

of

$$\left\langle \frac{1}{E+W} \right\rangle \left\langle \frac{W}{E+W} \right\rangle^{n} \underset{n\to\infty}{\sim} \exp\left(n \ln \left\langle \frac{W}{E+W} \right\rangle \right).$$

For the three cases above, this leads to the following results

(i) $0 < \mu < 1$ The small-E expansion of $\langle W/(E+W) \rangle$ reads

$$\langle W/(E+W) \rangle = 1 - A(E\tau_\xi)^\mu \pi / \sin \pi\mu \, .$$

Thus the Laplace transform of $\langle P(x,t) \rangle$ behaves as

$$\exp\left(- \frac{C}{\cos(\pi\mu/2)} n\xi (E\tau_\xi)^\mu \right), \quad C = A \frac{\pi}{2\sin(\pi\mu/2)} \, , \tag{3.96}$$

which shows that, in the scaling region, the average diffusion front can be expressed in terms of a Levy law of order μ, characterized by the two parameters $\beta = +1$ and C (given above),

$$\xi \langle P(x,t) \rangle \to \left(\frac{\tau_\xi}{t} \right)^\mu f_\mu \left(\frac{\tau_\xi^\mu}{\xi} \frac{x}{t^\mu} \right), \quad f_\mu(u) = \frac{1}{\mu} u^{-(1+1/\mu)} L^{(C)}_{\mu,+1}(u^{-1/\mu}) \tag{3.97}$$

The shape of the diffusion front (3.97) is depicted in fig. 3.6. The scaling function $f_\mu(u)$ is concentrated

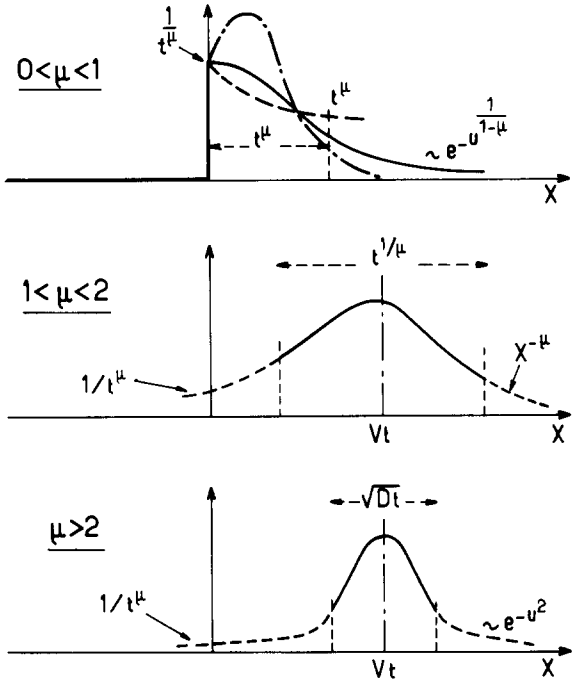


Fig 3 6 Scaling functions for the average diffusion front in the three cases $0 < \mu < 1$, $1 < \mu < 2$, and $2 < \mu$, plotted versus the associated rescaled variable The three curves in the top figure correspond to $0 < \mu < 1/2$, $\mu = 1/2$ and $1/2 < \mu < 1$, respectively

on $[0, +\infty[$ as a result of the bias One can compute the large-time behaviour of the *average* $\langle \overline{x(t)} \rangle$ in this directed model (making use of appendix B),

$$\langle \overline{x(t)} \rangle \underset{t\to\infty}{\sim} \xi \left(\frac{\mu \Gamma(\mu) C}{\cos(\pi\mu/2)} \right)^{-1} \left(\frac{t}{\tau_\xi} \right)^{\mu} \tag{3 98}$$

Remarks

* Equation (3 97) is a priori only correct in the scaling region ($x \to \infty$, $t \to \infty$, x/t^μ finite). However, the value of $\langle P(x=0, t) \rangle$ obtained from (3 97) reads $A\Gamma(\mu)(t/\tau_\xi)^{-\mu}$, which coincides with the *exact* result deduced from (3 93),

$$\xi \langle P(0, t) \rangle = \int_0^\infty \psi(W) \, \mathrm{e}^{-Wt} \, \mathrm{d}W \sim A\Gamma(\mu)(\tau_\xi/t)^\mu \tag{3 99}$$

* For a *fixed environment*, it is clear that*)

$$\overline{x(t)} = \xi_t(\omega) t^\mu \,, \tag{3 100}$$

where $\xi_t(\omega)$ is a random function of the environment (with a variance of order one). Thus $\overline{x(t)}$ is *not* asymptotically a sample independent quantity, and the diffusion front $P(x, t)$ cannot obey a generalized CLT when expressed in the scaled variable x/t^μ (whether a CLT holds when rescaled as a function of $x/\bar{x}$ is an open question). Analogously, one expects

$$\overline{x^2(t)} - \overline{x(t)}^2 \sim \xi'_t t^{2\mu} \tag{3 101}$$

(ii) $1 < \mu < 2$ The expansion of $\langle W/(E+W) \rangle$ now reads

$$1 - \langle 1/W \rangle E + A \frac{\pi}{\sin \pi\mu} (E\tau_\xi)^\mu + \cdots ,$$

leading to the limiting form of the diffusion front

$$\xi \langle P(x, t) \rangle \to \frac{B}{t^{1/\mu}} L_\mu \left(-B \frac{x - Vt}{t^{1/\mu}} \right) ,$$

$$V = \xi \langle 1/W \rangle^{-1} \,, \quad B = \frac{\xi^{1/\mu}}{\tau_\xi} V^{-(1+1/\mu)} \,, \tag{3 102}$$

where L_μ is a Levy law of zero mean, characterized by

$$\beta = +1 \,, \qquad C = A \frac{\pi}{2 \sin(\pi\mu/2)}$$

Its shape is drawn in fig 3.6, the dispersion with respect to the mean position is anomalous in this phase (as foreseen above) and sample dependent: $\overline{x^2(t)} - \overline{x(t)}^2 \sim \xi_t t^{2/\mu}$ (only the leading term Vt in the thermal average $x(t)$ is sample independent). This anomalous dispersion reflects the trapping of a fraction of walkers in very deep local traps Note, however (see fig. 3.6 and appendix B), that in this phase

*)The probability distribution of $\xi_t(\omega)$ in eq (3 100) has recently been characterized through its moments, see [Asl90a]

$\langle P(x,t)\rangle$ has a tail decaying as $x^{-\mu}$ for large x. This allows one to show that $\langle \overline{x^2(t)} - \overline{x(t)}^2 \rangle \sim t^{3-\mu}$, in accordance with a recent result of [Asl90b]. Hence the average and the typical position fluctuations behave differently in this phase.

(iii) $\mu > 2$. $\langle W^{-1}\rangle$ and $\langle W^{-2}\rangle$ being finite, the expansion of $\langle W/(E+W)\rangle$ is simply

$$1 - \langle 1/W\rangle E + E^2\langle 1/W^2\rangle + \cdots ,$$

and thus $\langle P(x,t)\rangle$ recovers a Gaussian limiting form,

$$\langle P(x,t)\rangle \to (4\pi D_{\mathrm{av}} t)^{-1/2} \exp[-(x-Vt)^2/4D_{\mathrm{av}}t] , \tag{3.103a}$$

where V has the same expression as above and D_{av} reads, for this directed model,

$$\frac{D_{\mathrm{av}}}{\xi^2} = \frac{1}{2}\left\langle \frac{1}{W}\right\rangle^{-3}\left(2\left\langle \frac{1}{W^2}\right\rangle - \left\langle \frac{1}{W}\right\rangle^2\right). \tag{3.103b}$$

Thus when $\mu > 2$, the first two cumulants of the position have a normal behaviour as a function of time. One should, however, keep in mind that the nth cumulant of the position remains anomalous as long as $\mu < n$ (since $\langle W^{-n}\rangle$ diverges). In other words, the limiting form (3 103a) of $\langle P(x,t)\rangle$ only holds in a "scaling region" in which $x - Vt$ is of the order of $\pm(D_{\mathrm{av}}t)^{1/2}$; outside this region $\langle P(x,t)\rangle$ has *algebraic tails*, which are responsible for the anomalous behaviour of the higher-order cumulants.

Remarks

(i) The physical meaning of the vanishing of the velocity for $\mu < 1$ is that the current J_L through the sample goes to zero with increasing sample size,

$$J_L \sim L^{1-1/\mu} , \quad L \to \infty$$

while it goes to a constant for $\mu > 1$

(ii) It is important to notice that the distribution of first passage times is itself a broad distribution, decreasing as $t^{-(1+\mu)}$ (for large t).

3.3.4.2. Results and conjectures for the random-force model. As is made clear by the arguments and calculations of section 3.3 3, the directed walk model with a broad distribution of hopping rates $\psi(W)$ does retain the essential features of the continuous random-force model, and more generally of all asymmetric one-dimensional models with bond disorder [Bou89a] [the exponent μ being given in general by condition (3.83)]. In particular, one expects the same diffusion behaviour and the same *analytic expressions* (3.97), (3.102) and (3.103) for the asymptotic form of the average diffusion front $\langle P(x,t)\rangle$. This has been proven rigorously in [Kes75] (although the proof concerns a discrete time model). However, the constant *parameters* entering these expressions (namely C, ξ, τ_ξ in the phase $\mu < 1$; C, B, V for $1 < \mu < 2$; and V, D_{av} for $\mu > 2$) obviously depend on the specific model considered (and were not predicted in [Kes75]). The general expressions of V and D_{av} (and of the diffusion constant D) have been established above (section 3.1), but no such general expressions exist for the anomalous phases $\mu < 2$ One should in fact note that once the units of space and time have been

chosen, only the single unknown parameter C is required to characterize fully the average front for $\mu < 2$

In the following, we review some conjectures that have been made [Bou89a] on the value of this parameter for the continuous random-force model (3 64). The idea is to insist on using the directed walk effective picture of the diffusion process at large scales in a *quantitative* manner Namely, one asks whether it is possible to find a lattice spacing ξ, and a value of A such that the directed model of section 3.3.4.1 reproduces quantitatively most of the diffusion properties of the random-force model (the time constant τ_ξ can be chosen to be equal to the natural time scale τ_1 without loss of generality) First, start from the directed model with a very small lattice spacing $a \to 0$, with the choice

$$\psi(W) = a\rho_a(W)\,, \tag{3 104}$$

where $\rho_a(E)$ is the average density of relaxation times of the random force model discretized on the same lattice (the continuous limit $a \to 0$ of ρ_a will be obtained in closed form in section 3.5). Then, by construction, $\langle P(0,t)\rangle$ coincides for both models for all t, since

$$\langle P(0,t)\rangle_{\mathrm{dir}} = \int_0^\infty \psi(W)\,\mathrm{e}^{-Wt}\,\mathrm{d}W = a\int_0^\infty \rho_a(W)\,\mathrm{e}^{-Wt}\,\mathrm{d}W \equiv a\langle P(0,t)\rangle \tag{3 105}$$

Let us first concentrate on the phase $\mu > 2$. Remarkably, this implies – owing to the small-E expansion (3.42) – that the continuous directed model has the same velocity V and diffusion constant D as the random-force model However, D_{av} does not coincide for the two models since $D/D_{\mathrm{av}} = 1 - 1/\mu$ for the random-force model while $D_{\mathrm{av}} = 2D - aV/2 \to 2D$ in the continuum limit of the directed model The correct D_{av} would be found for a lattice spacing ξ equal to $x_1/(\mu - 1)$, which is precisely twice the correlation length of the trapping times $\bar{\tau}(x)$, as found in section 3.3.3.2 With this choice of ξ, one can define a new directed model by grouping together ξ/a sites of the first one, in the limit $a \to 0$ Thus we have obtained the desired equivalence, and it is physically very sensible that the "effective lattice spacing" should be of the order of the correlation length between traps Remarkably, in the anomalous phase $0 < \mu < 2$, the value of C is found *not to depend on the choice of* ξ, and one can simply work with $\xi = a$. Hence one is prompted to identify A with $\lim_{E\to 0}(E\tau)^{1-\mu}\rho(E)$, which is calculated below (section 3 3.5). This leads to the conjecture that the parameters characterizing the average diffusion front of the random-force model are those summarized in table 3.1 [Bou89a] The same results were recently obtained in [Asl90b], where the identification with a directed walk is more carefully discussed It follows that in the phase $\mu < 1$, the position obeys, for large time, the law

$$\frac{\langle\overline{x(t)}\rangle}{x_1} \sim \frac{2^{\mu-1}\Gamma(\mu)\sin\pi\mu}{\pi\mu^2}\left(\frac{t}{\tau_1}\right)^\mu\,, \tag{3 106}$$

which, for small μ, becomes

$$\frac{\langle\overline{x(t)}\rangle}{x_1} \sim \frac{1}{2\mu^2}\left(\frac{t}{\tau_1}\right)^\mu$$

This μ^{-2} dependence of the prefactor for small μ is very important physically; indeed, one expects – as discussed in section 3 3.2 – that for times $t < \tau_0 = \tau_1\,\mathrm{e}^{1/\mu}$, the behaviour of the particle is insensitive to the bias, i e , $x \sim x_1 \ln^2 t/\tau_1$ The point is that this expression correctly crosses $\mu^{-2}(t/\tau_1)^\mu$ for $t = \tau_0$ and $x = x_0 = x_1/\mu^2$

Table 3 1
Scaling forms of $\langle P(x,t)\rangle$ and diffusion behaviour of the random-force model, as a function of the parameter $\mu = 2F_0T/\sigma$, $x_1 = 4T^2/\sigma$, $\tau_1 = x_1^2/2D_0$ $(D_0 = T/\gamma)$

$\mu = 0$	$\dfrac{1}{x_1 \ln^2(t/\tau_1)}\, f_0\!\left(\dfrac{x}{x_1 \ln^2(t/\tau_1)}\right)$
	$f_0(u) = \dfrac{8}{\pi}\sum_{k=0}^{\infty} \dfrac{(-1)^k}{2k+1} \exp[-\tfrac{1}{2}\pi^2(2k+1)^2\lvert u\rvert]$
	$\bar{x}_t/x_1 \sim \xi_t(\omega) \ln^2(t/\tau_1)$
$0<\mu<1$	$\dfrac{1}{x_1}\left(\dfrac{\tau_1}{t}\right)^{\mu} f_\mu\!\left(\dfrac{\tau_1^\mu}{x_1}\dfrac{x}{t^\mu}\right)$
	$f_\mu(u) = \dfrac{1}{\mu}\, u^{-(1+1/\mu)} L_{\mu+1}^{(C)}(u^{-1/\mu})$, $\quad C \overset{?}{=} \dfrac{\pi\mu}{2^\mu \Gamma(\mu)^2 \sin(\pi\mu/2)}$
	$\bar{x}_t/x_1 \sim \xi_t(\omega)(t/\tau_1)^\mu$
$1<\mu<2$	$\dfrac{B}{x_1 t^{1/\mu}} L_{\mu+1}^{(C)}\!\left(-B\dfrac{x-Vt}{t^{1/\mu}}\right)$
	$\dfrac{V}{V_0} = 1 - \dfrac{1}{\mu}$, $\quad B = \dfrac{x_1^{1/\mu}}{\tau_1} V^{-(1+1/\mu)}$, $\quad C \overset{?}{=} \dfrac{\pi\mu}{2^\mu \Gamma(\mu)^2 \sin(\pi\mu/2)}$
	$\bar{x}_t \sim Vt$, $\quad \overline{x_t^2} - \bar{x}_t^2 \sim x_1^2 \xi_t(\omega)(t/\tau_1)^{2/\mu}$
$\mu>2$	$\dfrac{1}{\sqrt{4\pi D_{\mathrm{av}} t}} \exp[-(x-Vt)^2/4D_{\mathrm{av}}t]$
	$\dfrac{V}{V_0} = 1 - \dfrac{1}{\mu}$, $\quad \dfrac{D_{\mathrm{av}}}{D_0} = \dfrac{\mu}{\mu-2}$
	$\bar{x}_t \sim Vt$, $\quad \overline{x_t^2} - \bar{x}_t^2 \sim 2Dt$, $\quad \dfrac{D}{D_0} = \dfrac{\mu-1}{\mu-2}$

Remark. As μ^{-2} is a decreasing function of the bias, one may point out a possible paradox with (3.106): it appears that $\langle \overline{x(t)} \rangle$ could be, due the prefactor, a decreasing function of the bias for sufficiently small times. However, it is easy to check that this is only the case for $t < \tau_0$, where the diffusion is logarithmic

3 3.5. Relaxation and probability of presence at the origin

3 3.5 1 Exact calculation of $\langle P(x,t|x,0)\rangle$ for the random-force model. A very important physical quantity, which may be related to the *relaxation properties* of the model, is the probability of presence of the particle *on its starting site*, $P(x,t|x,0)$. Its average over disorder, as we shall now briefly show, may be computed *exactly* [Bou87a] for the model (3.64) (for a more detailed presentation of the different calculation methods see [Bou89a, Asl90d])

The starting point of the method is to notice that $P(x,t|x_0,0)$ can be decomposed into the eigenstates of a Schrodinger equation [G5] (see also [Sch86, Tos88, Tei89]),

$$P(x,t|x_0,0) = \exp\!\left(\int_{x_0}^{x} \frac{F(z)}{2D_0}\,\mathrm{d}z\right) \sum_n \phi_n(x)\phi_n(x_0)\,\mathrm{e}^{-E_n t}\,.$$

Note that, using the closure relation one has indeed

$$H_S\phi_n = E_n\phi_n\,, \quad H_S = -D_0\frac{d^2}{dx^2} + \frac{F^2(x)}{4D_0} + \frac{1}{2}\frac{dF}{dx} \tag{3 107}$$

H_S can be written as $H_S = S^+S$, where

$$S = -1\sqrt{D_0}\,\frac{d}{dx} + \frac{1}{2\sqrt{D_0}}\,F(x) \tag{3.108}$$

This implies that the spectrum of H_S is positive, despite the fact that the "potential" $F^2/4D_0 + F'/2$ is not bounded from below From a physical point of view, it means that the relaxation times of the diffusion process ($\tau_n = E_n^{-1}$) are all positive, as expected

From (3.107), one obtains, for all times t,

$$P(x_0, t|x_0, 0) = \sum_n \phi_n^2(x_0)\,e^{-E_n t}\,,$$

or, introducing the density of states $\rho(E) = \lim_{L\to\infty} L^{-1}\,\Sigma_n\,\delta(E - E_n)$,

$$\langle P(x_0, t|x_0, 0)\rangle = \lim_{L\to\infty} L^{-1}\int_{-L/2}^{L/2}\langle P(x_0, t|x_0, 0)\rangle\,dx_0 = \int_0^\infty dE\,\rho(E)\,e^{-Et} \tag{3 109}$$

Thus, the density of states of H_S directly yields the average of the probability of presence at the origin Many methods may be used to obtain $\rho(E)$ The most natural one for a one-dimensional disordered Hamiltonian is the Dyson–Schmidt method [Dys53, Sch57, Lie66] (e.g. used in [Fri60, Hal65, Der84b] to obtain the energy spectrum in a white noise potential) It relies on the well-known "node counting" theorem of one-dimensional quantum mechanics, stating that the number of zeros of $\phi_E(x)$ per unit length is equal to the number of states below energy E, $N(E) = \int_0^E \rho(E')\,dE'$ Writing (3.107) in terms of a new variable u (this calculation is presented with a normalization $\sigma = 4$, $D_0 = 1$),

$$u(x) = [\ln\phi(x)]' - \tfrac{1}{2}F(x)\,,$$

one obtains a "Langevin equation" for u, for which x plays the role of the time,

$$\frac{du}{dx} = -(u^2 + 2\mu u + E) - 2u\xi(x) \tag{3.110}$$

[with $\xi(x) = \frac{1}{2}F(x) - \mu$ playing the role of the thermal noise]. Counting the number of zeros of $\phi_E(x)$ amounts to counting the divergences of u. The quantity $N(E)$ we want to calculate is thus exactly (in the large-L limit) the average current j which counts the escape "frequency" of u j is related to the "asymptotic" ($x\to\infty$) probability density $Q(u)$ associated with the stochastic equation (3.110) by $N(E) = j = \lim_{u\to\infty} Q(u)u^2$. One thus writes a "Fokker–Planck" equation for $Q(u, x)$, which in this case reads (see [Bou89a] for a precise discussion of the ambiguities associated with the stochastic calculus prescription to be used – which here turns out to be Stratonovich's)

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial u}\left(2u\,\frac{\partial}{\partial u}\,(uQ) + (u^2 + 2\mu u + E)Q\right)$$

Once the stationary solution ($\partial Q/\partial x = 0$) is known, the current $j(\mu, E)$ is obtained All these calculations can be performed explicitly [Bou87a, 89a] and one finally finds

$$j = \frac{2}{\pi^2}\left[J_\mu^2(\sqrt{E}) + N_\mu^2(\sqrt{E})\right]^{-1},$$

where J_μ and N_μ are Bessel functions of order μ. Reintroducing all the original physical quantities, one obtains

$$N(E) = \frac{\sigma}{2\pi^2 D_0^2}\left[J_\mu^2(\sqrt{\mathscr{E}}) + N_\mu^2(\sqrt{\mathscr{E}})\right], \quad \mathscr{E} = \frac{16 D_0^3 E}{\sigma^2} = 2E\tau_1 \tag{3.111}$$

The limiting behaviour of $N(E)$ for $E \to 0$ and $E \to \infty$ is of special interest. First, for all μ, $N(E) \sim (E/D_0)^{1/2}/\pi$ for $E \to \infty$, i.e., one recovers the free-particle spectrum, this is expected since it corresponds to the fact that at short times the particle diffuses freely ($x < x_1$). On the contrary, the low-energy behaviour of $N(E)$ strongly depends on μ,

$$x_1 N(E) \underset{E\to 0}{\simeq} \begin{cases} 2\ln^{-2} E, & \mu = 0, \\ (E\tau_1)^\mu / 2^{\mu-1}\Gamma^2(\mu), & \mu > 0. \end{cases} \tag{3 112}$$

These results allow one to determine exactly $\langle P(x_0, t|x_0, 0)\rangle$ through eq. (3.109), a log–log plot of which is displayed in fig. 3.7 for different values of μ. In particular, the long-time behaviour reads

$$x_1\langle P(x_0, t|x_0, 0)\rangle \sim \begin{cases} 2\ln^{-2} t, & \mu = 0, \\ [\mu/2^{\mu-1}\Gamma(\mu)](\tau_1/t)^\mu, & \mu > 0, \end{cases}$$

and thus fully confirms the physical analysis of section 3.3.3

More information on this problem can be found in [Bou89a], where, for example, the "replica method" is used to calculate $\rho(E)$ and the localization length $\lambda(E)$ associated with H_S, through its average Green function,

$$\langle P(x, x, -E + i0^+)\rangle = -(d/dE)\lambda^{-1}(E) + i\pi\rho(E).$$

3 3 5.2 Sample to sample fluctuations. For the disordered models considered here, $P(x, t|x, 0)$ *strongly fluctuates* from site to site [Del89, Bou89a] (or from sample to sample). This is most easily seen on the directed model of section 3.3.4.1, for which

$$P_n(t|n, 0) = e^{-W_n t}. \tag{3 113}$$

One may thus easily estimate $\langle [P(x, t|x, 0)]^q\rangle$,

$$\langle [P(x, t|x, 0)]^q\rangle \sim t^{-\mu} \neq \langle P(x, t|x, 0)\rangle^q \sim t^{-\mu q}, \tag{3.114}$$

which clearly displays the non-self-averaging nature of $P(x, t|x, 0)$. In particular

$$-\langle \ln P(x, t|x, 0)\rangle = \langle W\rangle t.$$

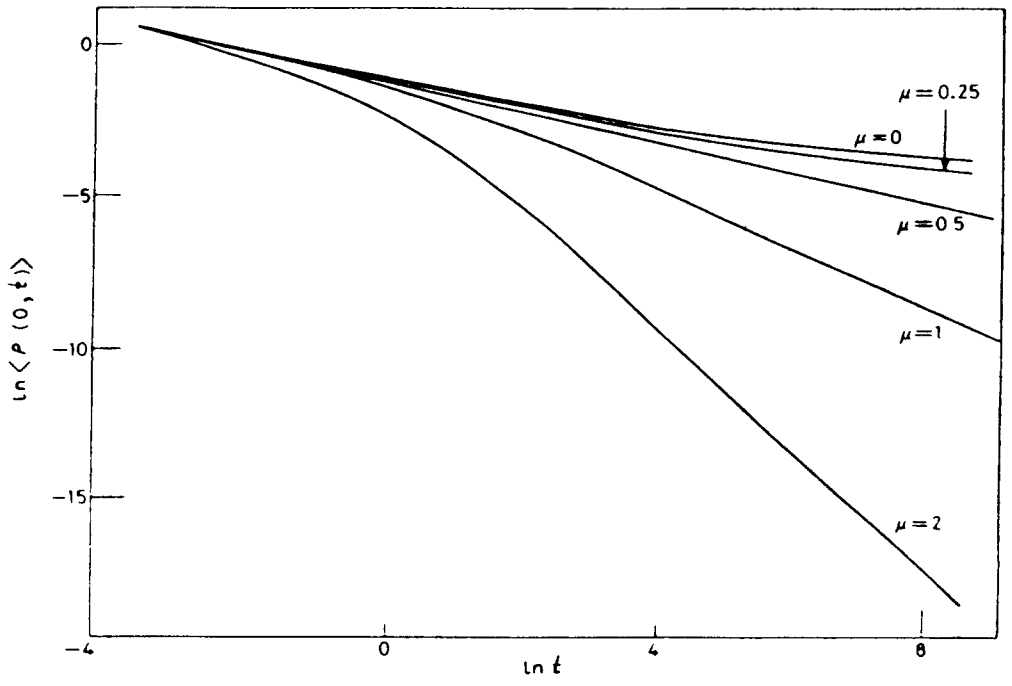


Fig 3 7 Average probability of return to the origin $\langle P(0,t)\rangle$ as a function of time t, in a log-log plot, and for different values of μ Note the cross-over from free diffusion (for short times) to disorder-dominated diffusion at large times

$P(x,t|x,0)$ thus decays exponentially for any *given* sample, but its *average* $\langle P(x,t|x,0)\rangle$ decays as a power law (see also [Del89])

Contrarily to many random systems the average $\langle P(x,t|x,0)\rangle$ has nevertheless a direct physical interpretation: If one starts with a particle population uniformly spread in space, it represents the *fraction of particles still at their initial point after time t*

3 3 6. Summary and discussion

Let us summarize some important physical consequences of the results obtained throughout section 3 3. Clearly, all the interesting non-Brownian features arise because of the induced broad distribution of trapping times, resulting from the combination of exponentially rare events contributing exponentially to the activation time

Such an exponential distribution (Poisson distribution) of energy barriers is certainly much more general (see, e.g., [Ram85, Tam87]). A trapping time distribution with a slow power law decay thus is not an exceedingly exotic possibility in physical systems. The following features are naturally associated with such broad distributions $\psi(\tau) \sim \tau^{-(1+\mu)}$

– diffusion fronts (and distribution of exit times) which involve well-characterized *Levy stable laws*;

– a succession of "phase transitions" (where different physical quantities become singular) as μ is varied – corresponding to the divergence of the successive moments of $\psi(\tau)$; very important is the transition between a "creep" phase, where the velocity is zero, and a "flow" phase in which a current may establish, note that in this case the velocity goes continuously to zero, which is at variance with other first-order depinning transitions,

– slow, algebraic relaxation of the system at long times; or equivalently, enhanced noise power spectrum at low frequencies.

The model considered above has also direct physical applications, either because the problem is truly one dimensional, or because it can be modelled as such in certain circumstances. We describe some of them in the next section.

3 3 7 *Application to physical problems*

3 3 7 1. Random-field Ising model. A magnetic domain wall, *if it has a large surface tension*, evolves in a disordered material in much the same way as a point particle (the centre of mass of the domain) under random forces, and may be modelled [Nee42] as a domain wall in a one-dimensional random-field Ising model, a problem which is indeed described by the model considered here. (For recent work on the dynamics of the random-field Ising model, see [Bru84, Gri84, May84, Nat88].)

Consider the one-dimensional Ising model in the presence of a random magnetic field; its Hamiltonian reads

$$\mathcal{H} = -J \sum_i S_i S_{i+1} - \sum_i h_i S_i , \tag{3.115}$$

with $S_i = \pm 1$ and

$$\langle h_i \rangle = h , \qquad \langle h_i h_j \rangle - h^2 = \sigma_h \delta_{ij} . \tag{3.116}$$

Then a *domain wall* between a region of up spins and down spins (see fig. 3 8) will evolve in the sample according to the following transition rates (assuming single spin flip dynamics):

$$W_{i\,i+1} = W_0 \, \mathrm{e}^{\Delta\mathcal{H}/2T} , \qquad W_{i+1,i} = W_0 \, \mathrm{e}^{-\Delta\mathcal{H}/2T} , \tag{3 117}$$

where $\Delta\mathcal{H}$ corresponds to the change in energy accompanying the spin flip $S_i \to -S_i$: $\Delta\mathcal{H} = 2h_i$. Those hopping rates thus precisely correspond to the model defined above, with, in this case,

$$\mu = hT/\sigma_h \tag{3.118}$$

Thus only if the external field h exceeds a certain critical value h_c will the domain wall acquire a finite velocity. For $h < h_c$, the domain wall "creeps" according to the law

$$\langle \bar{x}(t) \rangle / a \sim (T^2/\sigma_h)^{1-2\mu} (W_0 t)^{\mu} . \tag{3.119}$$

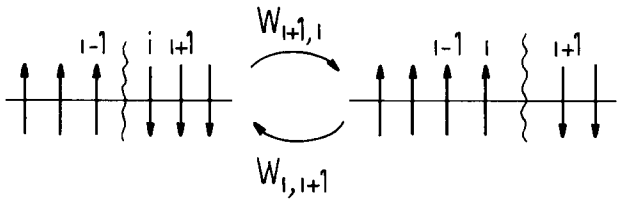


Fig 3 8 Domain wall in a one-dimensional random-field Ising model

This analysis allows us to discuss the following problem. Suppose that the sample has been prepared in the $S_i = -1$ state and that an external field $h > 0$ is suddenly applied. Then up regions will nucleate and expand according to the laws described above If $h \ll J$, the nucleation rate (corresponding to the creation of a + spin) is simply

$$\Gamma = W_0 \, e^{-J/T} \tag{3 120}$$

The dynamics of the whole system will thus exhibit two stages (see [Bou89g]):

First stage The system nucleates up spin regions up to a time t_c defined by

$$x \simeq a(W_0 t_c)^{\mu} (T^2/\sigma_h)^{1-2\mu} \quad (V t_c \text{ if } \mu > 1) , \tag{3 121a}$$

and

$$(x/a)\Gamma t_c \simeq 1 , \tag{3 121b}$$

or

$$W_0 t_c \sim \begin{cases} e^{J/T(1+\mu)} (T^2/\sigma_h)^{(2\mu-1)/(1+\mu)} , & \mu < 1 , \\ (V\Gamma)^{-1/2} , & \mu > 1 \end{cases} \tag{3 122}$$

t_c is simply the typical time for two growing "up" regions to meet, since eqs. (3.122) state that the nucleation probability within the distance spanned by the domain wall in a time t_c is of order 1. For $t \ll t_c$ and $\mu < 1$, one expects the mean magnetization to scale as $(\Gamma t)^{\mu} - 1$ (or $\Gamma V t^2 - 1$ for $\mu > 1$)

Second stage When t is of the order of t_c, M is nearly equal to its equilibrium value $M_{eq} = 1$ (in this discussion we assume $T \ll J$) and slowly relaxes towards it according to the law (see fig. 3 9)

$$|M - M_{eq}| \sim t^{-\mu}$$

This stage corresponds to the rare but difficult to overturn sequences of spins ($h_i < 0$ and large).

3 3 7 2 Configuration transition in disordered polymers De Gennes has suggested [dGe75] that the domain wall between the helix and coil phases of a "heteropolymer" - see fig. 3 10 (e g. a random

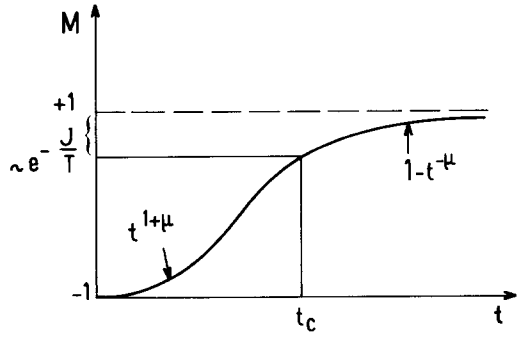


Fig 3 9 Evolution of the total magnetization as a function of time in a one-dimensional random-field Ising model prepared in a "down" state and with an "up" magnetic field suddenly switched on at $t = 0$

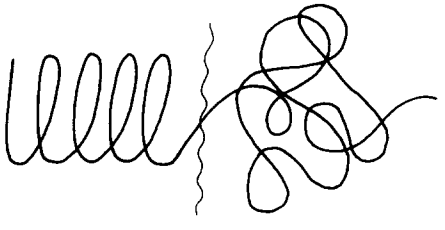


Fig 3 10 "Domain wall" between two configurations of a heteropolymer helix and coil "phases"

sequence of two species of monomers which do not have the same helix–coil transition temperature) would evolve quite similarly to the Ising domain wall described above. This is simply because the free energy variation $\Delta F(s)$ corresponding to a one-monomer displacement of the domain wall is a random function of the arc lengths, leading to transition rates quite similar to (3.118). The latent heat released by such a process thus resembles very much the magnetization discussed above Note that in the original treatment of de Gennes, the critical value $\mu = 1$ separating a creep ($V = 0$) from a flow phase appears, even if the diffusion law in the creep phase was incorrectly discussed.

3.3.7.3. Dislocation motion in disordered crystals The motion of a dislocation in a perfect crystal can be modelled as the dynamics of a string on an inclined washboard (see, e g., [Hir68]) Its downhill progression proceeds as follows: The string nucleates a kink–antikink pair (fig 3.11a), which are torn apart by the external stress, until they annihilate with the neighbouring nucleated pair (fig 3.11b), finally resulting in the translation of the full dislocation. The motion of the kink (or the antikink) is thus *one dimensional*, its progression results in an energy gain for the dislocation proportional to the distance travelled This is equivalent to saying that the kink's position x satisfies the Langevin equation (3.64) with $F(x) = F_0$, where $\eta(t)$ accounts for the thermal fluctuations.

If now the crystal contains foreign solute atoms randomly placed on the lattice, one must take into account the interaction energy between those solute atoms and the dislocation If one can neglect the motion of solute atoms (no "dynamical aging") and consider that they generate *quenched* disorder, one may argue [Pet71, Vin86] that the kink position follows eq (3.64), with F_0 proportional to the external stress and $\sigma = \langle F^2 \rangle - F_0^2$ related to the concentration of solute atoms and the strength of their interaction with the dislocation. This suggests that dislocation dynamics in disordered metals should reveal very rich and interesting peculiarities. In particular, the above discussion of the dynamics of a random-field Ising model may be partly transposed [Pet71, Vin86, Bou89g] One expects that the average translational velocity of the *whole dislocation* will behave as

$$V_{\mathrm{disl}} = a/t_{\mathrm{c}} , \tag{3 123}$$

where t_{c} is defined by eqs. (3.121) (Note that the whole discussion only makes sense for $W_0 t_{\mathrm{c}} \gg 1$, and that the kink energy J must be much larger than T, so that a "kink" is a well-defined object) The

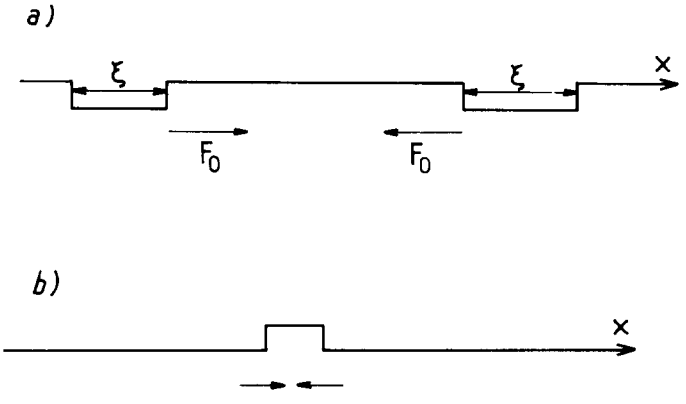


Fig 3 11 (a) Nucleation of a kink–antikink pair in a "washboard" potential (b) The kink and antikink get farther apart, corresponding to an overall translation of the dislocation

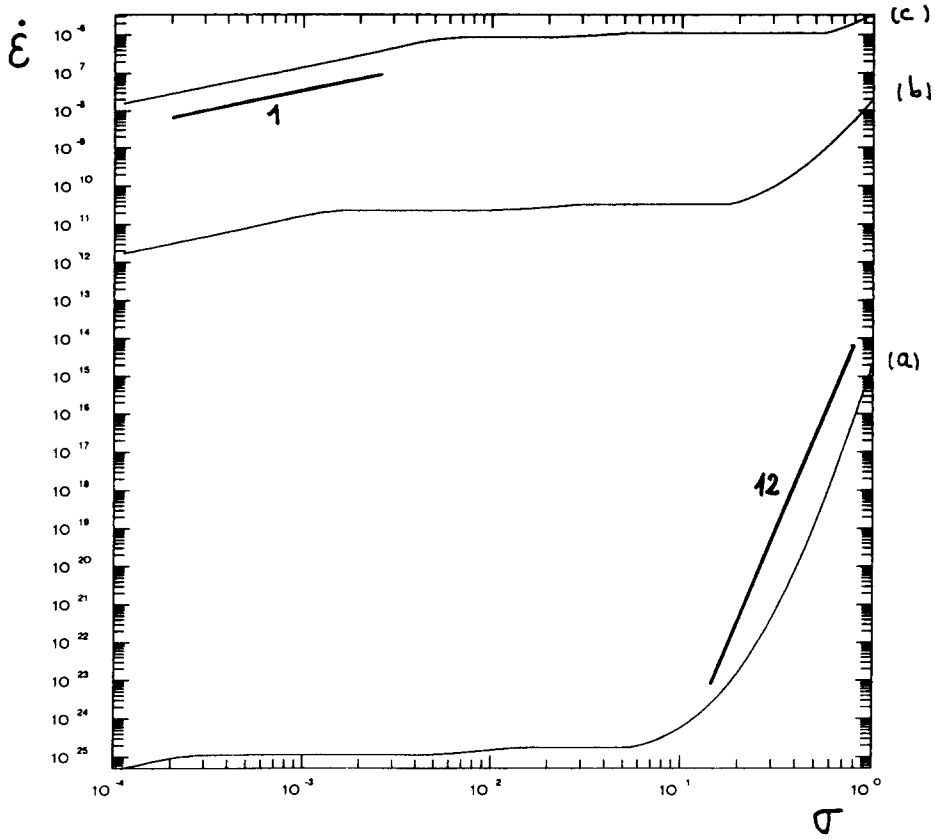


Fig 3 12 Typical stress–strain curve resulting from the creep motion of the kinks due to disorder, for three different temperatures (a) 150 K, (b) 300 K, (c) 450 K We have taken the kink–antikink energy equal to 10^4 K, and 1% impurities each causing a fluctuation of 1000 K in the kink's energy Note the linear regime for small stresses and high temperatures, and the non-linear regime for intermediate stresses and low temperatures, mimicking a power law $\varepsilon = \sigma^n$ with large n

resulting relation between the velocity of the dislocation (and hence in certain cases the rate of deformation of the sample [Hir68]) and the applied stress has the shape depicted in fig. 3 12 (see [Bou89g] for more details) The plateau region could explain a similar anomalous feature observed in gold-doped silicon

4. Anomalous diffusion in a field of random forces in more than one dimension

The diffusion behaviour of random barriers and random traps models has been investigated in chapter 2 It has been shown there that, in more than one dimension, anomalous diffusion only arises for the latter and, even in that case, requires an a priori broad distribution of local trapping times*) Various analytical techniques can be used to study the normal diffusion properties of these models in more than one dimension, mainly effective medium approximations (cf section 2 4.2.3) and systematic weak-disorder expansions The results of such an expansion for a general hopping model will be presented in section 4 2 3, and the reader is referred to [Der83a,b] for further information.

*) This holds true provided the hopping rates display no long-range correlations

In the present article, we rather wish to put the emphasis on *anomalous* diffusion *induced* on large scales by a local disorder In this respect, the most interesting model to be studied is the diffusion in a field of random forces ($\boldsymbol{F}(x)$) (type C model), a continuum description of which is the d-dimensional version of the Langevin equation*) of chapter 3,

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{\eta}(t)\,, \tag{4.1}$$

$$\overline{\eta_\mu(t)\eta_\nu(t')} = 2D_0\delta_{\mu\nu}\delta(t-t')\,, \tag{4 2}$$

to be supplemented with an appropriate description of the distribution of ($\boldsymbol{F}(X)$) It will be shown below that this model retains the essential physics of anomalous diffusion in the presence of (narrow) quenched disorder (and that, in particular, including short-range correlated randomness in the local diffusion constant $D(\boldsymbol{x})$ – the symmetric part of the hopping rates – does not lead to new effects)

It is the aim of this chapter to understand whether this model retains something of its rich one-dimensional phenomenology, and whether new behaviour can arise in $d>1$ An obvious statement in this respect is that the effect of quenched disorder tends to be weaker as the dimension is increased, since more and more paths connect two given positions. We shall mainly concentrate on the case of a zero average force $\langle \boldsymbol{F} \rangle = \boldsymbol{0}$ (the response to a small average bias will be investigated in section 4.3.3). Even in that case, the above question has been the subject of numerous (and somewhat controversial) investigations in the literature [Mar83, Obu83, Pal83, Luc83, Fis84, Aro84, Fis85, Kra85, 86a, Bou87b]

The original physical motivation [Mar83] was whether ultra-slow logarithmic diffusion "à la Sinai" can still arise in more than one dimension. The answer is, as we shall see, in the affirmative, but interestingly enough, turns out to depend on the type of geometrical constraints imposed on the random force Indeed, in $d>1$, the latter is not always the gradient of a potential and, as a result, different physically interesting anomalous diffusion behaviour can arise. From a technical point of view, this model requires the use of renormalization group techniques, to which a large part of this chapter is devoted.

4.1 Characterization of the random force field; physical motivation

4 1.1 Geometrical constraints; range of the correlations

The distribution of ($\boldsymbol{F}(\boldsymbol{x})$) will be taken to be Gaussian, characterized by its correlation function $G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{y})$ and mean $\boldsymbol{F}_0$,

$$\langle \boldsymbol{F}(\boldsymbol{x}) \rangle = \boldsymbol{F}_0\,, \qquad \langle [F_\mu(\boldsymbol{x}) - F_{0\mu}][F_\nu(\boldsymbol{y}) - F_{\nu 0}] \rangle = G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{4.3}$$

all higher-order correlation functions of $\delta F_\mu = F_\mu - F_{0\mu}$ being a sum of products of two-force correlations. As will be shown below, possible deviations from a Gaussian distribution would not affect the large-time diffusion behaviour. According to the physical situation at hand, one would like to impose different geometrical constraints on $\boldsymbol{F}(\boldsymbol{x})$. The most important cases in practice are (cf section 4 1 2)

(I) independent components $F_\mu(x)$ in each direction: $G_{\mu\nu} \propto \delta_{\mu\nu}$,

(II) "incompressibility" constraint: div $\boldsymbol{F} = 0$,

(III) potential force $\boldsymbol{F}(x) = -\mathrm{grad}\, U(x)$

*) γ has been included in the definition of the "force" in eq (4 1), and we have set $T/\gamma = D_0$, the "bare" diffusion constant

More generally, we shall allow $\boldsymbol{F}(\boldsymbol{x})$ to be an *admixture* of an incompressible (transverse) part and of a potential (longitudinal) part without cross-correlations. This is most easily expressed on the Fourier transform of $G_{\mu\nu}(\boldsymbol{x}-\boldsymbol{y})$,

$$G_{\mu\nu}(\boldsymbol{k}) = G_{\mathrm{T}}(\boldsymbol{k}^2)(\delta_{\mu\nu} - k_\mu k_\nu/k^2) + G_{\mathrm{L}}(\boldsymbol{k}^2)k_\mu k_\nu/k^2 \tag{4 4}$$

The particular cases (I), (II) and (III) correspond to $G_{\mathrm{T}} = G_{\mathrm{L}}$, $G_{\mathrm{L}} = 0$ and $G_{\mathrm{T}} = 0$, respectively Typical two-dimensional realizations of a random field satisfying one of these constraints (with $\boldsymbol{F}_0 = \boldsymbol{0}$) are depicted schematically in fig. 4.1 (taken from [Kra85]) A tracer described by the Langevin equation (4 1) moves in this quenched field by *convection* along "flow lines" of $F(x)$ and *molecular diffusion* between these lines under the action of the thermal noise $\eta(t)$ The competition of these two effects controls the diffusion properties of the tracer It is physically obvious that these properties can be very

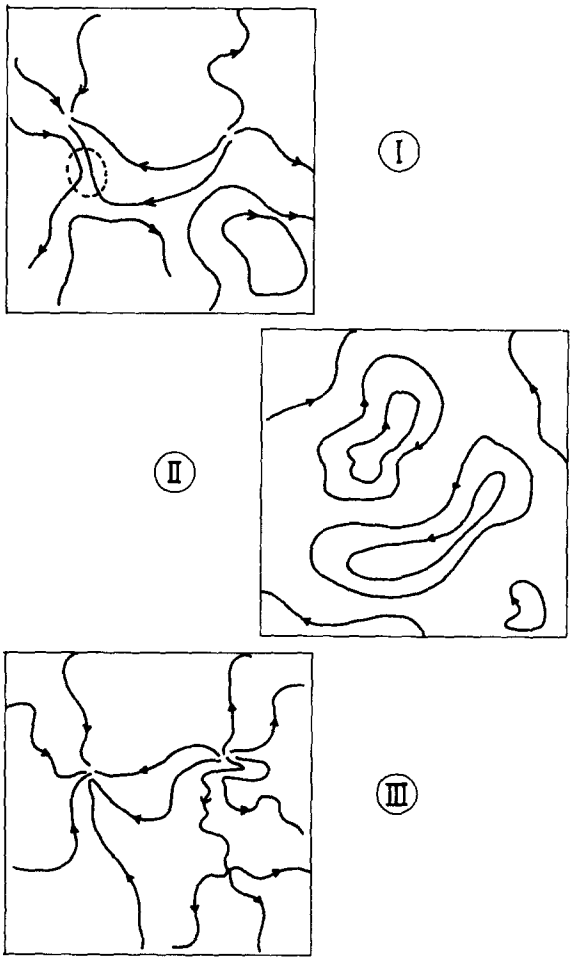


Fig 4 1 Typical configuration of a random force field in the following three particular cases (I) $G_{\mathrm{T}} = G_{\mathrm{L}}$ uncorrelated components, (II) $G_{\mathrm{L}} = 0$ or div $F = 0$ incompressible flow, for which the flow lines close (possibly at infinity), (III) $G_{\mathrm{T}} = 0$ or $F = -\mathrm{grad}\, V$ potential case, for which sinks act as traps for the thermal particle

different for different geometrical constraints, and that models II (incompressible) and III (potential) are two extreme cases: in the former, fast convective motion along closed flow lines is possible, while in the latter the tracer is convected into local wells, from which it can only escape by thermal activation. An admixture of G_T and G_L allows two nearby flow lines to be of opposite directions, and thus the tracer to escape more easily, leading to some intermediate diffusion behaviour. These qualitative statements will indeed be confirmed by calculations

The last point to be defined to characterize the model fully is the space dependence of $G_{\mu\nu}(\boldsymbol{x})$ Only the *long-distance* behaviour will turn out to be important for the asymptotic diffusion properties As will be demonstrated in the next section, the description of certain physical situations require spatial long-range correlations of $F(x)$ to be considered [Mar83, Pel85, Bou87b]. We shall characterize their decay by an exponent a (common to G_T and G_L),

$$G_{T,L}(\lambda(\boldsymbol{x}-\boldsymbol{y}))\sim\lambda^{-a}G_{T\,L}(\boldsymbol{x}-\boldsymbol{y}) \quad \text{for } \lambda\to\infty \tag{4.5}$$

– *Short-range* correlations correspond to an integrable correlation function (cf section 1 3 1) and thus to $a>d$. In this case one has

$$G_{T,L}(\boldsymbol{k}^2)\sim\sigma_{T,L}\,, \quad \boldsymbol{k}^2\ll\Lambda^2\,, \quad a>d \tag{4.6a}$$

– *Long-range* correlations arise when $a<d$ and are such that

$$G_{T\,L}(\boldsymbol{k}^2)\sim\sigma_{T,L}(\boldsymbol{k}^2)^{-(d-a)/2} \quad , \quad \boldsymbol{k}^2\ll\Lambda^2\,, \quad a<d \tag{4.6b}$$

In these expressions, Λ^{-1} denotes a short-distance length scale

4.1 2 Some physical motivations

4.1 2.1 Turbulent diffusion: a quenched description. One can think of describing the *relative**) diffusion of a pair of particles in a turbulent flow by the Langevin equation (4 1). $\boldsymbol{F}(\boldsymbol{x})$ should then be thought of as the difference between local velocities of two nearby points at a distance $\boldsymbol{x}$, and subjected to the incompressibility (type II) constraint $\operatorname{div}\boldsymbol{F}=0$. The limitation of this description is that the turbulent *relative* velocity field is considered to be *quenched* (time independent) – a point to which we shall come back below. Describing the statistical properties of the relative velocity field $\boldsymbol{F}(\boldsymbol{x})$ is a central problem in all theories of turbulence. A widely used form is (see, e.g., [Mon71])

$$G_T(R)\sim\bar{\varepsilon}^{2/3}R^{2/3}(R/l_0)^{\mu/3}\,, \tag{4.7}$$

for relative separations R in the range $l_d\lesssim R\lesssim l_0$, l_d being the dissipation length scale and l_0 the stirring length scale (below which the turbulent cascade is initiated). $\bar{\varepsilon}$ denotes the mean energy input. The value $\mu=0$ corresponds to the 1941 Kolmogorov theory, while more refined models [Man74, Fri78] taking into account intermittent corrections at small length scales lead to $\mu=d-d_F$, d_F being the fractal dimension of the active region As far as diffusion properties are concerned, the important feature of (4 7) is that the velocity field displays *long-range correlations* which are *increasing* with R in

*) *Relative* diffusion is considered in order to get rid of the overall motion of the fluid In particular $\langle F\rangle=0$

the range $l_d \ll R \ll l_0$, the exponent a defined by (4 5) being here

$$a = -\tfrac{2}{3} - \mu/3 \quad (<0)\,. \tag{4.8}$$

Turbulent diffusion has been the subject of numerous studies, both experimental and theoretical, which were initiated by Richardson's pioneering paper [Ric26] suggesting a hyperdiffusive law (in three-dimensional atmospheres)

$$\overline{R^2(t)} \sim t^3 \tag{4 9}$$

Anticipating on the results of this chapter (section 4.3), it turns out that the diffusion exponent resulting from (4 1) with long-range correlations ($a < d$, here $a < 0$ and $d = 3$) can be calculated exactly when the incompressibility constraint div $\boldsymbol{F} = 0$ is satisfied, and reads $\nu = 2/(2 + a)$ for $a < 2$ Inserting the value (4.8), this leads to

$$\overline{R^2(t)} \sim t^{2\nu}\,, \quad 2\nu = 3 + 3\mu/(4 - \mu) \tag{4.10}$$

This result is of course not new· it is well known that Richardson's law results from Kolmogorov's theory, and the μ-dependent intermittent corrections in (4 10) were derived by Hentschel and Procaccia [Hen84], who claimed that the data of Richardson's original paper are best fitted with a non-zero μ ($\simeq 0.36$), see fig. 4 2 Nevertheless, describing turbulent diffusion by the model (4 1), (4 2) with a quenched random velocity field is to our knowledge an original suggestion of the present paper*) It could in particular be a fruitful starting point for predicting the full *diffusion front*, an important issue for comparison with experiments

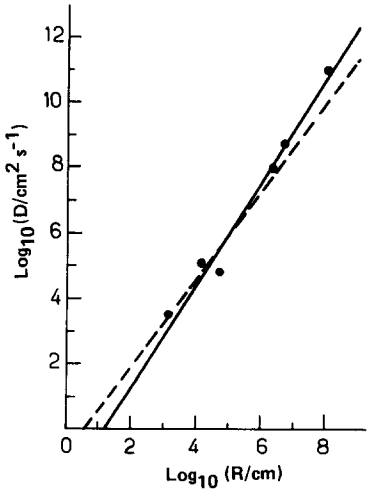


Fig 4 2 Richardson's data for the scale dependence of the effective diffusion coefficient $D(R)$ as a function of R for turbulent diffusion (in the atmosphere), showing that $R^2 \simeq t^3$ Note that the data are best fit by a slightly stronger exponent, which can be accounted for by the "intermittency corrections" to Kolmogorov's theory [Fri78, Hen84]

*) It is remarkable in this respect that the result (4 10) previously deduced basically from dimensional analysis turns out to be *exact* for this model

It should be noted that the assumption of a time independent relative velocity field is reasonable up to time scales of the order of the correlation time t_R, which can be estimated as [Hen84]

$$t_R \sim \bar{\varepsilon}^{-1/3} R^{2/3} (R/l_0)^{\mu/3}$$

Beyond this (R-dependent) time scale, model (4 1) no longer applies, in this regime, Hentschel and Procaccia have suggested a different diffusion exponent, $2\nu = 3 + 3\mu/(1-\mu)$ However, because of the R-dependence of t_R, these two regimes should not be viewed as short- and long-time regimes, and the "quenched" one, eq (4 10), turns out to be the relevant one in some experimental situations [Hen84]

Finally, one should mention that other descriptions of the turbulent diffusion problem have been proposed, e g using Levy flights (see [Shl86, 87] and references therein).

4 1 2 2. Relaxation in systems with complex energy landscapes One can think of (4.1) with $F = -\text{grad}\, U$ (model III) as a very idealized description of the relaxation properties of a system with a complex form of the energy $U(x)$ as a function of position x in configuration space. Relaxation is indeed connected to the properties of thermally activated diffusion in this complex energy landscape (fig. 4.3). With disordered magnetic materials (such as spin glasses [Bin86], for example) in mind, this description is of course quite far from reality for the following reasons:

- The topology of spin configuration space (hypercube of large dimension) is quite different from the euclidean space at hand here
- Describing the energy landscape by a (quenched) random potential ignores much of the complexity of the system.
- Describing the evolution of the system in terms of hopping between nearby configurations in phase space discards the possible collective modes which may contribute significantly to the relaxation properties (e g. the domain walls).

However, one can hope that some of its consequences are sufficiently "universal" to see this model as a useful guide. Indeed, strong evidence will be given below that, *when correlations are sufficiently long ranged*, model (4.1) with a potential force displays logarithmic diffusion "à la Sinai" *in arbitrary dimension* [Bou87b, Mar83] (for $\langle \boldsymbol{F} \rangle = \boldsymbol{0}$),

$$\langle \overline{x^2(t)} \rangle \sim (\ln t)^{4/(2-a)}, \quad a < 2. \tag{4.11}$$

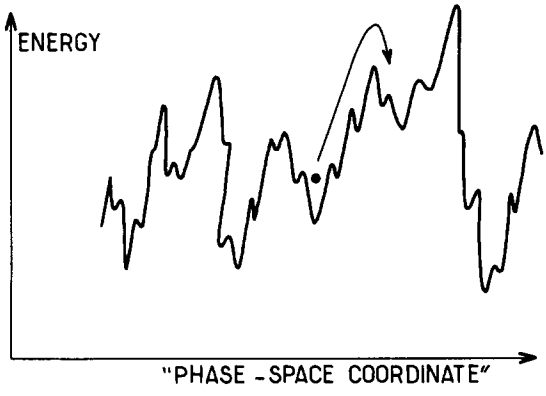


Fig 4 3 Schematic plot of the energy versus a "coordinate" describing the microscopic state of a complex system (e g a spin glass)

That (4.11) holds in this case was indeed suggested in [Mar83] and motivated several further works on this model. It is tempting to relate the very slow low-temperature relaxation [Oci86, Ref87a,b, Fer89] observed in systems such as spin glasses (below T_g) to such ultra-slow logarithmic diffusion in configuration space. It was pointed out in [Mar83] that the latter generates low-frequency "$1/f$ noise" in the autocorrelation function, and thus in the noise spectrum,

$$\lim_{T\to\infty} \frac{1}{T}\left|\int_0^T \mathrm{d}t\, \mathrm{e}^{\mathrm{i}ft} x_\mu(t)\right|^2 \sim \frac{(\ln f)^{4/(2-a)}}{f} \tag{4 12}$$

One should note that the condition $a<2$ simply means that potential correlations (and thus the *typical* energy barrier) *grow with distance*, since $\langle U(0)U(x)\rangle \sim x^{2-a}$ (see also [Fis88, Bra88a] for somewhat related ideas)

Whether this is a realistic assumption when describing the energy landscape of, e g , a spin glass is to a large extent an open question (in [Bou87b] it was argued that this is indeed the case of the energy correlation in the Sherrington–Kirkpatrick model of a spin glass, at least for a wide range of distances) One should note that even if potential correlations grow only up to a "length" scale L, the cross-over time below which logarithmic diffusion applies (inverse of the frequency above which $1/f$ noise is observed) can be huge at low temperature since

$$t_c \sim \exp(L^{1-a/2}/kT) \tag{4 13}$$

An interesting issue connected with this model is the effect of a non-zero bias $\langle F\rangle$ (e g , an applied magnetic field for a spin system) Whether successive phases exist as in the one-dimensional case (chapter 3) is an open question. Let us finally mention that some attempts have also been made to describe dynamical properties of spin glasses *above* T_g through diffusion properties in configuration space [Cam85, 86]. The reader is referred to, e.g., [Ste87] for a discussion of recent ideas on the modelling of dynamical properties of systems with a complex phase space

Other possible applications of model (4 1)–(4 3) have been suggested in [Fis84, 85, Kra85, 86a, Aro84]

4 2 Relevance of disorder; weak-disorder expansions and their failure

4.2 1 Statistical mechanism and relevance of weak disorder

4 2.1.1 Heuristic discussion The statistical mechanism which can lead to anomalous diffusion for this model (in the absence of an average bias) is the *induced correlation* in the temporal sequence of random forces seen by the walker This is possible because the same value of the random force will be seen each time the walker visits a given site (quenched disorder) This was already demonstrated on the example of a layered medium in section 1 3 2 (which is indeed a particular anisotropic limit of the model considered here)

A simple statistical argument can thus be used to understand in which cases anomalous diffusion will arise Consider first the case where no long-range *spatial* correlations are present a priori in the random force field ($a>d$), and ask how weak disorder will affect a *normal* Brownian walker in the medium The number of *different* values of the random force encountered by this walker in a time t is of order

$$t^{d/2} \quad \text{in } d<2\,, \qquad t/\ln t \quad \text{in } d=2\,, \qquad t \quad \text{in } d>2\,,$$

each one being encountered $t^{1-d/2}$, $\ln t$ and a constant number of times, respectively. Thus a long-range *temporal* correlation in the sequence of forces encountered is induced only *in less than two dimensions* More precisely, the extra displacement induced by weak disorder (treated perturbatively) is predicted by this argument to be of order

$$\delta x_t = \int^t \mathrm{d}\tau\, F(\tau) \sim \begin{cases} \sqrt{t}\,, & \text{in } d>2\,, \\ \ln t\,(t/\ln t)^{1/2} \sim \sqrt{t \ln t}\,, & \text{in } d=2\,, \\ t^{1-d/2}\sqrt{t^{d/2}} \sim t^{1-d/4}\,, & \text{in } d<2\,, \end{cases}$$

where the sum of correlated variables has been analyzed along the lines of section 1 3 1 Thus, weak disorder will modify only the diffusion constant without changing the diffusion law for $d>2$ [Pal83]. For $d \leq 2$, on the contrary, δx_t is found to be much larger than $\sqrt{t}$, indicating a failure of the perturbative approach and the occurrence of anomalous diffusion. This will indeed be confirmed by systematic weak-disorder expansions of the next section (section 4 2 2).

This line of argument is easily generalized to the case where long-range spatial correlations are present [Bou87b]. Then, following the remarks of section 1 3.1, the random forces present in a sphere of radius R can be grouped into R^d/N_{id} effectively independent "families" of N_{id} "almost identical" values, with

$$N_{\mathrm{id}} \sim \int^R \frac{r^{d-1}\,\mathrm{d}r}{r^a} \underset{R\to\infty}{\sim} \begin{cases} \text{const}\,, & a>d\,, \\ R^{d-a}\,, & a<d\,. \end{cases} \tag{4.14}$$

Thus, $a>d$ and $a<d$ correspond to the regimes of "short-range" and "long-range" *spatial* correlations, as expected The above analysis is unchanged in the former case ($a>d$), while, when $a<d$, one has $R^d/N_{\mathrm{id}} \sim R^a$ independent values of the force, and a simply replaces d in the above analysis One thus concludes [Pel85, Bou87b] that when long-range correlations are present a priori in the quenched random force field, *diffusion can be anomalous in any dimension provided* $a<2$ (The relevance of such correlations to actual physical situations has been emphasized above.)

The conclusions of the present analysis (and the more detailed results provided by the R.G method) are summarized in fig 4 4.

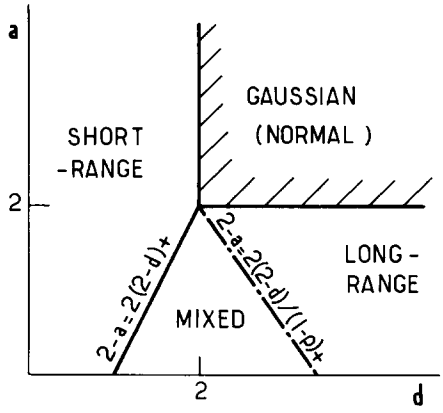


Fig 4 4 Regions of the (d, a) plane (d is the dimension of space, a the exponent governing the decay of correlations) where disorder is "relevant", i e , changes the diffusion exponent ν to a non-trivial value $\neq 1/2$ ε expansions can be performed near $a=2$ or $d=2$, note, however, that, when both a, d are close to 2, a simultaneous expansion in $2-d$ and $2-a$ is needed, yielding new regimes in the phase diagram ("mixed" phase)

4.2 1 2. Self-consistent approximation (à la Flory) of the diffusion law. Having identified the statistical mechanism as an induced long-range time correlation, one can try to push further the above argument, and devise a Flory-like approximation of the diffusion behaviour (much along the same lines as in section 1 3 3). Assuming a diffusion law $x^2 \sim t^{2\nu}$, the number of effectively independent values of the random force encountered by the walker in a time t reads

$$N_{\text{eff}} \sim \begin{cases} \min(t, t^{\nu d}) & \text{if } a > d\,, \\ \min(t, t^{\nu a}) & \text{if } a < d\,. \end{cases} \tag{4 15}$$

The overall displacement can thus be estimated in the usual way as

$$\langle \overline{x_t^2} \rangle \sim (t/N_{\text{eff}})\sqrt{N_{\text{eff}}}\,, \tag{4.16}$$

which, self-consistently, must be of order t^{ν} This leads to

$$\nu_{\text{approx}} = \begin{cases} 1/2\,, & d > 2\,, \\ 2/(2+d)\,, & d < 2\,, \end{cases} \quad a > d\,, \tag{4 17a}$$

$$\nu_{\text{approx}} = \begin{cases} 1/2\,, & a > 2\,, \\ 2/(2+a)\,, & a < 2\,, \end{cases} \quad a < d \tag{4 17b}$$

This approximation (according to which hyperdiffusion arises in all anomalous cases) will turn out to be quite poor in general, since it neglects all *trapping* effects (induced by the "potential" component of the force) They make the visited sites highly inequivalent, and the corresponding weights should be included in the sum (4 16). However, remarkably enough, (4 17) turns out to be *exact* for incompressible (type II) force fields div $\boldsymbol{F} = 0$ [Hon88a] (see also [For77]). This is not unexpected since in this case the stationary probability distribution is constant, and the visited sites are indeed equivalent.

4 2 2. Perturbative treatment of the disorder

4.2.2.1. Systematic expansion We now turn to more quantitative methods and show how a systematic perturbative treatment of the disorder can be made for model (4.1). We shall keep a non-zero $\langle \boldsymbol{F} \rangle = \boldsymbol{F}_0$ for completeness As in section 3 1, the basic quantity to be considered is the Laplace transform $P(\boldsymbol{x}, \boldsymbol{y}, E)$ of the probability of presence $P(\boldsymbol{x}, t|\boldsymbol{y}, 0)$, which is a Green function of the Fokker–Planck operator,

$$-D_0\,\Delta P + \boldsymbol{\nabla}\cdot(\boldsymbol{F}P) + EP = \delta(\boldsymbol{x} - \boldsymbol{y})\,. \tag{4 18}$$

Introducing the Fourier transform

$$P(\boldsymbol{k}, E) = \int \frac{\mathrm{d}^d \boldsymbol{x}}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} P(\boldsymbol{x}, 0, E)\,, \tag{4 19}$$

(4.18) is converted into an integral equation,

$$P(\boldsymbol{k}, E) = P_0(\boldsymbol{k}, E)\Big(1 - \mathrm{i}\boldsymbol{k}\cdot \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d}\, \delta\boldsymbol{F}(\boldsymbol{q}) P(\boldsymbol{k} - \boldsymbol{q}, E)\Big)\,, \tag{4.20}$$

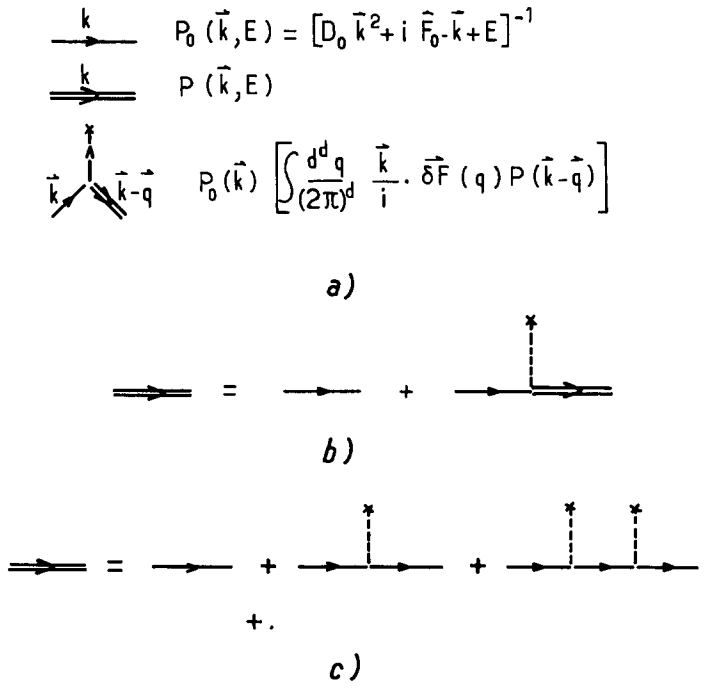


Fig 4 5 Diagrammatic rules for a fixed environment (a) Propagators and interaction vertex (b) Graphical representation of eq (4 20) δF is treated as an external source and an integration is carried out over the attached momentum (c) The perturbative expansion

where $P_0(k, E) = (D_0 k^2 + iF_0 \cdot k + E)^{-1}$ is the "free" Green function in the absence of disorder and $\delta F(x) = F(x) - F_0$ is the random part of the force.

Equation (4.20) is the starting point of a perturbative expansion in powers of δF, which is simply generated by iteration. This expansion is conveniently represented graphically, as shown in fig 4.5.

Deriving the diffusion behaviour from the large-time limit of the fixed configuration problem is rather hard, and it will be convenient to consider disorder-averaged quantities (keeping in mind the possible subtleties associated with fluctuations, mentioned in chapters 2 and 3). Owing to the Gaussian form of the distribution of δF, the average over disorder of the perturbation series of fig. 4.5 amounts to a pairing of external force lines in all possible ways (Wick's theorem), as depicted in fig. 4.6. This expansion can be conveniently reorganized in the form

$$\langle P(k, E)\rangle = [D_0 k^2 + iF_0 \cdot k + E - \Sigma(k, E)]^{-1}, \tag{4.21}$$

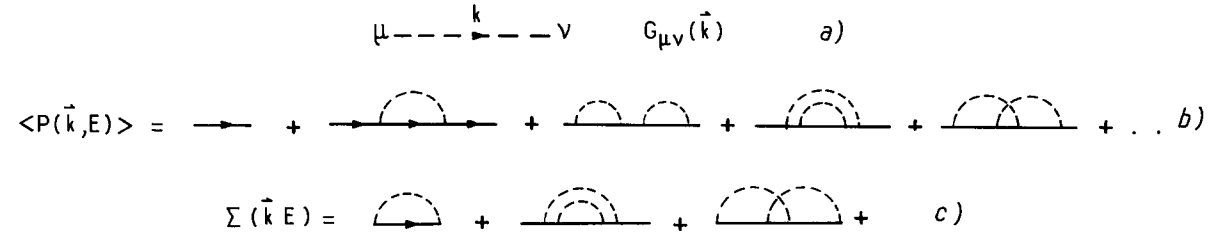


Fig 4 6 Diagrammatic rules for the *disorder-averaged* Green functions (a) Propagator of the random force, the indices μ and ν are contracted with the incoming momenta at the vertices (b) Perturbative expansion of $\langle P(k, E)\rangle$ (c) Perturbative expansion of the self-energy (one-particle irreducible graphs)

where the "self-energy" $\Sigma(\boldsymbol{k}, E)$ involves only diagrams which cannot be disconnected by cutting a P_0 propagator ("one-particle irreducible" diagrams, see fig 4.6c)

4 2.2.2. First-order corrections to the diffusion constant and the failure of naive perturbations. The above expansion can be used to obtain the disorder-induced corrections to the diffusion constant. For a zero external bias ($\boldsymbol{F}_0 = \boldsymbol{0}$), $D_{\rm av}$ is easily seen to be given by

$$D_{\rm av} = D_0 - \partial^2 \Sigma / \partial k^2 |_{k=E=0} \tag{4.22}$$

The first-order contribution to the self-energy is obtained from the first diagram in fig 4 6c as

$$\Sigma^{(1)}(\boldsymbol{k}, E) = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{k_\mu G_{\mu\nu}(\boldsymbol{q})(k_\nu - q_\nu)}{D_0(\boldsymbol{k}-\boldsymbol{q})^2 + \imath \boldsymbol{F}_0 \cdot (\boldsymbol{k}-\boldsymbol{q}) + E} \tag{4.23}$$

This expression is then expanded up to second order in $\boldsymbol{k}$ to yield, for $\boldsymbol{F}_0 = \boldsymbol{0}$ [Aro84]*),

$$\frac{D_{\rm av}}{D_0} = 1 + \left(1 - \frac{1}{d}\right) \frac{1}{D_0^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{G_{\rm T}(k^2)}{k^2} - \frac{1}{d} \frac{1}{D_0^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{G_{\rm L}(k^2)}{k^2} \tag{4 24}$$

Thus, the transverse (incompressible) part of the random force increases D, while the longitudinal (potential) part decreases it, this is expected on physical grounds, as emphasized in section 4.1.1.

The most important remark to be made on (4.24) is that the corrections to D/D_0 are found to *diverge* (for $d \leq 2$ when $a > d$, and for $a \leq 2$ when $a < d$). This is due to the small-k ("infrared") singularities of the integrals in (4.24), as is clear from the small-k behaviour of $G_{\rm T,L}(k^2)$ given by (4.6) This failure of naive perturbation theory is of course the signal of the occurrence of anomalous diffusion; this fully confirms the analysis of the previous section as summarized in fig. 4.4 More sophisticated methods have to be used to study this anomalous diffusion behaviour; this is the aim of the renormalization group reviewed in section 4 3

4.2 2 3. Connection with critical phenomena As emphasized in chapter 1, the large-time limit of a random walk process can in fact be viewed as a *critical phenomenon*. Indeed, in this limit, a CLT applies and $\langle P(\boldsymbol{x}, t) \rangle$ takes a scaling form characterized by a few relevant parameters within some attraction basin ("universality class") Table 4 1 summarizes this general connection In this framework, anomalous diffusion appears as a departure from mean-field (Brownian) behaviour and is signalled, as is well known for critical phenomena, by long-distance divergences in the naive perturbation approach Note that, at finite time (non-zero E), an infrared cut-off $k_{\rm min} \sim \sqrt{E/D_0}$ is present, which defines the critical (anomalous diffusion) regime at the mean-field level as

$$t \gg \Lambda^{-2}/D_0 \quad \text{(molecular diffusion time)} \tag{4 25}$$

In an analogous way, a non-zero average bias also introduces a cut-off $k_{\rm min} \sim F_0/D_0$, even if the infinite-time limit $E = 0$ is taken first. This is because the presence of an average bias *destroys the induced long-range time correlation* (cf. section 1 3 2 and section 4 2 1 1) Thus the limit $\boldsymbol{F}_0 \to \boldsymbol{0}$ at $E = 0$

*) It is understood that these integrals imply a short-distance cut-off $k_{\rm max} \sim \Lambda^{-1}$ Note that fluctuations are expected to be weak above the critical dimension, so that D is indeed obtained despite the averaging procedure ($D_{\rm av} = D$)

Table 4 1

Diffusion process	Phase transition
Probability of presence $P(x, E)$	Spin–spin correlation $\langle S(0)S(x)\rangle_T$
E	$T - T_c$
$t\to\infty$ $(E\to 0)$	$T\to T_c$
Central limit theorem $\langle P(x, E)\rangle \to \lvert x\rvert^{-\nu d} f[\lvert x\rvert/\xi(E)]$	Critical scaling form $\langle S(0)S(x)\rangle \sim \lvert x\rvert^{2-d-\eta} f[\lvert x\rvert/\xi(T)]$
Diffusion exponent $\xi(E)\sim E^{-\nu}$ $(\overline{\Delta x^2}\sim t^{2\nu})$	Correlation length critical exponent $\xi(T)\sim \lvert T - T_c\rvert^{-\nu}$
Probability conservation $\int d^d x\, P(x, E) = 1/E$	Divergence of susceptibility $\chi = \int d^d x\, \langle S(0)S(x)\rangle \sim \lvert T - T_c\rvert^{-\gamma}$
$1 = \nu(2-\eta)$ (one single independent exponent)	$\gamma = \nu(2-\eta)$
Brownian diffusion $\nu = \frac{1}{2}$, $\eta = 0$	Mean-field (Gaussian) behaviour $\nu = \frac{1}{2}$, $\eta = 0$
$P_0(k, E) = (D_0 k^2 + E)^{-1}$	$\langle S(0)S(k)\rangle = (k^2 + T - T_c)^{-1}$

is also a critical limit, defined by (at the mean-field level)

$$\mathrm{Pe} \equiv \Lambda^{-1} F_0/D_0 \ll 1 . \tag{4.26}$$

This dimensionless number is known as the *microscopic Péclet number* of the flow, and $\mathrm{Pe}\ll 1$ defines the regime in which diffusion effects dominate over convective effects

4 2 2 4 Formulation as a zero-component field theory the replica trick It is most convenient to have a more global representation of the above perturbation series To this aim, one can introduce the generating function

$$Z[J, \hat J] = \int \mathrm{D}\phi\, \mathrm{D}\hat\phi\, \exp\Big(\imath S(\phi, \hat\phi) + \int \hat J\phi + \int J\hat\phi \Big) , \tag{4 27}$$

where $\phi(x)$ and $\hat\phi(x)$ are independent fields à la Martin–Siggia–Rose [Mar73, dDo78], and the action S reads

$$S(\phi, \hat\phi) = \int d^d x\, \hat\phi[E\phi - D_0\, \Delta\phi + \nabla\cdot(F\phi)] \tag{4 28}$$

It is easily seen, by successive integration over ϕ and $\hat\phi$, that

$$P(x, y, E) \equiv -\imath \frac{\delta^2}{\delta J(x)\, \delta \hat J(y)} \ln Z[J, \hat J]\big|_{J=\hat J=0} \tag{4 29}$$

The perturbative expansion of (4 29) indeed coincides with the rules of fig 4 5 for a *fixed* configuration (δF)*)

Averaging over disorder of eq (4 29) is made difficult by the logarithm on the r h s A way to overcome this difficulty is to use the celebrated "replica trick", which relies on the identity

$$\ln Z = \lim_{N\to 0} [(Z^N - 1)/N]$$

Computing Z^N leads to the introduction of N copies $(\phi^a, \hat\phi^a)$, $a = 1, \ldots, N$, of the fields The average over the Gaussian force can then easily be taken, and one finally obtains [Kra85, 86a, Bou87b]

$$\langle P(x, y, E)\rangle = \lim_{N\to 0} \frac{1}{N} \sum_a \langle \phi^a(x)\hat\phi^a(y)\rangle_{S_{\mathrm{av}}} , \tag{4 30}$$

$$S_{\mathrm{av}}[\phi^a, \hat\phi^a] = \sum_a \int d^d x\, (E\phi^a\hat\phi^a + D_0\, \partial_\mu\phi^a\, \partial_\mu\hat\phi^a - F_{0\mu}\phi^a\, \partial_\mu\hat\phi^a)$$
$$+ \frac{1}{2} \sum_{a\,b} \int d^d x\, d^d y\, \phi^a(x)\, \partial_\mu\hat\phi^a(x)\, G_{\mu\nu}(x-y)\phi^b(y)\, \partial_\nu\hat\phi^b(y) \tag{4 31}$$

*) The Fokker–Planck operator is not a Hermitian operator, and the global convergence of the integration (4 27) could be questionable However, we shall only be concerned with its perturbative meaning

The perturbation expansion in $G_{\mu\nu}$ of this field theory coincides with the rules of fig 4 6 In particular, the zero-component limit taken order by order cancels all closed loops of P_0 propagators (which were indeed absent in fig 4 6), much in the same way as for the self-avoiding walk problem (section 1 3 3, and [dGe72]) A different field-theoretical representation of this problem can be used, e g , by keeping an explicit time dependence [Fis84, Pel85]

4 2 3 Weak-disorder expansions for general lattice hopping models

For the sake of completeness, we shall quote here the results of a weak-disorder expansion due to Derrida and Luck [Der83b], valid for a large class of hopping models on *discrete* lattices described by the master equation (2 1) The hopping rates $W_{rr'}$ are decomposed into a translationally invariant and a random part,

$$W_{rr'} = W_0(r - r') + \delta W_{rr'} \tag{4 32}$$

No assumption is made on the symmetry of $\delta W_{rr'}$, nor are the jumps restricted to nearest neighbours only However, it was assumed in [Der83b] (for the mere sake of simplicity) that the pairs $(\delta W_{xx'}, \delta W_{x'x})$ were independent random variables from link to link This, unfortunately, does not allow one to impose various geometrical constraints such as (4 4) (except when $G_L = G_T$) The technique used in [Der83b] is a generalization of the steady-state technique for a periodized sample introduced by Derrida for $d = 1$ in [Der83a] and briefly presented in section 3 1 1

Here, we only quote the results for the expansion of the velocity V_μ and diffusion tensor $D_{\mu\nu}$ up to the first non-trivial order in the disorder, i e , up to order $n = 2$ in the moments,

$$C_{np}(r - r') = \langle \delta W_{rr'}^{n-p} \, \delta W_{r'r}^{p} \rangle \tag{4 33}$$

The results of [Der83b] read*)

$$V_\mu = \sum_z z_\mu W_0(z) + \sum_z z_\mu [C_{20}(z) - C_{21}(z)] I(z) , \tag{4 34}$$

$$D_{\mu\nu} = \frac{1}{2} \sum_z z_\mu z_\nu W_0(z) + \frac{1}{2} \sum_z z_\mu z_\nu [C_{20}(z) + C_{21}(z)] I(z) + \frac{1}{2} \sum_{z,y} [z_\mu y_\nu W_0(y) + z_\nu y_\mu W_0(y)][C_{20}(z) - C_{21}(z)] J(z) , \tag{4 35}$$

where $I(z)$ and $J(z)$ are integrals over the Brillouin zone $B = [-\pi, \pi]$

$$I(z) = \int_B \frac{d^d q}{(2\pi)^d} \frac{e^{iqz} - 1}{\hat{W}_0(0) - \hat{W}_0(q)} ,$$

$$J(z) = \int_B \frac{d^d q}{(2\pi)^d} \frac{e^{iqz} - 1}{[\hat{W}_0(0) - \hat{W}_0(q)]^2} ,$$

with $\hat{W}_0(q) = \Sigma_z W_0(z) e^{iqz}$ The expansion of the velocity was in fact given up to $n = 4$ in [Der83b]

This expansion can be applied to the random barrier model (symmetric disorder $W_{xx'} = W_{x'x}$), for which it has been mentioned in section 2 4 2 2 that no exact expression of the diffusion constant is known in more than one dimension It is then found that the effective medium result $D = a^2/W_{EMA}$ with W_{EMA} given by (2 53) is exact in first order [Der83b]

When applied to asymmetric disorder, infrared divergences are found below $d = 2$ [Der83b], of the same nature as those described in section 4 2 2 2 for $G_L = G_T$ and short-range correlations As far as large-time diffusion properties are concerned, the continuum model (4 1) retains all the important features of the original lattice model, provided one identifies $\sigma_T = \sigma_L = \sigma$ with

$$\sum_r x_\mu x_\nu [C_{r0}(x) - C_{r1}(x)] \propto \sigma \delta_{\mu\nu} \tag{4 36}$$

[Note that eq (4 24) for D is recovered by taking the continuum limit of (4 35)] Remarkably, a "critical phase" with $V = 0$ (cf section 3 3 2) is correctly predicted by the expansion in one dimension [Der83b]

4 3 Anomalous diffusion behaviour from renormalization group methods

4 3 1 The renormalization group strategy

4.3.1 1. Basic ideas Quantitative analysis of the anomalous diffusion laws arising when disorder is

*) The lattice spacing has been set equal to one in these expressions

"relevant" is best handled by renormalization group (RG) methods. The main ideas on which these methods rely can be summarized as follows. A detailed introduction to renormalization group ideas in the context of equilibrium critical phenomena can be found in a number of textbooks. Application to dynamics along lines very similar to those followed here can be found in [Hoh77, Ma75, For77].

(i) Since large-time diffusion behaviour depends only on *large-scale features* of the problem at hand, one would like to define a "large-scale effective dynamics" by integrating out short-distance degrees of freedom To do so, it is convenient to work in Fourier space and to separate the modes in two shells, $0 < k < \Lambda/b$ and $\Lambda/b < k < \Lambda$, where b is some arbitrary scaling factor and Λ the short-distance cut-off Accordingly

$$P(\boldsymbol{k}, E) = P^{<}(\boldsymbol{k}; E) + P^{>}(\boldsymbol{k}; E)\,, \tag{4 37}$$

where $P^{<}$ ($P^{>}$) is non-zero in the shell $[0, \Lambda/b]$ ($[\Lambda/b, \Lambda]$) only. One would then like to express $P^{>}(\boldsymbol{k}, E)$ as a function of $P^{<}(\boldsymbol{k}; E)$ and $\boldsymbol{F}(\boldsymbol{k})$ by solving the integral equation (4 19) This would allow one to obtain an equation satisfied by $P^{<}(\boldsymbol{k}, E)$, of which all modes $k \in [\Lambda/b, \Lambda]$ can be eliminated by averaging over the corresponding modes of $\boldsymbol{F}(\boldsymbol{k})$.

(ii) This large-scale dynamics for $P^{<}(\boldsymbol{k}, E)$ is then compared to the original one by making a scale transformation,

$$\boldsymbol{k}' = b\boldsymbol{k}\,, \qquad E' = \alpha(b)E\,, \tag{4 38}$$

such that the range of k' is again $(0, \Lambda)$. At this stage $\alpha(b)$ is an arbitrary function The *renormalization group transformation* $\mathcal{R}_b$ of the parameters $\mathcal{P} = (D_0, \sigma_T, \sigma_L, \ldots)$ specifying the model is then obtained by requiring that

$$k < \Lambda/b\,, \qquad \langle P(\boldsymbol{k}, E; \mathcal{P})\rangle \equiv \alpha(b)\langle P(b\boldsymbol{k}, \alpha(b)E, \mathcal{R}_b(\mathcal{P}))\rangle \tag{4.39}$$

(The factor $\alpha(b)$ multiplying P on the r.h.s. is *dictated* by the conservation of probability (see table 4 1), which also insures that the "critical point" remains at $E = 0$ without being shifted.)

(iii) This procedure is carried out *in a recursive way* This in principle requires one to extend the parameter space $\mathcal{P}$ to include all new parameters generated by the process. (In fact, in favourable cases, only a finite number of relevant parameters need to be retained, see below) The function $\alpha(b)$ is then determined by the requirement that a *fixed point* $\mathcal{P}^* = \mathcal{R}_b(\mathcal{P}^*)$ is reached when the procedure is carried ad infinitum. This is equivalent to saying that $\langle P(\boldsymbol{k}, E)\rangle$ obeys a generalized *central limit theorem* Indeed, let us assume for simplicity that $\alpha(b)$ behaves as a power law for large b,

$$\alpha(b) \sim b^{1/\nu} \quad (b \to \infty)\,. \tag{4.40}$$

Then, taking the limit $b \to \infty$ in (4.39) in such a way that $\alpha(b)E$ remains constant [e.g , $\alpha(b)E = 1$] leads to

$$\langle P(\boldsymbol{k}, E, \mathcal{P})\rangle \underset{\substack{E\to 0\\ k\to 0}}{\longrightarrow} \frac{1}{E}\,\langle P(\boldsymbol{k}/E^{\nu}, 1, \mathcal{P}^*)\rangle\,, \tag{4.41}$$

which is, in Fourier–Laplace space, the very expression of a generalized CLT, ν being the diffusion

exponent. $\alpha(b)$ need not be a pure power law for large b, in which case k/E^{ν} is replaced by $k\alpha^{-1}(1/E)$ (α^{-1} being the reciprocal function) and the diffusion behaviour is not a pure power law of time. Note that the r.h.s of (4 40) involves only quantities *at the fixed point*, which insures the *universality* of the limiting scaling form of $\langle P(\boldsymbol{x}, t)\rangle$ within the attraction basin of $\mathscr{P}^*$.

(iv) A fourth step involves the analysis of the attraction basins and of the stability of the different possible fixed points. Normal diffusion corresponds to a "Gaussian fixed point", for which $\alpha(b) = b^2$ and such that the scaling function is Gaussian (normal CLT). Anomalous diffusion is associated with the occurrence of a non-Gaussian renormalization group fixed point

In practice, however, the limitation of the method lies in step (i). Indeed, solving for $P^{>}$ requires one to handle the integral equation (4 19) [if one could solve it exactly, there would be no need for any of the subsequent steps (ii)–(iv)!]. The crucial point is that perturbative methods can now be safely used, since one has to deal only with modes in the shell $[\Lambda/b, \Lambda]$, thereby avoiding infrared divergences. However, this implies that non-trivial fixed points can only be identified consistently by perturbative techniques when they depart from the Gaussian fixed point by an infinitesimal amount In fact, only a systematic expansion of the fixed points and associated diffusion exponents in the small parameter

$$\varepsilon = 2 - d\,, \quad \text{for } a > d\,, \qquad \varepsilon = 2 - a\,, \quad \text{for } a < d$$

(measuring the distance from the critical boundaries in fig 4 4) can thus be devised in general. This is, however, only a limitation of a technical nature, and the fundamental concepts of the renormalization group (together with their probabilistic meaning) are in no way limited to perturbative methods. Indeed, we shall see below that in some remarkable cases, the anomalous diffusion behaviour of model (4 1) can be analyzed beyond the framework of ε-expansions

Remark. The above presentation of the RG method is meant to emphasize its probabilistic content. It is also somewhat closer in spirit to Wilson's original ideas (see, e g , [Wil74]) and to real-space approaches. However, beyond one-loop order (and all the more if one wants to handle properties of all orders) it is most helpful to make use of a field-theoretic formulation along the lines of, e.g , [Bre76]. The replicated field theory (4 31) – or related formulations – is in this respect a very convenient starting point. This is the approach followed in most papers quoted above.

4 3 1 2 One-loop calculations Step (i) of the RG program is conveniently carried through perturbatively using the diagrammatic representation established in section 4 2 2 1 The main steps are illustrated by fig 4 7a–f The integral equation satisfied by $P^{>}$ (fig 4 7a) is used in the one satisfied by $P^{<}$ (fig 4 7b) to obtain an expression of the latter in which $P^{>}$ has been eliminated up to some desired order (fig 4 7c) One then averages over the modes $\Lambda/b < k < \Lambda$ of the random force The integral equation obtained in this way (fig 4 7d) is then compared to the original one One sees that they can be put in a similar form, provided the bare two-point function is corrected by a self-energy term, and interaction with the random force is corrected by the last two diagrams of fig 4 7d (the last one being a non-local correction)

This can be turned into a correction $\delta_b D$ to the diffusion coefficient and $\delta_b \sigma_{\mathrm{T}}$, $\delta_b \sigma_{\mathrm{L}}$ to the correlator of the random force, associated, at first (one-loop) order, with diagrams 4 7e and 4 7f, respectively All momentum integrations involved in these diagrams are restricted to the shell $\Lambda/b < k < \Lambda$, and are thus convergent As an example, the correction $\delta_b D$ can be simply read off from (4 23),

$$\frac{\delta_b D}{D} = = \frac{\Lambda^{d-2} c_d}{D^2}\left[(1 - d^{-1})\sigma_{\mathrm{T}} - d^{-1}\sigma_{\mathrm{L}}\right]\frac{1 - b^{-(d-2)}}{d-2}\,, \quad c_d = 2^{1-d}\pi^{-d/2}/\Gamma(d/2) \tag{4 42}$$

Replacing Λ^{d-2} by Λ^{a-2} and $b^{-(d-2)}$ by $b^{-(a-2)}$, one obtains the corresponding expression for $a < d$ It turns out to be useful to

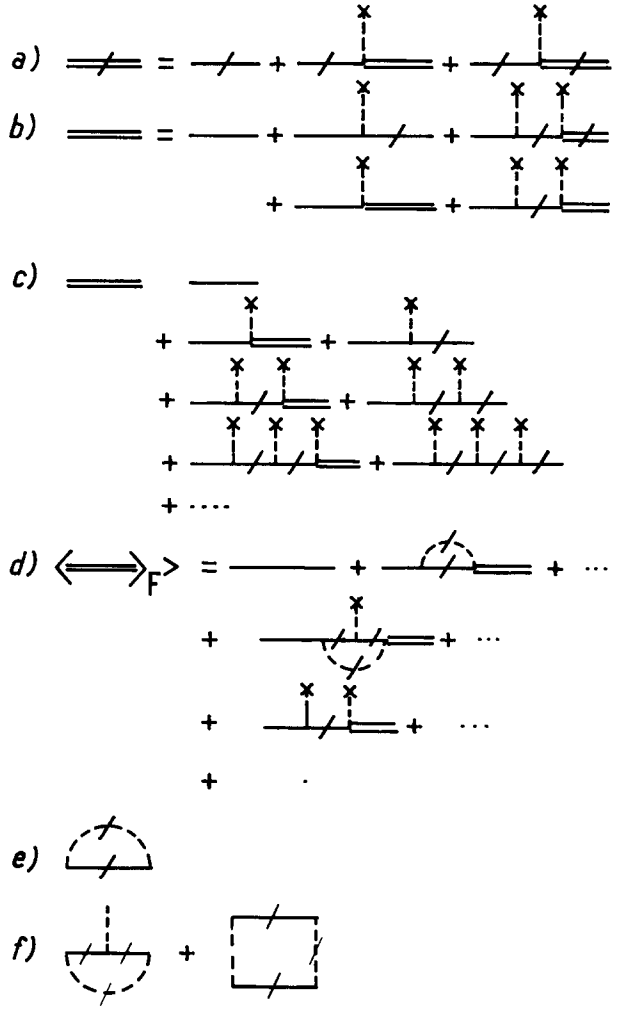


Fig 4 7 Renormalization procedure carried through perturbatively Moments in the integrated shell $[\Lambda/b, \Lambda]$ are indicated by a bar on the propagator See text for an explanation of (a)–(f)

define dimensionless coupling constants by

$$g_{T\,L} = \frac{\sigma_{T,L}}{D^2}\, c_d(1-d^{-1}) \times \begin{cases} \Lambda^{d-2}, & a > d\,, \\ \Lambda^{a-2}, & a < d \end{cases} \tag{4 43}$$

Performing the rescaling (4 37) of space and time is then a simple matter of dimensional counting Since $x'^2/t' = [\alpha(b)/b^2]x^2/t$ and $\Lambda^{d-2} = (\Lambda/b)^{d-2} b^{d-2}$, the RG transformation of the parameters D and $g_{T\,L}$ reads

$$\begin{aligned} &\mathcal{R}_b(D) = [\alpha(b)/b^2](D + \delta_b D)\,, \\ &\mathcal{R}_b(g_{T\,L}) = b^{2-d}(g_{T\,L} + \delta_b g_{T\,L}) \quad (a > d) \end{aligned} \tag{4 44}$$

(d being replaced by a in the last equation in the case $a < d$)

We shall now quote the result of explicit calculations of this RG transformation, up to two-loop order for short-range correlations ($a > d$) [Luc83, Fis84, 85, Aro84, Kra85, 86a] and up to one-loop order in the long-range case $a < d$ [Pel85, Bou87b, Hon88a] (the two-loop result was obtained recently for the latter case as well, see [Hon88a]) This is conveniently expressed in

differential form, by considering an infinitesimal scale transformation and retaining only first order It is helpful to set

$$b = e^{l}, \qquad \alpha(b) = \exp\Big(\int_0^l \frac{dl'}{\nu(l')}\Big),$$

and to denote the "running parameters" (4 44) by $D(l)$, $g_{T\,L}(l)$

In the short-range case ($a > d$), the results read

$$\begin{aligned} d(\ln D)/dl &= -2 + \nu^{-1}(l) + g_T - g_L - 2g_T g_T , \\ d(g_T)/dl &= \varepsilon g_T - g_T(2g_T - g_T) - g_T g_L(g_L - 3g_T) , \\ d(g_T)/dl &= \varepsilon g_T - g_T g_L + g_T g_T(g_T + g_L) , \end{aligned} \tag{4 45}$$

where $\varepsilon = 2 - d$

For long-range correlations $a < d$, one obtains at one loop

$$\begin{aligned} d(\ln D)/dl &= -2 + \nu^{-1}(l) + g_T - g_T/(d-1) , \\ d(g_T)/dl &= \varepsilon g_T - 2g_T^2 , \\ d(g_T)/dl &= \varepsilon g_L - 2g_T g_T , \end{aligned} \tag{4 46}$$

with $\varepsilon = 2 - a$ in this case

4.3 2 The anomalous diffusion behaviour: ε-expansion and exact results[*)]

The RG equations (4.45) and (4.46) behave in a drastically different way in the different regions of the (d, a) plane of fig 4 4·

(i) When $\varepsilon < 0$, namely for $d > 2$ in the short-range case ($a > d$) and for $a > 2$ in the long-range case ($a < d$), the Gaussian fixed point $g_T^* = g_L^* = 0$ is stable, and $g_{T\,L}(l)$ converge exponentially fast to zero when $l \to 0$. The function $\nu(l)$ is determined by the requirement that a fixed point is reached and thus that $d \ln D/d \ln l \to 0$ at large l, thus $\nu(l) \to 2$ for $l \to \infty$ and $\alpha(b) \sim b^2$, which is the Brownian result ($\nu = 1/2$) Diffusion is thus predicted to be normal in this region, as expected

(ii) When $\varepsilon > 0$, namely when $d < 2$ in the short-range case and $a < 2$ in the long-range case, the flow in the (g_T, g_L) plane corresponding to eqs (4 45) and (4 46) is depicted in figs 4 8a and 4.8b, respectively Non-trivial fixed points appear (even a *line* of fixed points in the long-range case!) The corresponding anomalous diffusion behaviour is obtained (in general, as an ε-expansion) from the behaviour of the function $\nu(l)$, which is determined by the same requirement as above If $\nu(l) \to \nu$ for $l \to \infty$, one has $\alpha(b) \sim e^{l/\nu} \sim b^{1/\nu}$ and hence ν is the diffusion exponent

(iii) For $\varepsilon = 0$ ($d = 2$ if $a > d$, $a = 2$ if $a < d$, corresponding to the boundary of the domain in fig 4 4), the Gaussian fixed point is only marginally stable. $g_T(l)$ and $g_L(l)$ again approach zero as $l \to \infty$, but more slowly than in case (i) (as a power law). As a result, logarithmic modifications of the Brownian diffusion behaviour are found

We now describe the types of diffusion behaviour, which for convenience are summarised in table 4 2

4 3 2 1 "Generic" models $\sigma_L \neq 0$, $\sigma_T \neq 0$ The behaviour of a generic situation is most different in the short-range and in the long-range correlated case. For short-range correlations, the unique fixed

*) It is clear that the diffusion exponent ν as calculated from (4 41) characterizes the *width* of the average diffusion front $\langle P(x, t) \rangle$ In the following, $x^2(t) \approx t^{2\nu}$ is understood in this sense

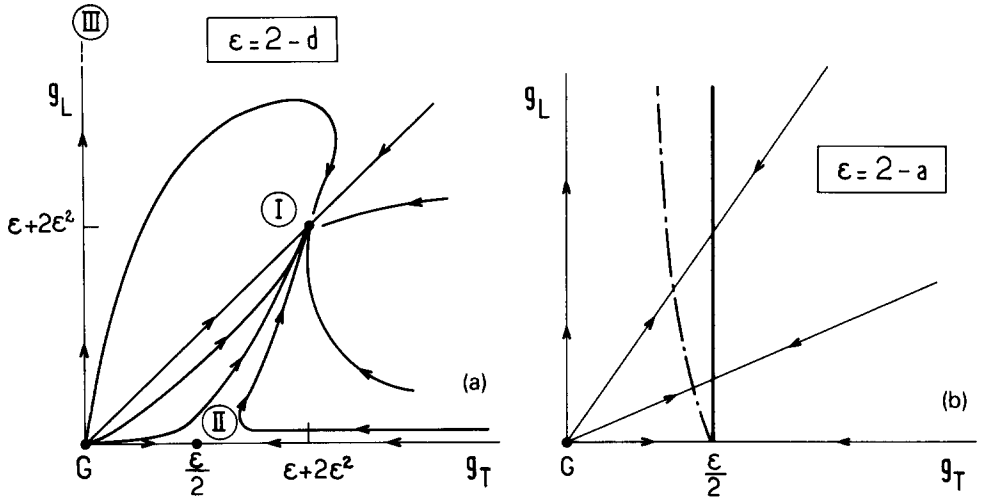


Fig 4 8 Renormalization group flows in the (g_T, g_L) plane (a) in the case of short-range correlations $(a > d)$, (b) in the case of long-range correlations $(a < d)$ Note that there appears a *line* of fixed points, which is shown as a solid line to order ε and as a dotted line to order ε^2

point (I) $g_T^* = g_L^* = \varepsilon + 2\varepsilon^2 + \cdots$ is reached starting from (g_T, g_L) with g_T *and* g_L non-zero At large scales, the possible correlations between each component of the random force at a given point are thus washed out. Corrections to normal diffusion only come in at two-loop order and are universal Using (4.45), they are found to be [Luc83, 84, 86, Fis84]:

$$\text{SR.} \quad \begin{cases} \overline{x^2(t)} \sim t^{1-\varepsilon^2} + \cdots, & d < 2, \\ \overline{x^2(t)} \sim 2D_0 t(1 + 4/\ln t), & d = 2. \end{cases} \tag{4.47}$$

This picture is drastically modified in the long-range case There, a line of fixed point arises [Bou87b(E), Hon88a] This can be shown to hold to all orders in the perturbative expansion [Hon88a] (the one-loop equations of the line $g_T^* = \varepsilon/2$ being of course modified)*). As a result, the anomalous diffusion exponent is *not universal*: it continuously depends on the ratio $\rho = \sigma_L/\sigma_T$ measuring the

Table 4 2
Behaviour of $\overline{x^2(t)}$

	Short range, $a > d$, $\varepsilon = 2 - d$		Long range, $a < d$, $\varepsilon = 2 - a$	
	$d = 2$	$d < 2$	$a = 2$	$a < 2$
"Generic" (σ_L, σ_T)	$2D_0 t(1 + 4/\ln t)$	$t^{1-\varepsilon^2+}$	$t(\sqrt{\ln t})^{(d-1-\rho)/(d-1)}$, $\rho \neq \rho_c$ [a]	$t^{1-(\varepsilon/4)(d-1-\rho)/(d-1)}$, $\rho \neq \rho_c$
			$2D_0 t\left(1 + \frac{4-d}{2(d-1)} \frac{1}{\ln t}\right)$, $\rho = \rho_c$	$t^{1-\varepsilon^2(4-d)/8(d-1)}$, $\rho = \rho_c$
Incompressible div $F = 0$ $(\sigma_L = 0)$	$t\sqrt{\ln t}$	$t^{4/(2+d)}$	$t\sqrt{\ln t}$	$t^{4/(2+a)}$
Potential $F = -\nabla U$ $(\sigma_T = 0)$	$t^{2\nu}$, $\frac{1}{2\nu} \equiv 1 + \frac{\sigma_L}{8\pi D_0^2}$	$(\ln t)^{4/(2-d)}$	$t^{2\nu}$, $\frac{1}{2\nu} = 1 + \frac{c_d}{2d} \frac{\sigma_L}{D_0^2} +$	$(\ln t)^{4/(2-a)}$

[a] $\rho_c = d - 1 +$

*) This results from the fact that the 1PI function $\Gamma_{\phi\phi\phi\phi}$ in the field-theoretical representation of section 2 2 4 does not diverge in the long-range case As a result $\delta\sigma_T = \delta\sigma_L$ to all orders and thus the relation $g_T\beta_L + g_L\beta_T = 0$ holds between β-functions

"compressibility" of the medium One obtains

$$\text{LR} \quad \begin{cases} \overline{x^2(t)} \sim t^{2\nu}, \quad 2\nu = 1 - \dfrac{\varepsilon}{4}\dfrac{d-1-\rho}{d-1} + O(\varepsilon^2), & a<2, \\ \overline{x^2(t)} \sim t(\ln t)^{(d-1-\rho)/(2d-2)}, & a=2 \end{cases} \tag{4 48}$$

This corresponds to hyperdiffusion for $\rho < \rho_c$ and hypodiffusion for $\rho > \rho_c$, where ρ_c is a critical value equal to $\rho_c = d - 1$ at one-loop order

Exactly at $\rho = \rho_c$, a two-loop analysis is required and yields [Hon88a]

$$\rho = \rho_c \quad \begin{cases} 2\nu = 1 - \varepsilon^2(4-d)/8(d-1) + \quad , & a<2, \\ \overline{x^2(t)} \sim 2D_0 t\left(1 + \dfrac{4-d}{2(d-1)}\dfrac{1}{\ln t}\right), & a=2 \end{cases} \tag{4.49}$$

The relevance of these results to turbulent diffusion (section 4 1 2.1) is intriguing Despite the manifest oversimplicity of the model, could some continuous dependence on compressibility be observed for pair diffusion in compressible turbulent flows?

Remark All the results quoted above are ε-expansions such that
- in the short-range case, $\varepsilon = 2 - d$ and a is kept constant, $a > d$,
- in the long-range case, $\varepsilon = 2 - a$ and d is fixed to a value $d > 2$

In the neighbourhood of the point $a = d = 2$, a *simultaneous expansion in both parameters* $2 - d$ and $2 - a$ must be performed, and the precise location of the boundaries of "short-range" and "long-range" universality classes must be determined This analysis has been performed at one-loop order in [Gev87, Hon88a], in the intermediate region, a third "mixed" universality class is found (whose boundaries are depicted in fig 4 4) In this region, different fixed points appear and it turns out furthermore that the diffusion laws can be modulated by oscillatory amplitudes [Gev87] As expected, the diffusion exponents are found to vary in a continuous way from one region to another (on this point, see also [Hon88b])

4 3 2 2 Incompressible models. exact diffusion behaviour. The diffusion behaviour of incompressible models, $\sigma_L = 0$, is associated with the fixed point (II): $g_L^* = 0$, $g_T^* \neq 0$ in figs 4 8a, b However, remarkably enough, an all-order analysis can be performed for these models, allowing one to obtain the exact diffusion behaviour for arbitrary values of a and d (not necessarily close to 2). This was first realized by Forster, Nelson and Stephen [For77] in a different context and reanalyzed recently [Hon88a] It stems from the fact that, due to symmetry properties [Hon88a], the RG equations (4.45), (4 46) involve only *one* independent function,

$$\mathrm{d}\ln D/\mathrm{d}l = -2 + \nu^{-1}(l) + \varphi(g_T), \qquad \mathrm{d}g_T/\mathrm{d}l = g_T[\varepsilon - 2\varphi(g_T)]. \tag{4 50}$$

$\varphi(g_T)$ is not known to all orders, but, from (4.50), its value at the non-trivial fixed point (if any) *has to be* $\varphi(g_T^*) = \varepsilon/2$ and the diffusion exponent reads

$$\nu = (2 - \varepsilon/2)^{-1} = \begin{cases} 4/(2+d), & 1<d<2 \quad \text{(SR)}, \\ 4/(2+a), & a<2 \quad \text{(LR)} \end{cases} \tag{4 51}$$

The self-consistent ("Flory-type") approximation devised in section 4.2.1 is thus *exact* for incompressible models (In one dimension, where the constraint $\mathrm{div}\, F = 0$ implies a constant F, the non-trivial fixed point presumably no longer exists) The implications of (4.51) for turbulent diffusion have been emphasized in section 4 1.2.1)

4 3.2.3. Potential case· strong disorder fixed point and logarithmic diffusion. For potential models ($\boldsymbol{F} = -\mathrm{grad}\, U$, $\sigma_T = 0$), it is seen from (4.45) and (4.46) that no corrections to the RG function $\mathrm{d}g_L/\mathrm{d}l$ arise up to two-loop order. This is indeed not fortuitous, and it has first been shown in [Bou87b] that it holds true to all orders of perturbation theory for both short-range and long-range correlations (see also the arguments of [Kra86b], and the recent detailed proof given in [Hon89a]) Thus $\mathrm{d}g_L/\mathrm{d}l = \varepsilon g_L$, and

$$g_L(l) = g_L^0 b^\varepsilon = g_L^0\, \mathrm{e}^{\varepsilon l}\,.$$

In the marginal situations ($d = 2$ for short-range, $a = 2$ for long-range), $g_L(l)$ thus remains equal to its bare value $g_L^0 = \sigma_L D_0^{-2} c_d (1 - d^{-1})$, and this results in a non-universal diffusion exponent continuously depending on the strength of disorder,

$$\begin{aligned} &\frac{1}{2\nu} = 1 + \frac{1}{8\pi}\frac{\sigma_L}{D_0^2}\,, \qquad d = 2 \quad (\mathrm{SR})\,, \\ &\frac{1}{2\nu} = 1 + \frac{c_d}{2d}\frac{\sigma_L}{D_0^2} + \cdots, \quad a = 2 \quad (\mathrm{LR})\,. \end{aligned} \tag{4.52}$$

(In [Bou87b(E)], it was suggested that these expressions could well have no corrections at higher orders in σ_L; this has indeed very recently been proven to hold in the SR case; see [Hon89b]. For the LR case, see [Der90].)

Remark The physical origin of this dependence of the diffusion exponents on the strength of disorder is the logarithmic increase of the typical potential barrier with distance (see also [Tos89]),

$$\Delta U(x) \sim \sqrt{\sigma_L \ln x} \tag{4 53}$$

This raises the question why the results (4 52) (which are derived from a RG analysis of the *disorder-averaged* quantity $\langle P(x,t)\rangle$) do not agree with a naive Arrhenius argument Indeed, (4 53) suggests that the typical diffusion time over a distance x is of order

$$t \sim x^2 \exp(\sqrt{\sigma_L \ln x}) \tag{4 54}$$

(a trial frequency of order x^2, corresponding to free diffusion, has been included) This argument suggests the following typical diffusion behaviour·

$$\overline{x^2(t)} \sim \sqrt{t}\, t^{-C\sqrt{\sigma_L/\ln t}}\,, \tag{4 55}$$

which is not a pure power law (though it does depend continuously on σ_L) In a restricted time interval (4 55) looks like an effective power law

$$t^{1/2 - C\sqrt{\sigma_L}} \tag{4 56}$$

When disorder is relevant (that is, for $d < 2$ in the short-range case and for $a < 2$ in the long-range case), $g_L(l)$ is driven to infinity, and one should know the large-l behaviour of the RG function $\mathrm{d}g_L/\mathrm{d}l$ to conclude. However, since an infinite disorder fixed point controls the physics at large scale, it is most likely that the activation argument leading to Sinai's behaviour in the one-dimensional case (section 3.3.2) still applies This suggests that *logarithmic diffusion* generalizing Sinai's finding *does exist in arbitrary dimension*, provided the typical potential barrier increases with distance,

$$\begin{aligned} &\overline{x^2(t)} \sim (\ln t)^{4/(2-d)}\,, \quad d < 2 \quad (\mathrm{SR})\,, \\ &\overline{x^2(t)} \sim (\ln t)^{4/(2-a)}\,, \quad a < 2 \quad (\mathrm{LR})\,. \end{aligned} \tag{4.57}$$

The last result has indeed been proven rigorously to hold for a closely related hopping model on a lattice, by Durrett [Dur86]

Numerical simulations of potential models have recently been performed by Pettini [Pet89, Bou89h] in $d = 2$ for different values of a Preliminary results are displayed in fig 4.9. They indeed provide evidence that diffusion is normal for $a > 2$, slower than any power law for $a < 2$ and depends continuously on σ_L (becoming slower when σ_L increases) for $a = 2$ No quantitative fit of the diffusion behaviour has yet been made for these results

4 3.3 *The effect of a small bias* F_0

In the presence of an average bias F_0, one expects that in general long-range time correlation can no longer be induced and that an asymptotic velocity V will characterize the large-time behaviour of the mean position How V depends on F_0 is the question one wants to answer. This can be studied using the weak-disorder expansion techniques introduced in section 4 2 2. A one-loop calculation of the self-energy (4 23) yields the correction to V at first order in $G_{T\,L}$,

$$\Sigma^{(1)}(\boldsymbol{k}) = \boldsymbol{k} \cdot \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\boldsymbol{q} G_L(q^2)}{D_0 q^2 - \mathrm{i}\boldsymbol{F}_0 \cdot \boldsymbol{q} + E} + O(\boldsymbol{k}^2) \tag{4 58}$$

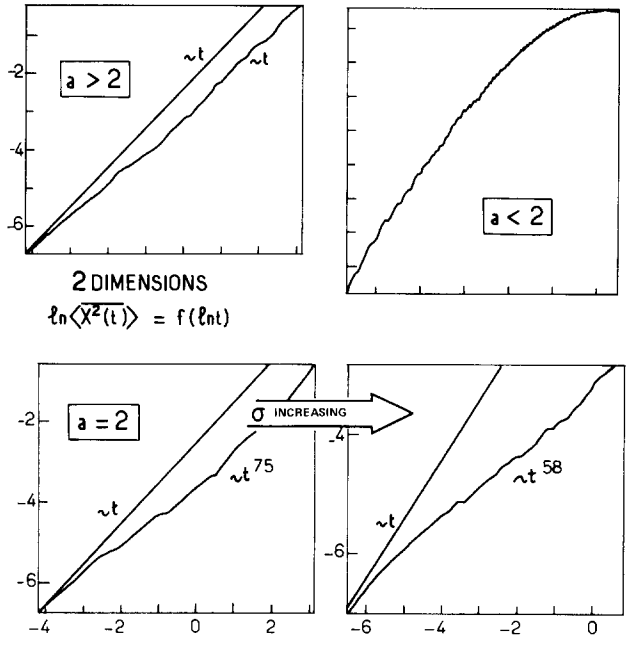


Fig 4 9 Numerical simulations performed by Pettini (unpublished) [Pet89, Bou89h] on the potential model in two dimensions exhibiting power law correlations These results are in qualitative agreement with the predicted behaviour (4 52)–(4 57) Note in particular the dependence of ν on σ for $a = 2$

For $d>2$ in the short-range case, and $a>2$ in the long-range one, this correction does not display any large-distance singularity when F_0 goes to zero, and one can safely expand to find the linear response in F_0,

$$V = F_0\left(1 - \frac{1}{dD_0^2}\int \frac{d^d q}{(2\pi)^d}\,\frac{G_L(q^2)}{q^2}\right).$$

As is clear from expression (4.24) of the diffusion constant to the same order, this expression does not in general satisfy the Einstein relation, $V=[D(F_0=0)/D_0]F_0$, except in the potential case $G_T=0$ (this is connected with the existence of a stationary state without steady-state current, see chapter 5).

When disorder becomes relevant, (4 58) behaves as a non-linear power of F_0 for small F_0, signalling failure of linear response theory; indeed, it has been emphasized in section 4 2.2.3 that the limit $F_0\to 0$ is also a critical limit Here again, RG methods have to be used [Luc83, Bou87b] to find the dependence of V on F_0 Along lines very similar to the above, one finds that linear response is violated only be logarithmic factors in marginal cases, and that for $d<2$ (SR) and $a<2$ (LR), a non-linear behaviour characterized by a new critical exponent φ is found,

$$V \sim F_0^{\varphi}. \tag{4.59}$$

As above, φ can be expanded in powers of $\varepsilon = 2-d$ (SR) or $2-a$ (LR); the results are displayed in table 4 3. Incompressible and potential models again turn out to be special cases. In the former case, no correction to the velocity of the pure model (and thus to linear response) is found at any order. $V=F_0$ (this follows from the fact that the steady-state current is simply proportional to $F(x)$ since $P=$ constant is the steady state in the case div $F=0$, see section 2.1.4) For potential models, a non-universal law is found in the marginal cases, while for $d<2$ (SR) or $a<2$ (LR), perturbative calculations do not allow a conclusion. Thus, the nature of the response to an external bias for potential models remains an open problem: it could well be that a zero-velocity phase (for F_0 below a threshold F_{0c}) does exist, as in one dimension (Let us remark that the correct one-dimensional behaviour does show up in (4.58) [Luc83])

In the presence of F_0, the weak-disorder expansion of the diffusion constant involves the integral

$$\int d^d q\,\frac{G_{T,L}(q^2)}{D_0 q^2 + i\boldsymbol{F}_0\cdot\boldsymbol{q} + E}, \tag{4.60}$$

Table 4 3
Response $V[F_0]$

	Short range, $a>d$, $\varepsilon=2-d$		Long range, $a<d$, $\varepsilon=2-a$	
	$d=2$	$d<2$	$a=2$	$a<2$
"Generic" (σ_L, σ_T)	$F_0/\ln(1/F_0)$	$F_0^{1+\varepsilon}$	$F_0(\ln 1/F_0)^{-\rho/2(d-1)}$, $\rho\neq\rho_c$	$F_0^{1+\rho\varepsilon/2(d-1)}$, $\rho\neq\rho_c$
Incompressible div $\boldsymbol{F}=0$ ($\sigma_L=0$)	$V=F_0$	$V=F_0$	$V=F_0$	$V=F_0$
Potential $\boldsymbol{F}=-\nabla U$ ($\sigma_T=0$)				

in which the limit $E \to 0$ should be taken before the small-F_0 behaviour is studied It is easily seen that. – in the SR case a finite "dispersion" constant is found for any dimension $d > 1$, with a non-linear dependence on F_0 for $1 < d < 2$,

$$D \sim F_0^{(1/\nu - 2)\varphi} \sim V^{1/\nu - 2} ; \tag{4.61}$$

– in the LR case, a finite D is found only for $a > 1$ (with again $D \sim F_0^{(1/\nu - 2)\varphi}$ for $1 < a < 2$, for $a < 1$, the $E \to 0$ limit gives rise to a divergence in (4.60) even at finite F_0, signalling an *anomalous dispersion regime*, first analyzed in [Koc89] (see also [G14, Dou89a]) for incompressible models, for which one can show that

$$\Delta x \sim t^{1 - a/2} \tag{4.62}$$

This is also discussed in section 5.7. In the general case, a RG expansion of ν in powers of $1 - a$ could be performed.

Thus we see that a variety of responses to a small bias can be observed in a disordered medium.
– linear response obeying the Einstein relation,
– linear response violating the Einstein relation,
– violation of linear response: $V \sim F_0^{\varphi}$,
– anomalous drift behaviour: $\bar{x} \sim t^{\alpha}$
Chapter 5 is devoted to an analysis of such problems in a more general framework.

5. Response to a bias, dispersion effects

We would like to discuss, in this chapter, how an external bias may perturb the diffusing particle. This situation has already been encountered in chapter 3 and section 4 3.3; we intend to give here a wider overview of the effect of a bias and of the resulting evolution laws. We first discuss the general features of the response to a *weak bias* in disordered media (section 5.1–5 4), and then turn to *dispersion* effects (5.6) *Strong bias* situations are considered in section 5 5.

5 1. The effect of a weak bias on a normal diffusion process: linear response and the validity of Einstein's relation

5.1 1. The case of a homogeneous medium

Consider first an unbiased random walker in a *homogeneous* medium, that we take for simplicity to be a one-dimensional lattice*) If one imposes a weak external force F_0, the hopping rates will read

$$W_{n,n+1} = W_0 \, \mathrm{e}^{-aF_0/2kT} = W_{\leftarrow} , \qquad W_{n+1\,n} = W_0 \, \mathrm{e}^{+aF_0/2kT} = W_{\rightarrow} \tag{5 1}$$

From the general relations (3.23), one directly obtains $\bar{x} = Vt$ with

$$V = a(W_{\rightarrow} - W_{\leftarrow}) , \tag{5 2}$$

*) One could have chosen to work with the continuous Langevin equation $\gamma \dot{x} = F_0 + \eta(t)$, for which it is easy to prove that $D = kT/\gamma$, a relation equivalent to (5 3)

or, for $F_0 \to 0$,

$$V = (W_0 a^2/kT)F_0 \equiv D_0 F_0/kT\,, \tag{5.3}$$

where $D_0 = a^2 W_0$ is the diffusion constant when $F_0 = 0$.

Thus, the response to a weak uniform external field exhibits three main features in the absence of disorder:

(i) a *velocity* appears: the mean position increases linearly with time;

(ii) this velocity depends *linearly* on the external force;

(iii) the mobility $m = V/F_0$ is equal to the diffusion constant without bias divided by kT (Einstein's fluctuation–dissipation relation).

Let us emphasize that on a discrete lattice, linear response only holds as long as the external bias is weak The full expression for V indeed reads

$$V = 2W_0 a \sinh(aF_0/2kT)\,. \tag{5 4}$$

5 1.2. Validity of Einstein's relation in a disordered medium

As is already clear from the results of section 3.3 and 4.3.3, disorder may ruin any of the above three properties However, when the unbiased diffusion process is *normal*, (i) and (ii) still apply in most cases; however, point (iii) can be in trouble, depending on the type of disorder at hand. We briefly summarize here the situation for models A, B and C when unbiased diffusion is normal. In some exceptional circumstances, properties (i) or (ii) can also be violated even when the unbiased diffusion is normal; such an example will be discussed in section 5.5.

* Type A models (symmetric random barriers) always obey Einstein's relation. In one dimension, this is easily proven from the general formulas (3.23) and (3 45). For the hopping rates

$$W_{n,n+1} = W_n\, \mathrm{e}^{-aF_0/2kT}\,, \qquad W_{n+1,n} = W_n\, \mathrm{e}^{+aF_0/2kT}\,, \tag{5.5}$$

one indeed obtains (for $F_0 \to 0$)

$$V = a^2 \langle 1/W \rangle^{-1} F_0/kT \equiv D_0 F_0/kT\,. \tag{5.6}$$

Note, however, that the field above which this relation is not valid may be much smaller than kT/a in disordered media [Ric89, Bou89e].

In higher dimensions, the same conclusion can be reached using, e.g., the method presented in [Der83b]· *The Einstein relation holds whenever detailed balance is obeyed.* (This is also true for model B.)

* Type B models (random traps) also obey Einstein's relation when $\langle \tau \rangle$ is finite. This is easily proven following the lines of section 2.4.1. The thermal average of the component of the position parallel to the bias reads

$$\overline{R_t} = a(N_{\rightarrow} - N_{\leftarrow})\,,$$

where $N_{\rightarrow}$ and $N_{\leftarrow}$ are the number of jumps along the bias and against the bias, respectively.

$N = N_{\rightarrow} + N_{\leftarrow}$ is related to the time t by

$$N = t/\langle \tau \rangle , \qquad N_{\rightarrow}/N_{\leftarrow} = e^{aF_0/kT} \simeq 1 + aF_0/kT ,$$

and thus

$$\overline{R(t)} = (t/\langle \tau \rangle) a^2 F_0/kT \equiv (D_0 F_0/kT) t , \tag{5.7}$$

where expression (2 36) of the diffusion constant has been used

* Type C (random force) models *do not obey* Einstein's relation in general, except when the random force is constrained to be the gradient of a potential This is the conclusion reached in section 4 3 3 on the basis of a weak-disorder expansion This is in no way surprising, and holds true in general; only in the potential case does a current-free equilibrium distribution exist (i e , if detailed balance holds) and may a fluctuation–dissipation theorem be proven (see appendix E for a general proof in one dimension [Fei88]). General results can also be reached when the random force is *divergenceless*, in which case the arguments of section 4 3 2 2 based on the known (current carrying) equilibrium distribution leads to

$$\overline{R(t)} = (D_0 F_0/kT) t , \tag{5.8}$$

where D_0 is the diffusion coefficient *in the absence of disorder* and *not* the diffusion coefficient of the disordered problem for zero bias.

These violations of Einstein's relation should, however, not be considered as a specific effect of disorder, indeed, it also holds true when a weak space dependent (but non-random) force is applied to a Brownian motion on a regular lattice

5 2 Linear response and "generalized Einstein relation" in the short-time (high-frequency) regime

We show in this section that, when diffusion is anomalous, linear response still applies at small enough time and that in this regime a generalization of Einstein's relation holds These remarks also apply to the case where the external bias has a harmonic time dependence, in the limit of large frequencies. The derivation presented below applies for a *fixed configuration* of disorder provided the modification due to the bias of the weight of a given trajectory is described by a Boltzmann weight,

$$P_0(\{\boldsymbol{r}_t\}_{t=1} \quad ,_t) \rightarrow P_0(\{\boldsymbol{r}_t\}_{t=1,} \quad _t) \exp\left(\sum_{t=1}^{t} \boldsymbol{r}_t \cdot \boldsymbol{F}_0/2kT \right) \tag{5 9}$$

(In particular, *temperature* must be well defined and constant in space.) In addition, the local force (including the external field) must be weak in order to avoid non-linear effects [the expansion $\exp(-aF_0/kT) \simeq 1 - aF_0/kT$ must be valid to insure $\Sigma_j W_{ij} = 1 + O(a^2)$ *at each site*]

Introducing the length scale ξ_F above which the potential energy gain outweighs the thermal energy (that is, $F_0 \xi_F = kT$), one must have

$$\overline{R(t)} = \overline{\left| \sum_{t=1}^{t} \boldsymbol{r}_t \right|} \ll \xi_F$$

Then it is easy to show that

$$\overline{R_{\parallel}(t)}_F = \frac{\Sigma_{\{r_i\}} R_{\parallel} P_0(\{r_i\}) \exp(\boldsymbol{F}_0 \cdot \boldsymbol{R}/2kT)}{\Sigma_{\{r_i\}} P_0(\{r_i\}) \exp(\boldsymbol{F}_0 \cdot \boldsymbol{R}/2kT)} \underset{F_0 \to 0}{=} \overline{R_{\parallel}(t)_0^2} \frac{F_0}{2kT} \tag{5 10}$$

[assuming $\overline{R(t)}_0 = 0$], where $R_{\parallel}$ denotes the component of $\boldsymbol{R}$ along $\boldsymbol{F}_0$.

This generalization of Einstein's relation to anomalous diffusion is generally valid only for *short times*, in a regime where the mean displacement $\overline{R(t)}_F$ is very small compared to the typical displacement $[\overline{R^2(t)}_0]^{1/2}$ (see fig. 5 1) (More generally, one should be careful that the two limits F small and t small can be non-commuting in some situations; (5 10) then applies provided the short-time regime is taken first, see section 5.4.)

A direct application of the result (5.10) concerns the response of a particle to an external field oscillating at frequency ω. If the field F_0 is such that $\xi_F^2 \gg \overline{R_0^2}(t = \omega^{-1})$ then the displacement will oscillate with an amplitude $A(\omega)$ given by $2kTA(\omega) = F_0 \overline{R_0^2}(\omega^{-1})$. The frequency dependent mobility $m(\omega)$ will thus scale as [Sch73]

$$m(\omega) = \omega A(\omega)/F_0 = \omega \overline{R_0^2}(\omega^{-1})/2kT \tag{5.11}$$

In particular, if free diffusion is characterized by $\overline{R_0^2}(t) \sim a^2 (t/\tau_0)^{2\nu}$, one will observe [Oht84, Gef83, G12]

$$m(\omega) = \frac{a^2}{2\tau_0 kT} (\omega\tau_0)^{1-2\nu} \tag{5.12}$$

in the short-time (high-frequency) regime $aF_0/kT \ll (\omega\tau_0)^{\nu}$

5.3. *The long-time regime: non-linear response, general rule and exceptions*

5.3.1. *Analysis à la Pincus*

In the long-time limit, the perturbative result $\bar{R} \sim F_0$ can break down, since eventually the energy gain due to the field is much greater than kT. In this case, one may borrow arguments from critical

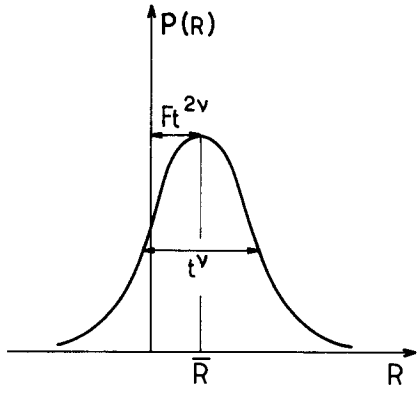


Fig 5 1 Qualitative shape of the diffusion front for a biased walker in the short-time limit where the generalized linear response (5 5) is still valid, the width of the probability distribution is then much larger than the offset of the mean position

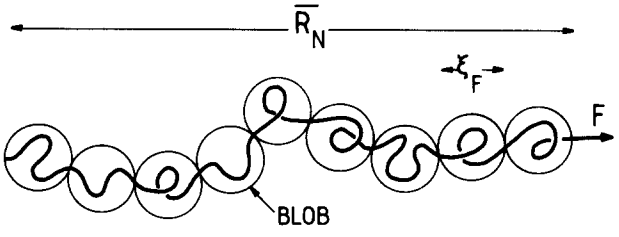


Fig 5 2 Pincus's blob picture of a walk stretched by an external force For length scales smaller than $\xi_F = kT/F$ the walk is almost unperturbed

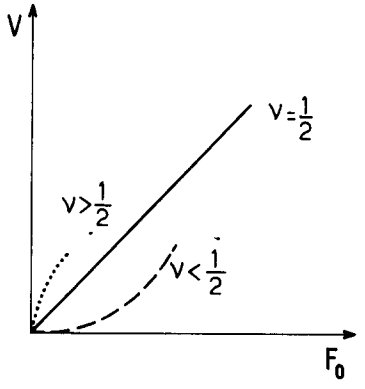


Fig 5 3 Velocity V versus applied force F_0 in the three cases $\nu > \frac{1}{2}$, $\nu = \frac{1}{2}$, $\nu < \frac{1}{2}$

phenomena and especially from polymer physics to obtain a general rule applying when well-identified conditions are satisfied These arguments rely on the following "blob" picture of the walk (originally introduced by Pincus for polymers [Pin76, G16]). At length scales *smaller than* ξ_F, the walk is considered to be unaffected by the field, while for length scales larger than ξ_F, the field dominates and directs the walk along its direction. ξ_F is the crossover length between *zero-field behaviour* and *infinite-field behaviour* (see fig. 5.2).

One may thus write, for $\overline{R(t)}_F \gg \xi_F$,

$$\overline{R(t)}_F = (t/t_\xi)\xi_F \,, \tag{5 13}$$

where t_ξ is the mean time needed for the unperturbed walk to cross the length ξ_F Thus, if t_ξ is finite, one may write (with $\xi_F \sim t_\xi^\nu$)

$$\overline{R(t)}_F \sim t\xi_F^{1-1/\nu} \sim t(F_0/kT)^{(1-\nu)/\nu} \,, \tag{5.14}$$

which shows that in the general case the response is *non-linear in the field* F_0, except in the case of normal diffusion, for which $(1-\nu)/\nu = 1$ Only in this case are the short- and long-time behaviour of $\overline{R(t)}_F$ identical (see fig 5 3)

Remarks

(i) Formula (5 14) might have been obtained by a scaling argument [G16], writing

$$\overline{R(t)}_F = t^{2\nu} F_0 f(\xi_F/t^\nu) \,,$$

with $f(u) \to 1$ for $u \to \infty$, then, demanding that for strong fields $R(t)$ should be linear in t fixes $f(u) \sim u^{2-1/\nu}$ for $u \to 0$, and (5 14) follows

(ii) For $\nu = \frac{1}{2}$, the argument may be made slightly more precise If $\xi_F = (D_0 t_\xi)^{1/2}$, then (5 13) reads

$$\overline{R(t)} = tD_0\xi_F^{-1} = t\,D_0F_0/kT \,,$$

so that one recovers Einstein's relation $m = D_0/kT$

(iii) *Validity of the law (5 14)* The arguments leading to (5 14) fail under two circumstances, which are quite often encountered in anomalous diffusion phenomena

(a) If the unbiased diffusion process is anomalous because of a broad distribution of trapping times, then for $\mu < 1$, (5 13) should read

$$\overline{R(t)}_F/\xi_F \simeq (t/t_\xi)^\mu \,, \tag{5 15}$$

where t_ξ is the typical time needed to cross the distance ξ_F Response in this case will be non-linear *both in time and in the applied field*, this is discussed in more detail below, in section 5 4 1 This would also be the case for Levy flights and for walks on fractals (see chapter 6)

(b) The above argument implicitly assumes that the (connected) correlations in displacements $\langle V(0)\cdot V(x)\rangle$ are negligible beyond ξ_F This may not be the case in highly correlated media, in the case of the stratified medium considered in section 1 3 2, it is easy to show that correlations are important up to scale $\lambda_F = \sigma_V^2/DF_0^3$, which is much larger than ξ_F Writing now $\overline{X(t)}_F = \lambda_F t/t_\xi$ with [see eq (1 70)] $\lambda_F = \sqrt{\sigma_V^{1/2}D^{-1/4}t_\xi^{3/4}}$ leads to the expected result $\overline{X(t)}_F = F_0 t$ Said differently, the enhanced diffusion law corresponds to a scale dependent temperature $T(\xi)$ defined as $\xi \sim [T(\xi)t]^{1/2}$, and the above arguments apply if λ_F is defined as $F_0\lambda_F \sim kT(\lambda_F)$

5.3.2. Non-linear response and the shape of the diffusion front

Yet another (fruitful) way to understand (5.14) [G16] is to write the response equation (5.10),

$$\overline{R(t)}_F = \frac{\int \mathrm{d}R\, RP_0(R,t)\, \mathrm{e}^{F_0R/2kT}}{\int \mathrm{d}R\, P_0(R,t)\, \mathrm{e}^{F_0R/2kT}} \,, \tag{5.16}$$

but instead of expanding the exponential, one looks for the *saddle point* of (5.16) (since F_0R/kT will be large). Writing

$$P_0(R,t) \sim t^{-\nu d} f(R/t^\nu) \,,$$

with $f(u) \sim \mathrm{e}^{-u^\delta}$ for $u \to \infty$, one has to minimize a "free energy", sum of an "elastic" part, $-R^\delta t^{-\nu\delta}$, and an external field part $F_0R/2kT$. This leads to $\overline{R(t)}_F = R_{\text{Saddle}} \sim t^{\nu\delta/(\delta-1)}F_0^{1/(\delta-1)}$ Imposing now that $\bar{R}(t)/t$ has a finite limit for $t \to \infty$ leads to

$$\delta = 1/(1-\nu) \,, \tag{5.17}$$

and to (5.14). The meaning of the equality (5 17) – which we have already encountered many times – is thus clear: the asymptotic shape of the diffusion front is connected to the response of the walk in a weak external field.

5 3 3 Application to the distribution of the magnetization in a ferromagnet

The analysis above applies to the probability distribution of the magnetization of a spin model at its critical point (see section 1 3 4) One can conjecture that it decays asymptotically as

$$P(M, L^d) \sim \exp[-(M/L^{d\nu})^{\hat\delta}] \,, \quad \hat\delta = 1/(1-\nu),\ \nu = (d-2+\eta)/2d \tag{5 18}$$

(This leads, for example, to $P(M,L) \sim \exp(-M^{16})$ for the two-dimensional Ising model!) Response to an external field will correspond to the saddle point of

$$\int \mathrm{d}M \exp(-M^\delta L^{-\nu d\delta} - MH) \,, \tag{5 19}$$

that is,

$$M_{\text{saddle}} \simeq L^d H^{(1-\nu)/\nu}$$

This can be transformed as $H \sim M^{\delta}$, which is the usual non-linear response law at criticality, with the usual relation

$$\delta = \nu/(1-\nu) \equiv (d+2-\eta)/(d-2+\eta)$$

The same argument leads to a prediction for the diffusion fronts on fractals, which we shall describe in the next chapter (section 6 2 3)

5 4. *Broad distribution of trapping times anomalous drift and violation of linear response, applications to the electrical properties of disordered materials*

5 4 1 *The effect of long trapping times*

The response of type-A models (random traps) to a weak bias is studied here, in the case where the distribution of trapping times is broad,

$$\psi(\tau) \simeq \tau^{-(1+\mu)} \quad (\tau \to \infty) \tag{5 20}$$

In this case, it has been shown in section 2 4.1 that the unbiased diffusion process is anomalous, with (for $\mu < 1$)

$$\nu = \mu/2 \quad (d>2), \qquad \nu = \mu/(1+\mu) \quad (d=1) \tag{5.21}$$

Let us apply a weak external bias, such that $aF_0/kT \ll 1$ The position of the walker is related to the number of jumps N by

$$\overline{R(t)}_F = aNaF_0/kT\,, \tag{5.22}$$

where, as in section 2.4 1,

$$t = \frac{N}{N_{\neq}} \sum_{i=1}^{N_{\neq}} \tau_i \simeq \tau_0 N N_{\neq}^{1/\mu - 1} \tag{5.23}$$

In this expression, τ_0 is a microscopic time, and $N_{\neq}$ stands for the number of *different* traps visited by the walker This number is itself affected by the bias, and its thermal average reads (for weak F_0)

$$N_{\neq} = (aF_0/kT)N + C_d \inf(N^{d/2}, N)\,, \tag{5 24}$$

where C_d is a constant which depends on the lattice type only Again, one has to distinguish between the two cases $d>2$ and $d<2$

– *If* $d>2$, one obtains

$$N_{\neq} \sim N \Rightarrow \overline{R(t)}_F \sim (a^2F_0/kT)(t/\tau_0)^{\mu} \tag{5 25}$$

Thus the (thermal average of) the position indeed depends linearly on the bias, but *non-linearly on time*. The exponent characterizing this anomalous drift is μ, and the generalized Einstein relation of section 5 2 is indeed satisfied even at large times

– If $d=1$, one has to compare the two terms in (5 24)

* If $aF_0/kT \ll C_1 N^{-1/2}$, i e , if one is interested in the *weak-bias limit* for large but fixed time, one gets

$$\overline{R(t)}_F \sim F_0 t^{2\mu/(1+\mu)} , \tag{5 26}$$

which, as expected, satisfies the "generalized Einstein relation", eq (5 10)
* In the opposite limit $aF_0/kT \gg 1$, one gets instead

$$\overline{R(t)}_F \sim F_0^{\mu} t^{\mu} , \tag{5 27}$$

which exhibits both a *non-linear dependence on the bias* and an *anomalous drift* behaviour The crossover between these two regimes simply reads

$$(aF_0/kT)(t/\tau_0)^{\mu/(1+\mu)} = 1 . \tag{5 28}$$

5 4 2 The frequency dependent conductivity of Hollandite

"Hollandite" ($K_{1\ 54}Mg_{0\ 77}Ti_{7\ 23}O_{16}$) is a one-dimensional ionic conductor (the charge carrier is K^+) As such, it is very sensitive to defects and impurities along the chain, which can be modelled as *energy barriers* to be crossed by the ion The model proposed by Bernasconi, Beyeler, Strassler and Alexander (BBSA) [Ber79] is a one-dimensional *symmetric* hopping model, with local hopping rates

$$W_{n\,n+1} = W_{n+1,n} = W_0 \, e^{-\Delta_n/kT} \tag{5 29}$$

The "lattice spacing" in this case is the typical distance between impurities a, and $W_0 = D_0/a^2$, where D_0 is the "bare" diffusion constant. BBSA propose an exponential distribution for local barrier heights (T_m is a characteristic temperature),

$$\psi(\Delta) = \begin{cases} e^{-\Delta/kT_m} , & \Delta_2 \le \Delta \le \Delta_1 , \\ 0 , & \text{if not} , \end{cases} \tag{5.30}$$

which induces a hopping rate distribution $\psi(W) \simeq W^{\mu-1}$ with

$$\mu = T/T_m . \tag{5 31}$$

As has been argued in chapter 2, this model is similar at long times to a trapping model with a local trapping time distribution $\psi(\tau) \sim W_0 (W_0 \tau)^{-1-\mu}$. The diffusion law in zero electric field is

$$\overline{R^2(t)}_0 = a^2 (W_0 t)^{2\mu/(1+\mu)} . \tag{5.32}$$

When an external a.c field F_0 of frequency ω is switched on, the dependence of the mobility $m(\omega)$ of the ion of frequency can be deduced from the above results, by replacing t^{-1} by $i\omega$ in the above formulas. The resulting regimes are shown in fig. 5 4.

Remarks

* The "phase shift" between current and voltage may be obtained using Kramers–Kronig relations [Mit89] – or more loosely by replacing $1/t$ by $i\omega$ in the above expressions Its value is shown in fig 5 4
* If the applied field is not small, the behaviour of the position reads

– in the short-time limit

$$\overline{R(t)}_F \simeq a \sinh(aF_0/kT)\,(W_0 t)^{2\nu}\,, \tag{5 33}$$

– in the long-time limit

$$\overline{R(t)}_F \simeq a \sinh(aF_0/kT)\,[\tanh(aF_0/kT)]^{\mu-1}(W_0 t)^{\mu}$$

The corresponding (non-monotonic) behaviour of the mobility versus the applied field for a given frequency is shown in fig 5 5

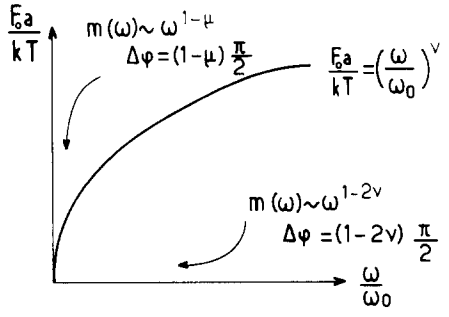


Fig 5 4 Frequency dependent mobility $m(\omega)$ and phase $\Delta\phi$ (between current and field) in the (ω, F) plane exhibiting a non-trivial crossover line between two different laws for $d<2$

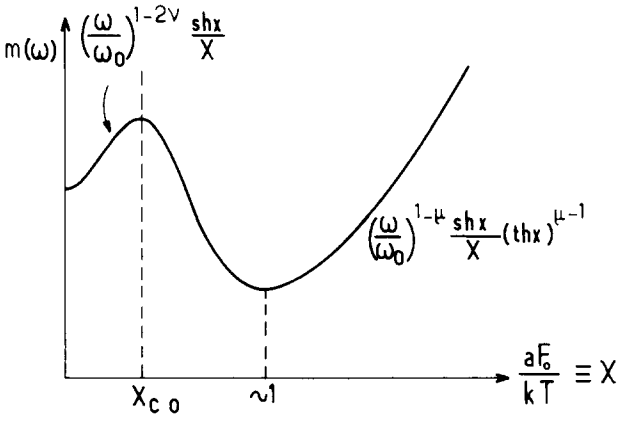


Fig 5 5 Dependence of the mobility $m(\omega)$ for a given frequency on the dimensionless variable aF/kT according to (5 33)

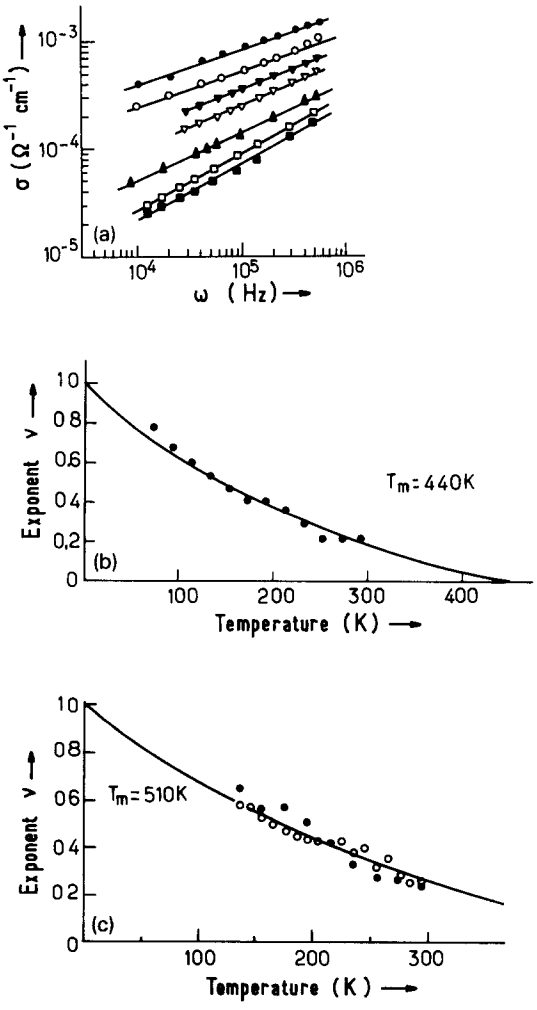


Fig 5 6 Experimental data for ionic conductors [Ber79, Bey81] (a) conductivity (mobility) versus frequency, showing a power law dependence, filled symbols Re σ, open symbols $-\mathrm{Im}\,\sigma$, (b), (c) the dependence of the exponent on temperature, and the fit (5 26) proposed by [Ber79] Note, however, that a purely linear dependence on temperature, as (5 27) would predict, is not a priori excluded

The complex mobility $m(\omega)$ has been measured for Hollandite [Ber79, Bey81] and follows quite accurately a power law,

$$m(\omega) = (i\omega)^{\alpha} , \tag{5.34}$$

with the required [Mit89] phase shift between its imaginary and real components ($\Delta\phi = \frac{1}{2}\pi\alpha$) The authors of [Ber79] implicitly assume that the applied field F_0 is sufficiently small so that the experimental results may be analyzed within linear response theory, eq. (5.26) The exponent α is thus identified with $(1-\mu)/(1+\mu)$, that is, $(T_m - T)/(T_m + T)$ The agreement with experiments at different temperatures is quite good (fig. 5 6) It would, however, be satisfactory to exhibit the crossover towards the non-linear regime by increasing the applied field, and to see the exponent α change from $(1-\mu)/(1+\mu)$ to $1-\mu$ (i e., $1 - T/T_m$) In particular, a straight line going through the experimental points of [Ber79, Bey81] is not a priori ruled out (fig 5 6), which underlines the need for more data. The existence of a crossover would furthermore validate the one-dimensional nature of the problem.

5.4 3 Photoconductivity of amorphous materials

Biased CTRWs with a broad distribution of trapping times have been used to explain anomalous electrical transport in some amorphous insulating materials (e.g. As_2Se_3) (see [Sch75, Pfi78] and references therein). Figure 5.7a shows a typical experimental set-up in which the transient current $I(t)$ through the sample is measured. The observed behaviour (fig 5 7b) is very different from the one expected if diffusion followed a biased Gaussian process. Indeed, $I(t)$ is well described by

$$I(t) \sim t^{-\alpha_1} , \quad t \lesssim \tau(L) ; \qquad I(t) \sim t^{-\alpha_2} , \quad t \gtrsim \tau(L) ,$$

where the sum of the exponents turns out to be close to -2 This is well accounted for if the charge carriers are assumed to perform a biased CTRW with a broad waiting time distribution, $\psi(\tau) \simeq \tau^{-(1+\mu)}$. The first regime corresponds to the motion of the centre of the packet $\overline{x(t)}/t \sim t^{\mu-1}$ For longer times [$\overline{x(t)}$ larger than the sample size], the current is due to the particles which have been trapped for a very long time, and thus decreases as the first passage time distribution, i.e., $t^{-(1+\mu)}$ Note that the sum of

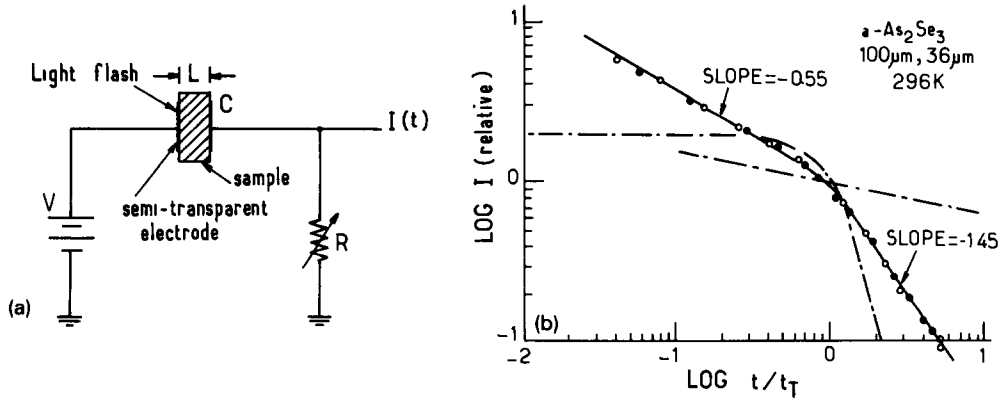


Fig 5 7 (a) Typical experimental set-up a potential drop is applied and a light flash creates mobile carriers (b) The induced transient current as a function of time Note the difference with the expected erf function (dot-dashed curve) if diffusion was normal

the exponents is exactly -2 in this model, which also predicts that the crossover time between the two regimes should scale as $\tau(L) \sim L^{1/\mu}$

5 5 *Strong violation of linear response. the combined effect of bias and geometric disorder*

Particular geometrical structures can induce [Bar83] a highly non-linear effect when an external field is applied Consider, for example, a disorder such that V-shaped obstacles (see fig. 5.8a) or "dead" dangling ends (fig 5 8b) are created Assume furthermore that the sizes l of those objects are distributed according to a certain probability distribution $p(l)$ It is quite obvious that the external field will drive the particle towards the bottom of those "fjords" which will act as traps with release time $t = \exp(F_0 l/kT)$ ($F_0 l$ is the typical energy scale inside a fjord of length l). Two cases can then be distinguished:

(a) $p(l)$ decays more slowly than exponentially with l In this case, as soon as the field is non-zero, the mean trapping time diverges, leading to a creep (anomalous drift) of the particle, corresponding to zero mobility (or more precisely to a time dependent mobility, vanishing as $t \to \infty$) This is to be contrasted with the fact that for a wide class of $p(l)$, the diffusion coefficient is finite when the external field is zero, this is one of the exceptional situations alluded to in section 5 2 There can of course be anomalous trapping (even for $F_0 = 0$) if $p(l)$ is decaying sufficiently slowly, or if *fluctuating local fields* $F(r)$ remain

(b) If $p(l)$ decays exponentially, say as $\exp(-l/l_0)$, then a field-induced dynamical phase transition will occur for a critical value F_c of the bias defined by the divergence of $\langle \tau \rangle$ [Bun86]

$$F_c = kT/l_0 \,, \tag{5 35}$$

This value separates a flow phase ($\bar{x} = Vt$) for $F < F_c$ from an anomalous drift phase $F > F_c$ of the type encountered in chapter 3,

$$\overline{R(t)}_F \sim t^{F_c/F} \tag{5 36}$$

For a sample of size L, the resulting current will have the shape depicted in fig. 5.9. Note that in this case the unbiased diffusion process is normal

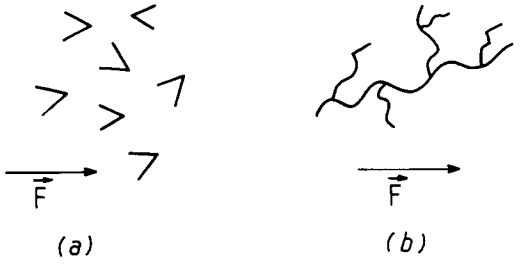


Fig 5 8 Two schematic instances where geometry combined with an external bias induces deep traps (a) V-shaped obstacles, (b) dangling dead ends or backbends

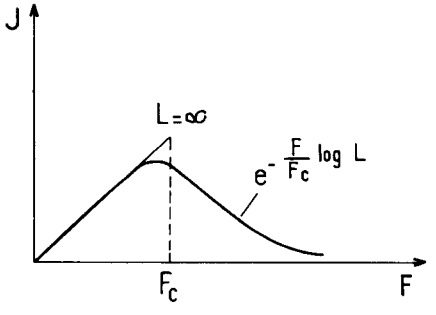


Fig 5 9 Current J versus applied force for a sample containing, say, dead ends with exponentially distributed lengths In this case, a "dynamical" phase transition occurs [Bun86, G12], which is rounded off for a finite-size sample (of length L)

If $p(l)$ decays faster than exponentially, the mobility will always be finite for an infinite sample, although it may be a *decreasing* function of the applied field.

5.6. *Problems with an average flow: dispersion*

We now turn to situations where a finite (not necessarily weak) velocity is *imposed*. Such situations are of great practical importance, e g. in the physics of porous media in which a packet of tracer particles is injected at the entry point of the sample and one observes the *dispersion* Δt of the exit time around its mean value (fig 5 10). Δt is related in a simple way (see below) to the longitudinal diffusion constant of the tracer, through

$$D_{\parallel} = \lim_{x \to \infty} U^3 \, \Delta t^2 / 2x \, . \tag{5 37}$$

Theoretically, one asks how the average velocity reflects the microscopic length and time scales or, said differently, what is the information contained in the $D_{\parallel}$ versus U curve [Saf59]? A classic reference on this subject is the compilation of data due to Fried and Combarnous [Fri71], reproduced in fig 5 11 Experimentally, this represents an important and useful tool to characterize the medium; quite a lot of work has thus been devoted to both aspects (for recent reviews see, e.g., [Koc85, Cha87a, b, Bra88b, Hul88, Bac89]).

5.6.1 *A toy model to understand dispersion laws*

An extremely simplified model [Bou88a] allows to discuss in an elementary way the different regimes to be expected in such situations. The medium is idealized as being one-dimensional (fig 5.12), and made of a "backbone" along which the particle is convected with velocity V At regularly spaced positions (separated by a distance ξ), the particle can leave the backbone for a while, with probability p;

– in one version of the model (fig 5 12a), it enters a trap, where it stays during a given time τ;

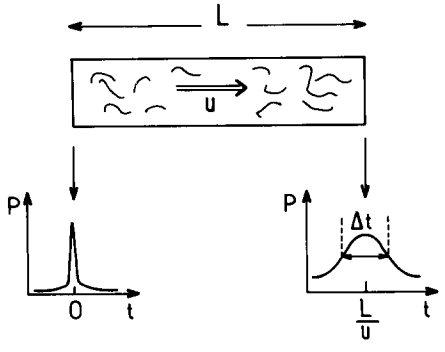


Fig 5 10 Typical *dispersion* experiment A concentration pulse is created at the entry point of a (porous) sample through which a flow is forced The dispersion of exit times Δt is then recorded at the other end of the sample Measurements at intermediate positions in the sample may also be performed, e g , by ultrasonic techniques, see [Bac87]

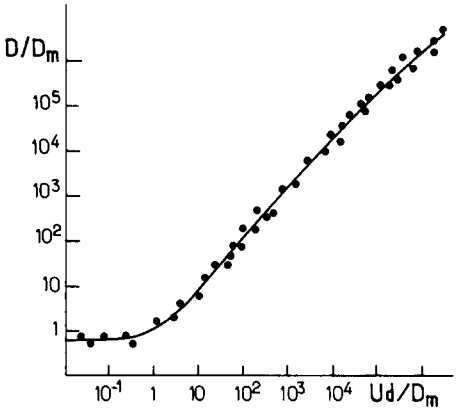


Fig 5 11 Compilation of experimental results [Fri71] showing the dependence of the dispersion coefficient on the flow velocity U

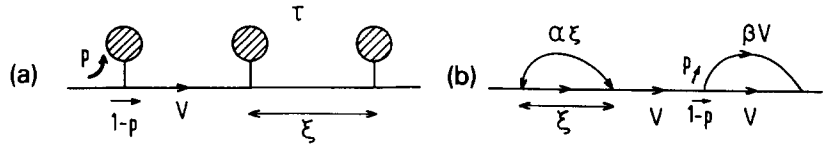


Fig 5 12 Simple model of dispersion [Bou88a] The particle evolves on the backbone with local velocity V Every ξ, it decides either to carry its way through, or to enter (a) a trapping region characterized by a release time τ, or (b) a region of locally slower velocity βV

– in another version (fig 5 12b), the traps are replaced by regions where the flow is locally slower (βV, $\beta < 1$)

All molecular diffusion effects have been neglected in these models, which are only meant to capture the main effects of dispersion For this reason, a complete analytic solution is straightforward We illustrate the calculations on the trapping model, the translation to slow velocity zones being most simple

The easiest quantity to compute is the probability for the particle to reach x at time t, knowing it was at $x = 0$ for $t = 0$ (first passage time or "residence time" distribution) With the notation $N = x/\xi$, one has

$$P(t, x) = \sum_{k=0}^{N} C_N^k p^k (1-p)^{N-k} \delta(t - x/V - k\tau) , \tag{5 38}$$

since no backsteps are allowed. The Laplace transform of $P(t, x)$ thus reads

$$P(E, x) = \mathrm{e}^{-Ex/V}(1 - p + p\,\mathrm{e}^{-E\tau})^N \tag{5 39}$$

Expansion for small E shows that for large t and x, $P(t, x)$ takes the following Gaussian shape:

$$P(t, x) = \frac{1}{\sqrt{2\pi\sigma_t x}} \exp\left(-\frac{(t-\bar{t})^2}{2\sigma_t x}\right) , \tag{5 40}$$

where the average and variance of the first passage time at x read

$$\bar{t} = x\left(\frac{1}{V} + p\,\frac{\tau}{\xi}\right) , \qquad \sigma_t = \overline{t^2} - \bar{t}^2 = x\,\frac{\tau^2}{\xi}\,p(1-p) \tag{5.41}$$

Since backsteps are forbidden (no molecular diffusion), the probability density $P(x, t)$ of finding the particle at x at time t is related to $P(t, x)$ by a "conservation" equation,

$$\int_x^\infty P(x, t)\,\mathrm{d}x = \int_0^t P(t, x)\,\mathrm{d}t . \tag{5 42}$$

It follows from this equation that $P(x, t)$ also takes a Gaussian asymptotic form,

$$P(x, t) = (4\pi D_\parallel t)^{-1/2} \exp\left[-\left(\frac{x - Ut}{2\sqrt{D_\parallel t}}\right)^2\right] , \tag{5 43}$$

where the velocity U and longitudinal diffusion constant $D_{\|}$ read

$$U^{-1} = V^{-1} + p\tau/\xi\,, \qquad D_{\|} = \tfrac{1}{2}p(1-p)U^3\tau^2/\xi\,. \tag{5.44}$$

The analogous expressions for the model with low-velocity bypasses (fig. 5.12b) follow immediately by replacing τ by $(\xi/V)(\alpha/\beta - 1)$ and ξ by 2ξ in these expressions,

$$U^{-1} = V^{-1}[1 + \tfrac{1}{2}p(\alpha/\beta - 1)]\,, \qquad D_{\|} = \tfrac{1}{4}p(1-p)(\alpha/\beta - 1)^2(U/V)^2U\xi\,. \tag{5.45}$$

A commonly used parameter in the physics of porous media is the fraction f of the total volume corresponding to traps (or to slow-velocity bypasses). Assuming the process to be ergodic due to the underlying molecular diffusion, f should be equal to the fraction of time spent in the traps As a function of the above parameters, f thus reads

for model (a). $f = p\tau U/\xi\,,$ (5.46)

for model (b): $f = \tfrac{1}{2}p(\alpha/\beta)U/V\,.$ (5.47)

Using f as a parameter, the above expressions for the diffusion constant can be cast into the form

(a) $D_{\|} = \tfrac{1}{2}fU^2\tau(1 - f\xi/U\tau)\,,$ (5 48)

(b) $D_{\|} = \tfrac{1}{2}fU\xi(\alpha/\beta)(1-\beta/\alpha)^2[1 - f(1+\beta/\alpha)]\,.$ (5.49)

Note that molecular diffusion has been neglected, which is valid provided:

$$\mathrm{Pe} \equiv V\xi/D_0 \gg 1 \quad \text{or} \quad D_{\|} \gg D_0 \tag{5.50}$$

These expressions have an important physical content·

– If trapping regions play the dominant role, with a well-defined release time (due to molecular diffusion, i.e , $\tau = a^2/D_{\mathrm{m}}$), then $D_{\|}$ scales like $U^2\tau$ for sufficiently large velocities; this is the case, for example, in chromatographic or filtration processes. This mechanism also plays some role in porous media at intermediate velocities [Mag89].

– If, on the contrary, the velocity nowhere vanishes but low-flow regions dominate ($\beta \ll 1$), then $D_{\|}$ is *linear* in the velocity,

$$D_{\|} = Ul_d \tag{5.51}$$

This is the law observed in almost all porous media at large velocities (cf fig 5.11); as this toy model clearly emphasizes, it simply follows from the inhomogeneities of the velocity field in the medium ("geometric dispersion"). As is also clear from (5.51), in the presence of very low velocity channels, the dispersion length $l_d = D_{\|}/U$ can be much larger than any "microscopic" length (e g. pore size), and even than the correlation length of the velocity field, since, for small β,

$$l_d \simeq f(1-f)\alpha\xi/2\beta \gg \xi \tag{5.52}$$

Indeed, large values of l_d are often encountered for sintered samples (see, e.g., [Cha87a,b, Bac89]).

Remark An interesting generalization of the above trapping model is when the trapping times are not all equal, but have some distribution $\psi(\tau)$ At each visit of a trap, a value τ is chosen according to that distribution (one has thus a biased CTRW in the sense of section 1 2) The above calculations are easily generalized to that case, since $P(t; x)$ reads

$$P(t, x) = \sum_{k=0}^{N} C_N^k p^k (1-p)^{N-k} \int \cdots \int \prod_i \mathrm{d}\tau_i \, \psi(\tau_i) \delta(t - x/V - \Sigma_{i=1}^{k} \tau_i) \,, \tag{5 53}$$

and thus, in Laplace transform,

$$P(E, x) = \mathrm{e}^{-Ex/V} [1 - p + p\hat{\psi}(E)]^{x/\xi} \tag{5 54}$$

Expressions (5 44) are generalized as follows

$$U^{-1} = V^{-1} + p\langle\tau\rangle/\xi \,, \qquad D_{\|} = (pU^3/2\xi)(\langle\tau^2\rangle - p\langle\tau\rangle^2) \tag{5 55}$$

In particular, for small p and as a function of the fraction f,

$$D_{\|} = \tfrac{1}{2} f U^2 \langle\tau^2\rangle / \langle\tau\rangle \,. \tag{5 56}$$

It is important to notice that it is the *ratio* $\langle\tau^2\rangle/\langle\tau\rangle$ which fixes the time scale

5 6 2 Anomalous dispersion

5 6.2.1 Trapping. If the second moment of the trapping time distribution diverges, it is apparent from (5 55) that $\Delta x^2(t)$ grows faster than linearly with time It is easy to see that if $\psi(\tau) \sim \tau^{-(1+\mu)}$ with $1 < \mu < 2$ (this case was already encountered in chapter 3), then

$$\overline{x^2(t)} - \overline{x(t)}^2 \simeq t^{2/\mu}$$

In this case the diffusion front never becomes Gaussian (it is rather a Levy law of index μ), and the apparent $D_{\|}$ shows a strong dependence on the size of the sample,

$$D_{\|}(L) \simeq L^{2/\mu - 1} \tag{5 57}$$

5 6.2.2. Long-range correlations of the velocity field. Anomalous dispersion can also arise if the velocity field is strongly correlated [Koc88, 89, G14], i e , if

$$\langle \boldsymbol{V}(0) \cdot \boldsymbol{V}(\boldsymbol{r}) \rangle - V^2 \simeq r^{-a} \tag{5 58}$$

As the mean velocity U of the particle is assumed to be non-zero, the dominant part in the *time* correlation of the velocity is obviously

$$\langle \boldsymbol{V}(0) \cdot \boldsymbol{V}(t) \rangle \sim (Ut)^{-a} \,, \tag{5 59}$$

from which the typical displacement Δx around Ut follows, again using the methods of chapter 1 (see,

however, [Dou89a]),

$$\Delta x(t) \sim \begin{cases} t^{1-a/2}, & a<1, \\ t \ln t, & a=1, \\ \sqrt{t}, & a>1, \end{cases} \tag{5 60}$$

resulting in a scale dependent dispersion coefficient,

$$D_{\|}(L) \sim L^{1-a}. \tag{5.61}$$

In this case, the diffusion front exhibits *two* maxima [Koc88, 89] This "scale effect" is experimentally known in hydrogeology and oil recovery, but apparently the mechanism responsible for it (trapping or correlations) has not been identified. Analysis of the diffusion fronts, which are very different in the two cases, should provide useful information

6. Anomalous diffusion on fractal structures and related transport properties

Anomalous diffusion can also arise from the very geometrical structure of the "substrate" on which the particle is evolving: "culs-de-sac" and loops do slow down diffusion. If they are *present at all length scales*, they may eventually change the diffusion exponent ν. This is what happens on fractals and has been extensively studied since De Gennes' paper on "ants in labyrinths" [dGe76] A comprehensive review on this subject by Havlin and Ben Avraham [G12] has recently been published and we refer the reader to this work for some information and references (see also [JSP84]). For the sake of completeness, we nevertheless summarize some fundamental aspects of "fractology" and give a few examples and open problems.

6.1. *Fractals*: *three characteristic dimensions*

A fractal is characterized by several different dimensions, related to different physical properties Let us quote three of them.

(*a*) *The fractal* (*or mass*) *dimension* relates the mass contained in a sphere to its size ε,

$$M_{\varepsilon} \underset{\varepsilon \to 0}{\sim} \varepsilon^{d_F}. \tag{6.1}$$

For an inhomogeneous structure one should rather look at the "multifractal spectrum", associated with the different *moments* of the fluctuating mass (see, e.g., [Pal88, Fed88] for a review),

$$\langle M_{\varepsilon}^{q}(\boldsymbol{r}) \rangle^{1/q} \sim \varepsilon^{d_q}. \tag{6.2}$$

d_F is not an intrinsic [Van84] property of the fractal but rather describes how the object is embedded in the outer space For example, a straight line has a "fractal" dimension $d_F = 1$, but if it is folded so as to give, e.g., a self-avoiding walk in two dimensions it becomes a fractal object of dimension $d_F = 4/3$ (see, however, [Kou89]).

(*b*) *The spectral dimension* [Ale82, Ram83] can be defined, for example, through the mean volume

occupied by a diffusing dye droplet initially concentrated on a given site After a long time t, this volume grows by definition as

$$V_S(t) \sim t^{d_S/2} \tag{6 3}$$

This volume is clearly invariant under a fractal deformation, d_S is an *intrinsic property of the fractal* For the example considered above $V_S(t) \sim t^{1/2}$ and thus $d_S = 1$ whatever the overall shape of the line on which the particle diffuses.

Alternatively, one could consider the probability of presence at the initial site after a time t; this will decay as

$$P(0, t) \sim V_S^{-1} \sim t^{-d_S/2} \tag{6 4}$$

As was set out in detail in chapter 2, $P(0, t)$ is the Laplace transform of the density of states of the Laplacian operator on the structure The low-energy density of states thus behaves as

$$\rho(E) \underset{E \to 0}{\sim} E^{d_S/2-1} \tag{6 5}$$

The typical distance $R(t)$ travelled by the particle in a time t is obtained as

$$V_S(t) \sim R(t)^{d_F} \Rightarrow R(t) \sim t^{d_S/2d_F} . \tag{6 6}$$

One may thus define an effective, scale dependent diffusion constant,

$$D(\xi) = \xi^{2-1/\nu} , \quad \nu \equiv d_S/2d_F \tag{6 7}$$

The probability distribution $P(R, t)$ is expected to obey a CLT at long time,

$$P(R, t) = [V_S(t)]^{-1} f(R/t^{\nu}) . \tag{6 8}$$

The asymptotic behaviour of $f(x)$ for large x is, as usual,

$$f(x) \underset{x \to \infty}{\sim} x^{\sigma} \, \mathrm{e}^{-x^{\delta}} \tag{6 9}$$

The value of δ is discussed below (section 6.3.1)

Note that the problem of a random walker (for which the weight of one walk is determined step after step) on a fractal does not define the same statistical measure as the "ideal polymer" problem [Mar89] (for which the weight is determined globally [G16]) This remark also holds in Euclidean space for a walk in a potential $V(\boldsymbol{R})$ ($\neq \boldsymbol{E} \cdot \boldsymbol{R}$).

(*c*) *The spreading (or chemical) dimension* (see, e.g., [JSP84, Sta84]) is related to the total volume accessible to t steps (fully "stretched") walks,

$$\hat{V} \sim t^{\hat{d}} \tag{6.10}$$

$\hat{d}$ is an intrinsic property of the structure ($\hat{d} = 1$ for the straight line). The maximal distance (in the

embedding space) $R_{max}(t)$ travelled by the walker is thus given by

$$[R_{max}(t)]^{d_F} = \hat{V} \Rightarrow R_{max}(t) \sim t^{\hat{d}/d_F} \tag{6.11}$$

Obviously, this means that $\hat{d} \leq d_F$, whatever the structure. The "tortuosity" of the fractal thus considerably slows down the walker, who, although walking "straight" in his own space, appears half drunk from the outside! The same argument also leads to the *minimal* time required to cross a given distance R,

$$t_{min} \sim R^{d_F/\hat{d}} . \tag{6.12}$$

(*d*) *Bounds*. We have already mentioned the obvious inequality $\hat{d} \leq d_F$. One can also show that $d_S \leq \hat{d}$, which can be understood as follows: Consider two points A, B separated by a distance R; isolate the minimal path linking those two points along the fractal (this path is $R^{d_F/\hat{d}}$ long) and delete all links not belonging to this shortest path. Diffusion is thus *speeded up* on this restricted linear structure, on which $s(t)$ behaves as $t^{1/2}$ (s is the arc length). The time t_{AB} needed to reach B starting on A on the full fractal will hence be *longer* than $s^2 = R^{2d_F/\hat{d}}$. $t_{AB} \sim R^{2d_F/d_S}$ then yields $d_S \leq \hat{d}$. Thus

$$d_S \leq \hat{d} \leq d_F \tag{6.13}$$

6.2. *Diffusion-related transport properties of fractals*

Many properties of general fractals are shared by the simple "folded chain" structure (for example, a random walk of N steps). If the current (or the walker, the phonon, etc.) cannot cross contact points, the structure is really internally one dimensional and thus $\hat{d} = d_S = 1$, while for the example of a random walk $d_F = 2$. We thus illustrate three important transport properties of fractals on this transparent "toy" model; as we shall see, some of them are quite interesting (see in particular section 6.2.2, which contains new results).

6.2.1 *Electrical properties of fractals*

Combination of the general theorem on resistor network [chapter 2, eq. (2.15)] and of eq (6.7) allows one to argue that the *d.c. conductivity* of a fractal network is scale dependent and behaves as

$$\sigma(\xi) \sim \xi^{2-1/\nu} \tag{6.14}$$

The resistance $\mathcal{R}$ between two points A, B separated by a distance ξ is obtained by arguing further that the current distribution extends over a "volume" ξ^{d_F} made of ξ^{d_F-1} "channels" of length ξ in parallel (see fig. 6.1). The resistance $\mathcal{R}(\xi)$ thus scales as

$$\mathcal{R}(\xi) \sim \xi/[\xi^{d_F-1}\sigma(\xi)] \sim \xi^{1/\nu - d_F} . \tag{6.15}$$

For the example of a "folded chain", $\nu = 1/2d_F$ and $\mathcal{R}(\xi) \sim \xi \sim s$: the end to end resistance of a linear structure is obviously proportional to the number of links. The problem becomes far more interesting if one lets the current flow through (quasi-) contact points. One must then distinguish two cases:

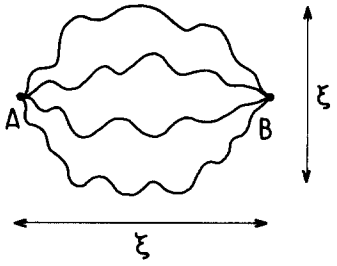


Fig 6 1 The paths contributing to the conductance between two points A and B which are ξ apart lie within a volume ξ^d, hence there are $\simeq \xi^{d-1}$ channels of length $\simeq \xi$ in parallel

(a) The fraction f of links belonging to no loops (i.e., the number of "hot" bonds carrying all the input current) is *finite*. In this case, only the prefactor of (6.15) changes, but the exponent remains equal to d_F

(b) This fraction f is equal to zero. Cross-links in this case may deeply affect the topological nature of the chain and thus change the spectral dimension to $d_S > 1$ This would correspond to an enhanced diffusion along the structure, and an end to end resistance growing as $\mathcal{R}(\xi) \sim s^{2/d_S - 1} \ll s$ Numerical experiments have been performed for random walks in three [Mov88] dimensions, indeed suggesting $d_S(d = 3) = 1\ 14$ (for the same problem in $d = 2$, see [Man89]) A value of d_S for *self-avoiding chains* in two and three dimensions with cross-links was proposed in [Bou87d], assuming $f = 0$. In this case d_S was related to the statistics of large loops within a polymer [dC180] Recent numerical work on SAWs [Sen88, Yan90, Bar90], however, seems to indicate that f is non-zero. A detailed study of the *distribution* of f would be highly interesting: does it peak for infinite chains?

6.2.2 Biased diffusion on fractals: linear and non-linear response

6 2 2 1 Static field. According to the general discussion of chapter 5, in the *short-time* regime (where $R(t) \lesssim kT/F_0$) the response of the walker is linear in the applied field (see also [Oht84], where this result was obtained for the percolation cluster),

$$\overline{R(t)}_F \simeq (F_0/kT)t^{2\nu} \tag{6.16}$$

The crossover time is thus

$$t^* = (kT/F_0)^{1/\nu} \tag{6 17}$$

For longer times, the diffusion behaviour is strongly dependent on the structure and anisotropy of the fractal Take any configuration of the folded chain that we consider, provided it is globally *isotropic* (that is, the mean value of the curve's tangent is zero, $(\langle \partial \boldsymbol{r}/\partial s \rangle = \boldsymbol{0})$, and assume that the external field is directed along z To reach a point B characterized by an abscissa z_B, the particle will typically have to go through points on the chain of abscissa of order $-z_B$. The potential barrier that the particle has to cross is hence of order $F_0 z_B$, and this will take a time

$$t \sim t^* \exp(F_0 z_B/kT) \tag{6 18}$$

The typical distance spanned by the particle is thus [Bar83]

$$\overline{R(t)} \sim \overline{z(t)} \sim (kT/F_0) \ln(t/t^*) \tag{6.19}$$

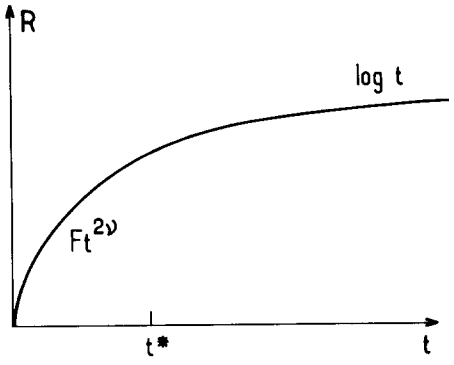


Fig 6 2 Time dependence of the average position of a biased walker on an isotropic structure, for long times, the fact that energy barriers of height $\simeq FR$ must be crossed induces a logarithmic progression of the walker [Rom88, G12, Rou87a]

(fig. 6.2) *independently* of, e g , *the fractal dimension* of the structure. This result was obtained in [Rom88] through a somewhat more involved analysis. If the structure is a simple random walk $\boldsymbol{R}(s)$ characterized by

$$\langle \partial \boldsymbol{R}/\partial s \rangle = \boldsymbol{V}_0 , \qquad \langle \partial \boldsymbol{R}/\partial s \, \partial \boldsymbol{R}/\partial s' \rangle_c = D_0 \delta(s - s') , \tag{6.20}$$

then the problem at hand is *equivalent* to the one-dimensional random force model considered in chapter 3; indeed, one has $V(s) = \boldsymbol{F}_0 \cdot \boldsymbol{R}(s)$ and hence, the local force along the structure (defined as $f(s) = -\partial V/\partial s = -\boldsymbol{F}_0 \cdot \partial \boldsymbol{R}/\partial s$) has a Gaussian distribution characterized by

$$\langle f \rangle = \boldsymbol{F}_0 \cdot \boldsymbol{V}_0 \equiv F_0 V_0 \cos\theta , \qquad \langle f(s) f(s') \rangle_c = F_0^2 D_0 \delta(s - s') \tag{6.21}$$

The walker will thus evolve according to the laws derived in chapter 3. In particular, the parameter μ is equal to

$$\mu = \frac{2kTF_0V_0 \cos\theta}{F_0^2 D_0} = \frac{kT}{F_0 l_d} \cos\theta , \tag{6 22}$$

where $l_d = D_0/2V_0$ is the "diffusion length" of the structure. For $\mu < 1$, the particle will thus evolve as $R(t) \sim t^{\mu}$, since $s \sim t^{\mu}$. Note that the stronger the field the weaker is μ, and thus the slower is the drift motion!

More generally, if the fractal is such that the minimum of the potential barrier over all paths joining A to B grows as $F_0 R^{\alpha}$ ($\alpha \leq 1$), the resulting diffusion behaviour will be

$$\overline{R(t)} \sim (\ln t)^{1/\alpha} \tag{6 23}$$

The "skewness exponent" α can be less than one if the fractal is anisotropic and/or if the "minimal paths" are sufficiently "straight" (for example, on the Sierpinsky gasket). Numerical simulations suggest $\alpha \simeq 1$ for percolation clusters [Hav86, G12, Bun87]. The picture developed in section 6.3.3 indeed leads to $\alpha \equiv 1$.

6.2 2.2. The finite-frequency case [Bou90]. We have learned from the above example that the dramatic effect of a static field on a random structure is to create high effective potential barriers, which considerably slow down the progression of the particle This effect is felt by the particle for times longer than t^* If the external field *oscillates* at a frequency such that $\omega^{-1} < t^*$, the trapping mechanism cannot operate and linear response theory is correct: the amplitude of motion in a field $F(t) = F_0 \sin(\omega t)$ is simply obtained as $A(\omega) \sim F_0 \omega^{-2\nu}$, for $\omega^{\nu} \gg F_0$ [Oht84] For lower frequencies (longer times) two lapses of time can clearly be distinguished during each period.

An "active" phase, when the field $F(t)$ is small, for which the law (6 16) is valid,

$$\overline{R(t)}_F \sim [F(t)/kT] t^{2\nu} ,$$

or, with $F(t) \sim F_0 \omega t$,

$$\overline{R(t)}_F \sim (F_0/kT) \omega t^{1+2\nu} \tag{6 24}$$

This is consistent until $F(t)\overline{R(t)}_0$ is of order kT, that is,

$$F_0 \omega t_1^{(1+\nu)} \sim kT \tag{6.25}$$

"A passive" phase, in which the field becomes too strong and the particle is effectively trapped in "backbends" and essentially does not move (in fact it does progress logarithmically). This phase takes place for times $t_1 < t < 2\pi/\omega - t_1$.

The maximal displacement is thus entirely built up during the active phase $[0, t_1]$, and therefore reads

$$A(\omega) \simeq \overline{R(t = t_1)}_F \sim (F_0/kT) \omega t_1^{1+2\nu} ,$$

or, using (6 25),

$$A(\omega) \simeq (kT/F_0\omega)^{\nu/(1+\nu)} \tag{6.26}$$

(for $\omega \ll t^{*-1}$, which corresponds to $t_1 \ll t^*$). Note that again the amplitude *decreases* with increasing field. For a given frequency, the low-field (6.16) and strong-field (6.26) regimes can be summarized by a scaling expression,

$$A(\omega, F_0) = (F_0/kT) \omega^{-2\nu} f(F_0/F^*) , \tag{6 27}$$

with

$$F^* \simeq \omega^{\nu} ,$$

$$f(x) \underset{x \to 0}{\longrightarrow} 1 , \qquad f(x) \underset{x \to \infty}{\longrightarrow} x^{-\alpha} , \quad \alpha = (1 + 2\nu)/(1 + \nu) \tag{6.28}$$

Harder et al [Har86] have numerically investigated this problem on the two-dimensional percolation cluster, for which $\nu \simeq 0.35$. They find precisely the scaling form (6.27) with $F^* \sim \omega^{0\,3}$, while they estimate $\alpha \simeq 1.5$. Our prediction for α is $\alpha = 2(d_F + d_S)/(2d_F + d_S) = 1.26$ (see fig. 6 3) This agrees quite well with their results.

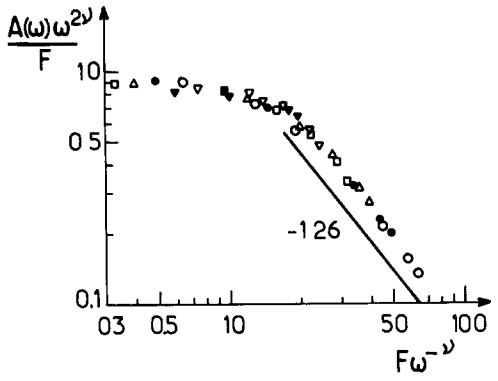


Fig 6 3 Biased motion on the percolation cluster numerical results of [Har86] for various values of the frequency, rescaled according to (6 27) $A(\omega)\omega^{2\nu}/F$ versus $F\omega^{-\nu}$ The asymptotic behaviour of the scaling function, as predicted by our formula (6 28), is shown by the solid line

For the "walk on a random walk" example (which should be much easier to study numerically) we expect

$$F^* = \omega^{1/4}, \qquad \alpha = 6/5 . \tag{6.29}$$

This subtle interplay between trapping and finite-frequency effects may serve as a toy model for pulsed electrophoresis of long molecules in gels, where anomalous effects are known to appear [Noo87]. Of course, the point particle studied here does not possess the internal degrees of freedom of a long macromolecule which are thought to play an important role in the later process [Vio88, Deu89, and references therein].

6.2.3. Diffusion front and localization on fractals

6.2.3.1. The shape of the diffusion front. The *diffusion front* on a disordered fractal may be discussed most clearly on the random chain structure that we cherish. Indeed, $P(s, t)$ is simply a Gaussian, $P(s, t) \sim t^{-1/2} e^{-s^2/t}$, and so is $P(\boldsymbol{R}, s)$. One may thus compute the *average* diffusion front (in real space) as

$$\langle P(\boldsymbol{R}, t)\rangle = \int ds\, P(s, t) P(\boldsymbol{R}, s) , \tag{6.30}$$

which, upon a saddle point approximation, yields in the regime $R \gg t^{1/4}$,

$$\langle P(\boldsymbol{R}, t)\rangle \sim \exp[-(R/t^{1/4})^{4/3}]\,(R^2 t)^{-d/6} . \tag{6.31}$$

The exponents appearing in (6.9) are thus given here by

$$\delta_{av} = \frac{1}{1-\frac{1}{4}} = \frac{1}{1-\nu}, \qquad \sigma = -d/3 . \tag{6.32}$$

The *typical* diffusion front (obtained without averaging over starting points, or else as $\exp\langle \ln P(\boldsymbol{R}, t)\rangle$),

however, is obtained as

$$\langle \ln P(\boldsymbol{R}, t)\rangle = Z^{-1}\int \frac{s^2}{t}\, e^{-R^2/s}\, ds \simeq R^4/t \tag{6 33}$$

The exponent δ_{typ} is now equal to 4 instead of 4/3 This shows the importance of correctly specifying the averaging procedure and the quantities of physical interest in disordered systems. More generally, Harris and Aharony have recently shown [Har87] (see also ref [G12] and references therein) that on a general fractal

$$\delta_{\text{average}} = 2d_F/(2d_F - d_S)\,, \tag{6.34}$$

while

$$\delta_{\text{typical}} = 2d_F/(2\hat{d} - d_S)\,. \tag{6.35}$$

(The above example is such that $d_F = 2$, $d_S = \hat{d} = 1$.) In particular, since $\hat{d} \le d_F$, one has $\delta_{typ} \ge \delta_{av}$ In the next section we shall present a simple physical derivation of (6.34) and (6 35). Present numerical simulations [G12, Yak89] seem not to agree with (6 35). We suggest, however, that, in order to test the reliability of numerical investigations of very rare event effects such as (6 9), one should deal first with the simpler random chain structure (6 32), (6 33)

6 3 2 2 Localization of waves The above probability distribution and the structure of a localized wave on the structure are two facets of the same problem, i e , the Laplacian on a fractal space Imagine that the eigenstate $\psi_E(R)$ of this Laplacian for an energy E decays asymptotically as

$$\psi_E(R) \sim \exp\{-[R/\lambda(E)]^{\eta}\} \tag{6 36}$$

The exponents η and δ are related to each other by

$$\delta = 1/(\eta^{-1} - \nu) \tag{6 37}$$

This relation has been obtained by Levy and Souillard [Lev87], and can be recovered as follows Using eq (3 107) one can express $P(\boldsymbol{R}, t)$ as

$$P(\boldsymbol{R}, t) \sim \int \rho(E)\, e^{-Et}\psi_E(0)\psi_E(\boldsymbol{R})\, dE$$

Assuming that $\lambda(E)$ behaves as a power law for small E, and performing a saddle point integration, one obtains

$$\lambda(E) \sim E^{-\nu}\,, \tag{6 38}$$

$$P(R, t) \sim \exp[-(R/t^{\nu})^{\eta'(1-\eta\nu)}] \tag{6 39}$$

Equation (6 39) is the announced result (6 37), and (6 38) simply expresses the fact that

$$N(E) = \int_0^E \rho(\varepsilon)\, d\varepsilon \sim E^{d_S/2} \sim 1/[\lambda(E)]^{d_F}\,,$$

i e , that the number of states *per site* is finite Combining (6 37) and (6 34), (6 35), one finds that the *average* behaviour of the wave function is

$$\langle \psi_E(R)\rangle \sim e^{-R/\lambda}\,, \tag{6 40}$$

while the *typical* wave function is "superlocalized" (that is, decays faster than exponentially),

$$\psi_{\rm typ}(R) \sim \exp[-(R/\lambda)^{d_{\rm F}/d}] \sim {\rm e}^{-s/s_0}\,, \tag{6 41}$$

which has an obvious meaning the wave functions are exponentially localized *along the structure*

Again, (6 40) and (6 41) are easy to obtain for the linear chain structure Assume that, e g , an attractive impurity is placed at the origin Then the bound states will decay (along the chain) as

$$\psi(s) \sim {\rm e}^{-s/s_0}$$

Now, in real space,

$$\langle \psi(R) \rangle = \frac{\int {\rm d}s\, P(R,s)\psi(s)}{\int {\rm d}s\, P(R,s)} \sim {\rm e}^{-R}\,,$$

while

$$\langle \ln \psi(R) \rangle = -\frac{\int {\rm d}s\, (s/s_0) P(R,s)}{\int {\rm d}s\, P(R,s)} \sim -R^2$$

6 2 4 Response to a constant internal bias [Bou89b]

Let us consider here the response of a particle diffusing on a fractal to a constant "internal field" (i e , such that the potential energy drops as $-F_0 s$) Generalizing somewhat eq (6 33), one may write

$$\langle \ln P(R,t) \rangle = \int {\rm d}s\, P(R,s) \langle \ln P(s,t) \rangle \sim -\int {\rm d}s\, (s/t^{\nu})^{\delta} P(R,s)\,,$$

where $\hat{\nu}$ is the *internal* diffusion exponent (defined by $s \sim t^{\nu}$) and $P(R,s)$ reads [G12]

$$P(R,s) \sim (s^{\hat{d}/d_{\rm F}})^{-d} f(R/s^{\hat{d}/d_{\rm F}})$$

This yields

$$\langle \ln P(R,t) \rangle \simeq -(R^{d_{\rm F}/\hat{d}}/t^{\nu})^{\delta} \stackrel{\rm def}{=} -(R/t^{\nu})^{\delta_{\rm typ}}\,,$$

or

$$d_{\rm F}\nu = \hat{\nu}\hat{d}\,, \tag{6 42}$$

$$\hat{\delta} = (\hat{d}/d_{\rm F})\delta_{\rm typ} = 2\hat{d}/(2\hat{d} - d_{\rm S}) \equiv 1/(1-\hat{\nu}) \tag{6 43}$$

This provides another derivation of (6 35) [Bou89b] Formula (6 43) has indeed the expected shape in internal space, which insures "linear response" to a constant "internal field" The arguments of section 5 3 lead in this case to the following equation of motion

$$s \sim tF_0^{(1-\nu)/\nu} \sim tF_0^{2\hat{d}/d_{\rm S}-1}\,,$$

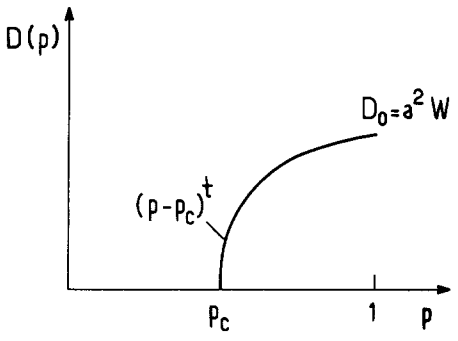
or

$$\overline{R(t)} \sim t^{\hat{d}/d_{\rm F}} F_0^{\alpha}\,, \quad \alpha = (2\hat{d} - d_{\rm S})\hat{d}/d_{\rm S}d_{\rm F} \tag{6 44}$$

6.3. *The percolation problem: crossover from fractal to Euclidean behaviour*

6.3.1. *Percolation: a short toolguide* [Sta85]

Let us consider a random walker on a disordered lattice, where the hopping rates $W_{n,n+l} = W_{n+l,n}$ can be either 0 or W. *Bond* percolation assumes that this choice is independently made on each bond, and



rig 6 4 Dependence of the average diffusion constant $D(p)$ on the concentration in the percolation problem As soon as an infinite cluster is formed, diffusion is possible and $D(p)$ grows as $(p-p_c)^t$ for $p-p_c$ sufficiently small

that $W_{n,n+l}=W$ with probability p and 0 with probability $1-p$ *Site* percolation is generated by deciding if a given site is "occupied" ($\theta=1$) with probability p or "empty" ($\theta=0$) with probability $1-p$; then $W_{n,n+l}=\theta(n)\theta(n+l)W$. The diffusion constant $D(p)$, defined as

$$D(p)\equiv\lim_{\substack{V\to\infty\\ t\to\infty}}\frac{1}{2dV}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\boldsymbol{R}_0}[\boldsymbol{R}(t)-\boldsymbol{R}_0]^2\,,$$

is non-zero only if the system "percolates", i.e , when an infinite cluster spans the whole sample. This happens when p is larger than a certain threshold p_c, the value of which depends on the type of percolation (bond or site), lattice and dimension The $D(p)$ curve has a typical "critical phenomenon" shape (fig 6 4), growing as

$$D(p)\sim(p-p_c)^t\,,\quad \text{for } p>p_c \tag{6 45}$$

Right at threshold, the infinite cluster is a fractal, characterized by dimensions d_F, d_S and $\hat{d}$, which depend only on the dimension of space d (table 6.1). The fraction $P_\infty(p)$ of present sites belonging to

Table 6 1
Exponents and characteristic dimensions for percolation Most of them have been taken from ref [G12], where error bars and appropriate references can be found

	$d=2$	$d=3$	$d=6$
ν_p	4/3	0 88	1/2
d_F	91/48	2 51	4
$\hat{d}/d_F$	0 88	0 725	2
d_S	1 30	1 32	4/3
backbone d_F^B	1 62	1 74	2
d_S^B	1 20	–	1
$\mu=2/d_S-1$	0 54	0 515	1/2
$\mu d_F/\mu^B d_F^B$(num)	0 9	–	1
$t_{min}=(d-2)\nu_p+1$	1	1 88	3
t	1 30	2 02	3
$t_{max}=\nu_p(d-2+d_F/\hat{d})$	1 51	2 07	3

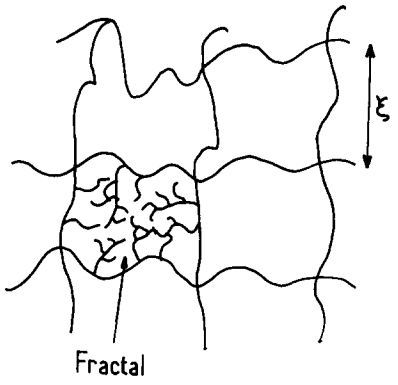


Fig 6 5 Schematic structure of the percolation cluster For all physical phenomena taking place at length scales smaller than ξ, this cluster appears as an infinite fractal, while for length scales larger than ξ it appears as a weakly disordered lattice of mesh size ξ, one is driven to the homogeneous fixed point $p = 1$

the infinite cluster grows as

$$P_\infty(p) \sim (p - p_c)^\beta . \tag{6.46}$$

A length scale ξ appears (the correlation length), which separates "fractal behaviour" (for all processes taking place at length scales $r < \xi$) from "Euclidean behaviour" for $r > \xi$ (fig. 6 5) As $p \to p_c$, ξ diverges like $(p - p_c)^{-\nu_p}$. An example of how to use such a simplifed picture of the structure is the following: At short times, the medium seen by a diffusing particle has a fractal structure and thus the particle evolves as

$$R \sim t^\nu \quad \text{with } \nu = d_S/2d_F .$$

This is valid until R is of order ξ. For longer times, the particle sees an effectively Euclidean lattice of lattice size ξ. Each cell acts as a "trap" with release time t_ξ, such that $\xi = t_\xi^\nu$. Hence for $t > t_\xi$, the particle diffuses *normally*, with a diffusion constant

$$D_\infty(\xi) \sim \xi^2/t_\xi \sim \xi^{2-1/\nu} . \tag{6.47}$$

$D_\infty(\xi)$ characterizes the motion of a particle *starting on the infinite cluster*, independently of this starting point Hence

$$D(p) = \lim_{\substack{V\to\infty \\ t\to\infty}} \frac{1}{2dV} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{R_0} [\boldsymbol{R}(t) - \boldsymbol{R}_0]^2 \equiv P_\infty D_\infty + (1 - P_\infty)0 \tag{6.48}$$

(since the particles starting on a finite cluster will not contribute to the average diffusion coefficient). Thus, using (6.45) and (6.47), one obtains [dGe76, Ale82]

$$t = \beta + \nu_p(1/\nu - 2) . \tag{6.49}$$

Percolation is thus the archetype of a "connectedness" transition, separating a connected phase – in

which physical excitations can be transmitted from one end of the sample to the other (mass or electrical current, mechanical stress, "forest fires", viruses, information, etc) – from a disconnected phase As such, it defines a universality class through which a large number of systems may be described (see, e.g , [Sta85, Rou89]) As we emphasized in the first chapter, two mechanisms may change the universality class of usual (bond or site) percolation

(a) *Strong correlation* between the basic units (e.g attraction), which do not occupy sites independently If those correlations are "sufficiently strong", not only the percolation threshold changes but also the critical exponents

(b) *Broad distributions*, for example of hopping rates· If one chooses $W_{n\,n+l}$ with probability $p\delta(W) + (1-p)\rho(W)$,

$$\rho(W) \underset{W\to 0}{\sim} W^{\mu-1}, \quad \mu \leq 1,$$

then the local mean "trapping time" $\langle 1/W \rangle$ diverges and one expects deep changes in the value of the exponents t, ν

This model has been introduced by Halperin, Feng and Sen [Hal85a] to retain the important features of "continuum percolation", such as the "Swiss cheese model", where one drills holes at random in a continuous medium Narrow "bottlenecks" such as the one represented in fig 6.6 will correspond to a high local resistivity, i.e , a low local transition rate Indeed, such a broad disorder affects the value of t: it has recently been shown [Mac88, Dou88], that, for $\mu < 1$,

$$t = \max\{t_0, (d-2)\nu_p + \mu^{-1}\},$$

where t_0 is the "pure" percolation exponent.

6 3.2. Electrical properties of the percolation cluster

6.3 2 1. d.c properties From the general theorem of chapter 2 and the above discussion, we immediately know how the conductivity grows when p crosses p_c. Indeed, from (2.15), the d.c. conductivity σ is *identical* to D if the local hopping rates are chosen as $W_{n\,n+l} = (a^2/C)\sigma_{n\,n+l}$ Hence

$$D \equiv \sigma \sim (p - p_c)^t, \quad t = \beta + \nu_p(1/\nu - 2),$$

or, introducing $d_F = d - \beta/\nu_p$ and $\nu = d_S/2d_F$,

$$t/\nu_p = d - d_F + 2d_F/d_S - 2 \tag{6 50}$$

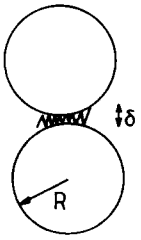


Fig 6 6 Narrow "bottleneck" through which current *must* pass near the continuous percolation threshold The possible dependence of the local "conductivity" on the width δ may induce a broad distribution of waiting times and hence change the universality class of percolation [Hal84, Mac88]

A very important lower bound on t can be found using the exact result on "red bonds" derived by Coniglio [Con81]. He found that the number of links carrying all the current ("hot bonds") is, per "cell" of size ξ (see fig. 6 5),

$$N_{\mathrm{rb}} = (p - p_{\mathrm{c}})^{-1} . \tag{6 51}$$

Setting the resistance of all other bonds equal to zero, one finds an upper bound on the conductivity, as

$$\begin{aligned} &\mathcal{R}(\xi) \equiv L^{2-d}\sigma^{-1}(\xi) > (\xi/L)^{d-1}(L/\xi)N_{\mathrm{rb}} , \\ &\text{or} \quad \sigma(\xi) < \xi^{2-d}N_{\mathrm{rb}}^{-1} \quad \text{or} \quad t > (d-2)\nu_{\mathrm{p}} + 1 \end{aligned} \tag{6 52}$$

(which in fact is exact in dimension 6, the upper critical dimension for percolation)

An analogous argument allows one to give also an upper bound (of geometrical origin) on t [Pik81],

$$t < (d-2)\nu_{\mathrm{p}} + \nu_{\mathrm{p}} d_{\mathrm{F}}/\hat{d} , \tag{6 53}$$

which has the following interpretation: Single out the shortest path spanning the distance ξ, and set all other resistances to infinity. One thus obviously has

$$\mathcal{R}(\xi) < (\xi/L)^{d-2} S_{\xi} ,$$

or, with $S_{\xi}^{\hat{d}} \sim \xi^{d_{\mathrm{F}}}$,

$$\sigma(\xi) > \xi^{2-d-d_{\mathrm{F}}/\hat{d}} ,$$

which yields the above result on t. Bounds (6.52) and (6 53) are quite narrow, as can be seen from table 6.1. In particular, they coincide for $d > 6$. Note that (6.53) may also be written as $\hat{d} < d_{\mathrm{S}}/(2 - d_{\mathrm{S}})$ [see the remark after eq. (6.59) below]

It is fairly obvious that on the percolation cluster one should distinguish links which participate in (electrical) transport, and links which do not (dead ends) Removing all those "cold" bonds generates a new object, the "backbone", which has its own characteristic dimensions $d_{\mathrm{F}}^{\mathrm{B}}$, $d_{\mathrm{S}}^{\mathrm{B}}$, $\hat{d}^{\mathrm{B}}$. As the conductivity obviously remains unchanged, one has the relation [see eq. (6.50)] [Sta84]

$$d_{\mathrm{F}}(2/d_{\mathrm{S}} - 1) \equiv d_{\mathrm{F}}^{\mathrm{B}}(2/d_{\mathrm{S}}^{\mathrm{B}} - 1) \tag{6 54}$$

(which is not very well satisfied by numerical results, see table 6.1) Said differently, the *average* diffusion coefficient D is unaffected, while the diffusion coefficient for particles restricted to the backbone is enhanced – since the delays induced by the dead ends are removed. One has precisely

$$D_{\infty}^{\mathrm{B}} = (P_{\infty}/P_{\infty}^{\mathrm{B}})D_{\infty} , \tag{6 55}$$

where P_{∞}^{B} ($\ll P_{\infty}$) is the probability of being on the backbone. Equation (6 55) has a simple meaning: Writing $D_{\infty} = \xi^2/t_{\xi}$, one finds $t_{\xi} \sim P_{\infty}$. Since the process is ergodic, the walk spends, in each region of space, a time proportional to the number of accessible sites in this region.

Note that the shortest path between two far away points *must belong to the backbone*, and hence

$$S_\xi \sim \xi^{d_F/\hat{d}} \sim \xi^{d_F^B/d_B} \Rightarrow d_F/\hat{d} = d_F^B/\hat{d}^B \tag{6.56}$$

6.3 2 2 a c properties. The a c properties of the percolation clusters call for a few clarifying comments, which echo those formulated in section 2.2.1 on the "electrical Einstein relation". First, one must clearly distinguish

(i) the problem of a random walker on the structure, driven by an oscillating external field, which is dealt with in section 6.2.2, 6 3 4 and 5.2, from

(ii) the problem of the a.c. conductivity – generally implicitly defined through the *input* impedance – of an electrical network

The latter problem must then be further specified, since for non-zero frequencies capacitance effects come into play [Con89, Cle90]

Hence, one may, for example, consider two completely different physical situations

(i) Each bond carries either a resistor with probability p or a capacitance with probability $1-p$. This is a standard model for a metal–insulator mixture. In this case, it is known [Efr76, Dao88, and references therein] that the macroscopic a.c. properties near p_c are related *not only* to the exponent t introduced above, but also to the exponent s describing the divergence of the conductivity of a conductor–superconductor mixture (i.e., a percolation network in which each missing bond is considered to be of zero resistance),

$$\sigma \sim (p - p_c)^{-s}$$

In particular, the high-frequency behaviour of $\sigma(\omega)$ near p_c is given by

$$\sigma(\omega) \sim \omega^{t/(s+t)}$$

($\simeq \omega^{0.72}$ in three dimensions) This behaviour has been experimentally confirmed on three-dimensional metal/insulator mixtures [Nik87]

(ii) A second situation of interest may be the following: Missing bonds have zero capacitance *but* each *node* is connected *to the ground* by a finite capacitance The random walk analogy (section 2.2.1) may be used to estimate the penetration depth $L_p(\omega)$ of the alternating input current in the sample. In a time ω^{-1}, the particle in the diffusion problem probes a region of depth $L_p(\omega) \sim \omega^{-\nu}$ The admittance $A_L(\omega)$ thus becomes independent of the total depth L of the sample, and can be obtained through

$$A_L(\omega) = \frac{\Sigma\sigma(\omega = 0)}{L} f(L/L_p) \sim \Sigma L^{-t/\nu_p - 1} f(L\omega^\nu),$$

$$f(x) \underset{x\to 0}{\longrightarrow} 1, \qquad f(x) \underset{x\to\infty}{\sim} x^{t/\nu_p + 1}$$

(Σ is the surface of the electrode) Hence,

$$A_L(\omega) \simeq \Sigma\omega^{\nu(t/\nu_p + 1)}, \quad \text{for } L_p \ll L.$$

This result has been obtained in slightly different terms by Rigord and Hulin [Rig88], and checked

experimentally on a model system. Note that it does *not* coincide with the extension to non-zero frequency of Einstein's relation proposed by Gefen, Aharony and Alexander [Gef83], which leads to

$$A_L(\omega) \sim \sigma(\omega) \sim P_\infty(\omega) D(\omega) \sim \omega^{\beta\nu/\nu_P} \omega^{1-2\nu}$$

For more work on this subject, we refer the reader to [Con89, Cle90] and references therein.

6.3.3 Dead ends as deep traps: diffusion on the percolation cluster for $d > 6$

The mechanism leading to anomalous diffusion on the percolation cluster can be very clearly identified in high dimensions, where the geometrical structure of the cluster is fairly simple. Indeed, in "sufficiently high" dimensions, the *steric* constraints are asymptotically irrelevant and hence the percolation cluster can be thought of as a *backbone* (which is a linear random walk, $d_F^B = 2$, $\hat{d}^B = 1$, $d_S^B = 1$) with a finite density of *dead ends* branching out. As the structure is statistically self-similar, those dead ends are themselves random walks with branches, and so on. It is easy to show (working, e.g., on the Bethe lattice [Sta85]) that the fractal dimension of the percolation cluster is $d_F = 4$ for $d > 6$ (when $d_F + d_F^B = 4 + 2$ becomes larger than d, that is, when $d < 6$, the structure and intersections between the dead ends and the backbone cannot be so simple)

From eqs. (6 54)–(6.56) one thus readily obtains

$$\hat{d} = 2\,, \qquad d_S = 4/3\,, \tag{6.57}$$

and hence $R^2 \sim t^{1/3}$ in high dimensions The value $d_S = 4/3$ has been conjectured (on a numerical basis) to be "superuniversal" by Alexander and Orbach [Ale82], i.e., independent of the dimension. ε-expansion ($\varepsilon = 6 - d$) [Har84, Wan86] and very precise numerical simulations for $d = 2$ [Nor88], however, seem to rule out this conjecture.

We would like to show how (6.57) may be recovered in a way which underlines (i) the basic mechanism responsible for this slow diffusion, that is, *trapping in the dead ends*, and (ii) the connection between the value of d_S and purely geometric features It would be interesting to find a generalization to $d < 6$ of the following arguments.

The idea is to model the structure as a random comb with a "crumpled" backbone. The spikes of this comb are in fact clusters containing n sites with probability $\psi(n)$. It is easy to show that, as the intersection with the backbone can be anywhere in the cluster, $\psi(n) \sim n^{1-\tau}$, where τ is the cluster distribution exponent in Stauffer's [Sta79] notation ($\tau = 5/2$ for $d > 6$). Hence $\psi(n)$ decays as $n^{-(1+\mu)}$ with $\mu = 1/2$

The problem of a random walk between "traps" (spikes) of fluctuating size is not trivial since it a priori combines a *quenched* aspect (the particle visiting twice a given spike sees twice the same environment) and an *annealed* aspect, since the exit time of a given region of space is distributed – due to thermal disorder. A "mean field" approach would consist in neglecting the quenched aspect and to use the effective trapping time distribution $\psi(t)$ to obtain the diffusion law much as in chapter 2. $\psi(t)$ is obtained as

$$\psi(t) = \int_1^\infty \mathrm{d}n\, \psi(n) P_n(t)\,,$$

where $P_n(t)$ is the probability of *first return* to the starting point, related to the probability of presence

P_0 on this site For Laplace transforms (appendix A) $P_n(E) = 1 - P_0(E)^{-1}$ Hence

$$P_n(t) \sim t^{-(2-\nu d_F)} \quad (t^{\nu d_F} < n), \qquad P_n(t) \sim 0 \quad (t^{\nu d_F} > n),$$
$$\Rightarrow \quad \psi(t) = \int_{t^{\nu d_F}}^{\infty} t^{\nu d_F - 2} n^{-(1+\mu)}\, dn \sim t^{-[1+1+\nu d_F(\mu-1)]} \tag{6.58}$$

ν is the exponent we are in fact looking for, since it also governs diffusion in the dead ends (the structure is self-similar)

The number of "traps", $N(s)$, encountered along the backbone must be such that

$$\sum_{i=1}^{N(s)} n_i = N(s) \int^{n_{\max}(N(s))} \frac{n\, dn}{n^{1+\mu}} \sim (s^{\hat{d}/d_F})^{d_F},$$

so that the correct number of sites in a sphere of size $s^{\hat{d}/d_F}$ is recovered Hence $N(s) \sim s^{\mu\hat{d}}$ This, in particular, imposes that $\mu\hat{d} \leq 1$; for $d = 6$, one has $\mu\hat{d} = 1$ The *time* needed to travel a distance s along the (one-dimensional) backbone is given by [cf. eq (6 58)]

$$t \sim [N(s)]^{2/[1+\nu d_F(\mu-1)]}$$

And since $R \sim s^{\hat{d}/d_F} \sim t^{d_S/2d_F}$, one finally obtains

$$d_S = 2/(1+\mu) \tag{6.59}$$

for any $\hat{d}$, d_F. In particular, for $d > 6$, $\mu = \frac{1}{2}$ and one indeed recovers $d_S = 4/3$ Table 6.1 shows that $\mu \simeq \frac{1}{2}$ in all dimensions

Remarks

(a) Note that $\hat{d}\mu \leq 1$, together with (6 59), is *equivalent* to (6 53)

(b) This picture suggests that for $p = p_c$ and in the presence of a static field,

$$\overline{R(t)} \sim (\ln t)^{1/\alpha}$$

with $\alpha \equiv 1$ Indeed, the local trapping time is in this case expected to be

$$t = \exp(F_0 a n^{1/d_F}/kT)$$

Hence, the distribution

$$\psi(t) \sim \frac{1}{t(\ln t)^{1+\mu d_F}}$$

Using arguments of [Hav86, Bun87, G12], one finally obtains

$$(\ln t)^{\mu d_F} \sim N(s) \sim s^{\mu d} \quad \text{or} \quad R \sim \ln t$$

(c) One may wonder whether the preceding "annealed" analysis is exact or not An analogous model (with purely one-dimensional spikes) was considered in [Hav87], and numerical simulations are in very good agreement with the annealed prediction We strongly believe that this analysis is exact, as the same result may be reached through the following argument

Assume that the *longest* trapping time encountered in N steps is $t_{\max}(N)$ Clusters must be divided in classes
- totally explored clusters, such that $t_{\max}^{\nu d_F} > n$,
- partially explored clusters, such that $t_{\max}^{\nu d_F} < n$

As the particle makes a one-dimensional random walk along the backbone, the number of different spikes encountered is $N^{1/2}$, each of which is visited $N^{1/2}$ times Out of those $N^{1/2}$ only

$$\sqrt{N} \int_{t_{\max}^{\nu d_F}}^{\infty} \frac{dn}{n^{1+\mu}} \sim \sqrt{N} t_{\max}^{-\mu \nu d_F}$$

are partially explored and *thus effectively infinite* (the quenched aspect, related to the finite size of those spikes, is thus irrelevant for those spikes) The particle visits

$$N_\infty = N^{1/2} N^{1/2} t_{\max}^{-\mu \nu d_F}$$

times infinite traps with a release time distribution $t^{-(2-\nu d_F)}$ The longest trapping time is therefore (cf ch 1)

$$t_{\max} = N_\infty^{1/(1-\nu d_F)} \quad (\text{for } 1 - \nu d_F \leq 1)$$

Hence

$$t_{\max}^{1+\nu d_F(\mu-1)} = N\,, \qquad t \simeq t_{\max}\,, \quad s \sim \sqrt{N}\,,$$

which gives back the above result Note that the totally explored clusters get hold of the particle during a time ($n_{\max} = t^{\nu d_F}$)

$$t_1 = N \int^{n_{\max}} n\, dn\, n^{-(1+\mu)}\,,$$

since they each act as a trap of mean release time proportional to n Therefore, t_1 is also of order $t_{\max}$

The results of this section are thus as follows:

(a) We have shown that an annealed approach to the "random comb" problem is essentially exact, thereby confirming the numerical simulations of ref. [G12].

(b) The diffusion exponent on the percolation cluster for $d > 6$ is related (through the trapping of the particle in deep dead ends) to geometrical characteristics of the structure. It would be nice to relate (if at all possible) – in the same spirit but in lower dimensions – the spectral dimension to other, mainly static exponents.

6.3.4. Biased diffusion and dispersion on the percolation network

6.3.4.1. Uniform (oscillating) external field. The analysis presented in section 6 2.2 must be adapted to the percolation problem since for $p > p_c$ diffusion becomes normal for $R > \xi$. It is fairly easy [Bou90] to determine the behaviour of the amplitude of motion $A(\omega, F_0)$ in the whole (ω, F_0) (frequency, field amplitude) plane for *non-zero* frequencies (for $\omega \equiv 0$, see [Rou87a]); see fig 6.7. Note that $A(\omega, F_0)$ is always a decreasing function of F_0 for large F_0

6.3.4.2. Pressure (or voltage) drop across the sample ("internal bias"). Suppose now that a pressure drop $\Delta\Pi$ is applied at the boundaries of, say, a non-wettable porous medium in which a liquid has been injected with a pressure slightly larger than the break-through pressure ($\Delta\Pi_c$) [dGe78]. The invading fluid has the structure of a percolation network It obeys an effective Darcy law for length scales larger than ξ, in which the permeability is the strict analogue of the electrical conductivity (e.g. [Guy87]),

$$P_\infty U = k(\xi) \nabla \Pi\,, \tag{6.60}$$

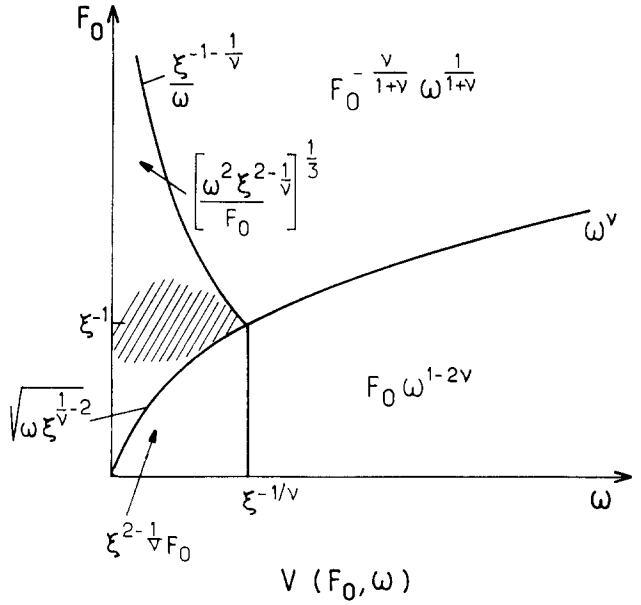


Fig 6 7 Full "phase diagram" [Bou90] for biased diffusion on the percolation cluster, in the (ω, F_0) plane, for finite ξ The laws govern the "velocity" $V(\omega) = \omega R(\omega)$ Note that V is always a decreasing function of F_0 for large forces Solid lines are crossover lines The velocity V is related to the amplitude of motion $A(\omega, F_0)$ through $V = \omega A$

with $k(\xi) \sim (p - p_c)^t$. This is valid for $|\nabla \Pi| / \Delta \Pi \sim L^{-1} \ll \xi^{-1}$, i.e , for samples with size much larger than ξ For $L \sim \xi$, the time t_ξ needed to cross the fractal network must clearly be intrinsic, i.e. invariant under "fractal deformation". But

$$t_\xi \sim \frac{\xi}{U} \sim \frac{\xi P_\infty}{k(\xi)|\nabla \Pi|} \sim \frac{P_\infty \xi}{P_\infty \xi^{2-1/\nu}} |\nabla \Pi|^{-1} .$$

Now, under a fractal deformation, the pressure drop $\Delta \Pi$ is obviously unchanged, but *not* the pressure gradient; and indeed (replacing $\nabla \Pi$ by $\Delta \Pi / \xi$), one has

$$t_\xi \sim \xi^{2d_F/d_S} (\Delta \Pi)^{-1} , \tag{6 61}$$

which is intrinsic since ξ^{d_F} is the (conserved) mass, and d_S is intrinsic.

If a dye is injected in the flowing fluid, one may observe *dispersion* due to trapping in the dead ends, and thus define a dispersion constant $D_\parallel(U, \xi)$ This problem was first studied by De Gennes [dGe83], his result can be recovered very easily through the simple model presented in section 5.7 and the associated formula (5.56),

$$D_\parallel(U, \xi) = \tfrac{1}{2} f U^2 \langle \tau^2 \rangle / \langle \tau \rangle$$

(f is the volume fraction of "traps", which is 1 for the percolation problem, since most sites belong to dead ends, $d_F^B < d_F$) $\langle \tau^2 \rangle$ and $\langle \tau \rangle$ can be computed using (6.58) as the distribution of waiting times, cut off above $t_{max} = \xi^{1/\nu}$, and with an exponent $\mu \simeq 1/2$. One finds $\langle \tau^2 \rangle \sim \xi^{1/\nu} \langle \tau \rangle$ (this is, however,

not true for arbitrary μ!). Finally

$$D_{\|} \sim U^2 \xi^{1/\nu}, \tag{6.62}$$

with U given by (6.60). In this formula, U is the *average velocity* and *not* the velocity of the flow on the backbone, as emphasized by De Gennes [dGe83].

Remarks

(a) The time $\xi^2/D_{\|} = (\xi/U)^2 \xi^{-1/\nu} = \xi^{1/\nu}\, \Delta\Pi^{-2}$ is invariant under a fractal deformation keeping $\Delta\Pi$ fixed – contrary to a worry expressed by Vannimenus [Van84]

(b) This problem has been quite carefully investigated numerically (in two dimensions) and the law (6 62) is found to be satisfactorily obeyed [Kop88]

7. Conclusion

It seems to us that a paper like this should end with a list of open problems, great or small, which are suggested here and there in the body of this review, or which are natural generalizations of the models considered above. We shall list them chapter by chapter as they appear in the article, regardless of their relative interest or difficulty.

1. A general theory of the limit distributions for sums of *correlated* random variables is still, to a large extent, lacking (attraction basins, dependence of the limit law on the structure of the correlations); as mentioned in section 1.3, tools inspired from the renormalization group are probably well suited for investigating this question. In the same spirit, an analytic calculation of the full average diffusion front in Matheron–de Marsily's layered model (section 1.3.2 and appendix C) would also be welcome. Self-consistent approaches "à la Flory" to various problems involving "relevant" correlations, when interpreted along the ideas of section 1.3.3.2, also raise a number of questions: Is Flory's approximation really an upper bound to the true value of ν, as suggested in section 1.3.3; similarly, is $4/(4+d)$ a lower bound in the case of linear polymers? Finally, could one devise approximations "à la Flory" for the critical exponents of more general critical phenomena [e.g., the O(n) model]?

2. The sample to sample fluctuations discussed in section 2.1.2 deserve further study, in particular regarding the equivalence between different averaging procedures. An interesting question in this respect, which we believe to be connected with some physical issues in relaxation processes, is that of ergodicity: in precisely what circumstances does the histogram of the positions of one particle in a given sample (correctly rescaled) coincide with the average over disorder of $P(x, t)$? This question is particularly intriguing in Sinai's problem.

Other, more specific, unsolved questions encountered in ch. 2 are the shape of the scaling function describing the diffusion front for random traps (in $d < 2$) and random barriers (in $d = 1$) with broad distributions of inverse hopping rates (including the mere existence of a CLT for a fixed sample in the random traps case) and the exact calculation of the prefactors of the diffusion law.

3. A proof that the diffusion coefficient is self-averaging for a general one-dimensional asymmetric hopping model (a physically likely property) is still lacking (it has been provided in [Asl89a] for the directed limit only).

A number of features of the remarkably rich one-dimensional random-force model of section 3.3 (and of the similar asymmetric hopping models) still deserve investigation, in particular in the $\mu < 1$ phase. As discovered by Golosov [Gol84] and reviewed in section 3.3.2.2, two particles starting at the

same point and subjected to different thermal noises do not separate with time, one may thus ask the following complementary question what happens to two particles initially separated by a distance L? Our guess is that for $\mu = 0$ they meet after a finite time t [typically of order $\exp(L^{1/2})$] and remain close ever thereafter What happens for $\mu > 0$? Clearly, the $\mu < 1$ phase shares a number of similarities with glassy dynamics. Does this phase exhibit ageing phenomena (i.e non-stationary evolution), like the dynamics of spin glasses below T_g [Lun85, Alb86, 87], as suggested in [Fei88]?

The physical situations to which this model applies also raise questions: numerical simulations of the relaxation of the random field Ising model in a $+H$ field, initially prepared in a down-spin configuration, would test the predictions made in section 3.3.7.1, 3, and allow one to investigate more precisely the interplay of domain wall creep and nucleation; a more detailed quantitative study of dislocation motion and comparison with experiments is also an interesting path to follow [Bou89g]

4 A number of open problems and questions are also raised by the random-force model in higher dimensions·

- The effect of a bias in the potential case, in particular, when unbiased diffusion is logarithmic ($d < 2$ for $a > d$ and $a < 2$ for any $d > a$): could there be a zero-velocity ("creep") phase as in $d = 1$?
- Does the Golosov phenomenon still hold for this logarithmic diffusion?
- The nature of the sample to sample fluctuations (i.e., the possible differences between the diffusion behaviour for a given sample and the average one), in particular in the long-range correlated case
- Almost nothing is known on the shape of the diffusion front in this model for $d > 1$
- A more detailed study of the effect of the anisotropy in the random-force correlations (see [Dug89] for a most recent first study) is required, in particular of the crossover between isotropic situations and the layered models of section 1.3 2

Here again, possible applications to physical situations raise perhaps the most interesting questions:
- How well does the hydrodynamical case $\operatorname{div} F = 0$ with long-range correlations (namely a close to $-2/3$) describe turbulent diffusion? In particular, what is the asymptotic diffusion front for this model and does it compare favourably with experiments? Could the diffusion exponent for turbulent diffusion continuously depend on the compressibility of the fluid, as the line of fixed points of section 4.3 2 suggests?

5. It has been shown in section 5.4 that the response to a weak external oscillating field exhibits (especially in one dimension) non-trivial crossover behaviour. For a given frequency, the response is linear in the field only for very weak fields $F < \omega^{\nu}$ Could the non-linear mobility predicted for higher fields be observed, either in one-dimensional ionic conductors [Bey81], see sections 5 4 2, or in a biased version of the Cardoso–Tabeling [Car88] experiment on diffusion among convection rolls, see section 1 2.3.3?

6 Finally, we have obtained in section 6 3.4 1 the full "phase diagram" for the behaviour of a particle subjected to an oscillating field on the percolation network. Numerical simulations at the percolation threshold agree well with our theory, simulations off threshold would also be most welcome.

Finally, a few general questions are the following.
- In most diffusion models considered in this article, the effect of inertia is neglected. For the random-force model, for example, it is natural to wonder what happens to the laws obtained in chapters 3 and 4 if the inertia of the particle is taken into account, that is, if the Langevin equation is written as

$$m\boldsymbol{X}_t + \gamma \boldsymbol{X}_t = \boldsymbol{F}(\boldsymbol{X}_t) + \boldsymbol{\eta}(t) .$$

Our guess is that at long times the behaviour is not modified by a non-zero mass m

– Similarly, what are the diffusion laws if disorder is not quenched but evolves with a given time autocorrelation function? Brownian diffusion is probably recovered at times much larger than the correlation time of the disorder.

We think, however, that the most interesting and fruitful path to follow is the dynamics of lines (polymer, dislocation, vortices, step on a crystalline surface, etc.) or surfaces (domain wall, etc.) in random media, which applies to a rich variety of physical situations (spin glasses, disordered type II superconductors, etc.) In other words, one tries to understand the role of the internal degrees of freedom in the overall motion of the "defect" Such an investigation has of course already been the subject of several works; we hope that the general ideas and methods developed here could help to make some progress in the qualitative understanding of these problems

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Appendix A. Some useful results and techniques in the theory of random walks

This appendix is only a brief (and lacunar) summary; the reader is referred to refs [G1, G5–G9, G13] for further information.

A.1 Random walk on a regular lattice, master equation, probability of presence

A discrete-time random walk process on a regular lattice is defined by the probability $p(\boldsymbol{e})$ to jump from site $\boldsymbol{X}$ to site $\boldsymbol{X}+\boldsymbol{e}$ at each time step The probability $P(\boldsymbol{X}, t)$ to find the walker on site $\boldsymbol{X}$ at time t obeys the master equation (which simply expresses the conservation of probability)

$$P(\boldsymbol{X}, t+1)=\sum_{\boldsymbol{e}} p(\boldsymbol{e}) P(\boldsymbol{X}-\boldsymbol{e}, t) \tag{A.1}$$

(the time step has been normalized to unity). Equation (A.1) has to be supplemented with initial conditions, e.g.,

$$P(\boldsymbol{X}, t=0)=\delta_{\boldsymbol{X},\boldsymbol{0}} \tag{A.2}$$

Translation invariance allows one to solve (A.1) by Fourier transforming. Defining

$$P(\boldsymbol{k}, t)=\sum_{\boldsymbol{X}} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}} P(\boldsymbol{X}, t)\,, \tag{A.3}$$

the solution of (A 1), (A.2) reads

$$P(\boldsymbol{k}, t) = [p(\boldsymbol{k})]^t , \tag{A 4}$$

where $p(\boldsymbol{k})$ is the structure function characteristic of the lattice,

$$p(\boldsymbol{k}) = \sum_{\boldsymbol{e}} p(\boldsymbol{e})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{e}} . \tag{A.5}$$

For a d-dimensional hypercubic lattice with only nearest-neighbour jumps,

$$p(\boldsymbol{k}) = \frac{1}{d} \sum_{\mu=1}^{d} \cos(a k_\mu) \tag{A 6}$$

$P(\boldsymbol{X}, t)$ is recovered by integration over the Brillouin zone of the reciprocal lattice,

$$P(\boldsymbol{X}, t) = L^d \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}} P(\boldsymbol{k}, t) \tag{A.7}$$

In the large-time limit this integral is dominated by the vicinity of $\boldsymbol{k} = \boldsymbol{0}$ (since $|p(\boldsymbol{k})| < 1$); hence $P(\boldsymbol{X}, t)$ is asymptotically a Gaussian distribution [$D = (\det D_{\mu\nu})^{1/d}$],

$$P(\boldsymbol{X}, t) \to (4\pi D t)^{-d/2} \exp\left(-\frac{1}{4t} (X_\mu - V_\mu t) D^{-1}_{\mu\nu} (X_\nu - V_\nu t) \right) , \tag{A.8}$$

where the velocity V and diffusion tensor $D_{\mu\nu}$ are given by

$$\begin{aligned} V_\mu &= \sum_{\boldsymbol{e}} e_\mu\, p(\boldsymbol{e}) \equiv \langle e_\mu \rangle , \\ D_{\mu\nu} &= \tfrac{1}{2} [\langle e_\mu e_\nu \rangle - \langle e_\mu \rangle \langle e_\nu \rangle] \end{aligned} \tag{A.9}$$

This illustrates the remarks made in section 1 1 on the central limit theorem: In the regime where (A.8) applies, only the first two moments of $p(\boldsymbol{e})$ remain and the detailed structure of the lattice is washed out. For the simple hypercubic example,

$$D_{\mu\nu} = (a^2/2d)\delta_{\mu\nu} \tag{A 10}$$

Restoring the time step τ_0, these expressions for V and D must be divided by τ_0. Substracting $P(\boldsymbol{X}, t)$ on both sides of (A.1), the continuum limit $\tau_0 \to 0$ is taken by keeping $W(\boldsymbol{e}) = p(\boldsymbol{e})/\tau_0$ fixed, leading to the continuous-time master equation,

$$\frac{\partial P}{\partial t} = \sum_{\boldsymbol{e}} W(\boldsymbol{e}) [P(\boldsymbol{X} - \boldsymbol{e}, t) - P(\boldsymbol{X}, t)] . \tag{A.11}$$

$W(\boldsymbol{e})$ is the hopping rate per unit time

A.2. Distribution of first passage time and number of visited sites

Let us denote by $P_1(\boldsymbol{X}, t)$ the probability that a walker starting on $\boldsymbol{X} = \boldsymbol{0}$ at $t = 0$ first reaches $\boldsymbol{X}$ at time t It obeys the following relation·

$$P(\boldsymbol{X}, t) = \sum_{t'=1}^{t} P_1(\boldsymbol{X}, t') P(\boldsymbol{X}, t | \boldsymbol{X}, t') ,$$

which expresses the fact that, in order to be on site $\boldsymbol{X}$ at time t the walker has to reach this site for the first time at any time $t' = 1, 2, \quad , t$ and then to be again on $\boldsymbol{X}$ at time t On a regular lattice, translation invariance allows us to put this relation in the form

$$P(\boldsymbol{X}, t) = \sum_{t'=1}^{t} P_1(\boldsymbol{X}, t') P(\boldsymbol{0}, t - t') . \tag{A 12}$$

This can be solved by introducing the generating functions

$$\tilde{P}(\boldsymbol{X}, \lambda) = \sum_{t=0}^{\infty} P(\boldsymbol{X}, t) \lambda^t , \qquad \tilde{P}_1(\boldsymbol{X}, \lambda) = \sum_{t=1}^{\infty} P_1(\boldsymbol{X}, t) \lambda^t \tag{A.13}$$

(note the difference in the range of summation, due to the fact that $P_1(\boldsymbol{X}, 0)$ is non-zero only for $\boldsymbol{X} = \boldsymbol{0}$) Formula (A.12) then reads

$$\tilde{P}(\boldsymbol{X}, \lambda) - \tilde{P}(\boldsymbol{X}, 0) = \tilde{P}_1(\boldsymbol{X}, \lambda) \tilde{P}(\boldsymbol{0}, \lambda) ,$$

which finally yields

$$\tilde{P}_1(\boldsymbol{X}, \lambda) = \tilde{P}(\boldsymbol{X}, \lambda) / \tilde{P}(\boldsymbol{0}, \lambda) , \quad \boldsymbol{X} \neq \boldsymbol{0} , \qquad \tilde{P}_1(\boldsymbol{0}, \lambda) = 1 - \tilde{P}(\boldsymbol{0}, \lambda)^{-1} \tag{A.14}$$

Using (A.4), $\tilde{P}(\boldsymbol{k}, \lambda)$ reads

$$\tilde{P}(\boldsymbol{k}, \lambda) = [1 - \lambda p(\boldsymbol{k})]^{-1} \tag{A.15}$$

Several interesting quantities can be obtained from $\tilde{P}_1(\boldsymbol{X}, \lambda)$. The probability to visit site $\boldsymbol{X}$ at least once reads

$$\sum_{t=1}^{\infty} P_1(\boldsymbol{X}, t) = \tilde{P}_1(\boldsymbol{X}, \lambda = 1) , \tag{A.16}$$

which, for $\boldsymbol{X} = \boldsymbol{0}$, is the probability to come back at least once at the initial site,

$$p_0 = \tilde{P}_1(\boldsymbol{0}, \lambda = 1) = 1 - \tilde{P}(\boldsymbol{0}, 1)^{-1} . \tag{A.17}$$

In the simple case of a hypercubic lattice with nearest-neighbour jumps $\tilde{P}(\boldsymbol{0}, \lambda)$ reads

$$\tilde{P}(\boldsymbol{0}, \lambda) = \left(\frac{a}{2\pi}\right)^d \int \mathrm{d}^d k \left(1 - \frac{\lambda}{d} \sum_{\mu=1}^{d} \cos(a k_\mu)\right)^{-1} . \tag{A.18}$$

This integral is infra-red divergent for $\lambda = 1$ if $d \leq 2$ and converges for $d > 2$. As a result, the walker always comes back to its initial position in $d \leq 2$ ($p_0 = 1$) while p_0 is finite for $d > 2$ ($p_0 = 0.3405$. for a cubic lattice) The divergence of $\tilde{P}(\mathbf{0}, \lambda)$ for $d < 2$ is characterized by

$$\begin{aligned} &\tilde{P}(\mathbf{0}, \lambda) \sim (1-\lambda)^{d/2-1}, && d < 2, \\ &\tilde{P}(\mathbf{0}, \lambda) \sim (1/\pi) \ln[1/(1-\lambda)], && d = 2, \end{aligned} \qquad \lambda \to 1 . \tag{A.19}$$

Using (A 14), this yields the time dependence of the distribution of the first passage times at the initial site, $P_1(0, t)$, at large time,

$$\begin{aligned} &P_1(\mathbf{0}, t) \sim C_d / t^{2-d/2}, && d < 2, \\ &P_1(\mathbf{0}, t) \sim c / \ln t, && d > 2 \end{aligned} \tag{A.20}$$

Another quantity of physical interest is the distribution of the number S_t of visited sites. This is a difficult problem, of which no full explicit solution is known. (Note that the probability that $S_t = t$ is proportional to the number of self-avoiding walks of t steps on the lattice!) The *average* value of S_t can, however, be obtained from $P_1(\boldsymbol{X}, t)$, since

$$\overline{S_t} = 1 + \sum_{\boldsymbol{X} \neq \mathbf{0}} \sum_{t'=1}^{t} P_1(\boldsymbol{X}, t) \tag{A 21}$$

($\Sigma_{t'=1}^{t} P_1(\boldsymbol{X}, t)$ is the probability that $\boldsymbol{X}$ has been visited between $t' = 1$ and $t' = t$.) The generating functions are thus related by

$$\overline{\tilde{S}(\lambda)} = (1-\lambda)^{-2} \tilde{P}(\mathbf{0}, \lambda)^{-1} \tag{A.22}$$

Using (A.19) together with

$$\begin{aligned} &\tilde{P}(\mathbf{0}, \lambda) \sim A_d + \chi_d (1-\lambda)^{d/2-1} + \cdots, && 2 < d < 4, \\ &\tilde{P}(\mathbf{0}, \lambda) \sim A_d + B_d(\lambda - 1) + \chi_d (1-\lambda)^{d/2-1} + \cdot, && d > 4, \end{aligned} \tag{A.23}$$

leads to

$$\bar{S}_t \sim \begin{cases} t^{d/2} + , & d < 2, \\ t/\ln t + , & d = 2, \\ t/A_d + \alpha_d t^{2-d/2} + \cdot, & 2 < d < 4, \\ t/A_d + B_d/A_d^2 + , & d > 4 \end{cases} \tag{A 24}$$

The average number of visits of a site thus diverges at long time for $d < 2$, while it is finite for $d > 2$ This is of great importance to understand the effect of disorder on random walks (see in particular ch. 4)

A.3 Continuous-time random walks

A general continuous-time random walk (CTRW) is defined by the probability $\psi(\boldsymbol{e}, t)\,\mathrm{d}t$ that the walker remains for a time t at a given site before performing a jump of length $\boldsymbol{e}$ between t and $t + \mathrm{d}t$. ψ is normalized by $1 = \Sigma_{\boldsymbol{e}} \int_0^\infty \mathrm{d}t\, \psi(\boldsymbol{e}, t)$ The probability that the walker remains at the same site between 0 and t thus reads

$$\phi(t) = 1 - \sum_{\boldsymbol{e}} \int_0^t \psi(\boldsymbol{e}, t')\,\mathrm{d}t' \,. \tag{A 25}$$

Let us consider a single walker with initial condition $P(\boldsymbol{X}, t=0) = \delta_{\boldsymbol{X},\boldsymbol{0}}$ and denote by $Q(\boldsymbol{X}, t)\,\mathrm{d}t$ the probability that the walker has arrived on site $\boldsymbol{X}$ between t and $t + \mathrm{d}t$ and has not moved since. $P(\boldsymbol{X}, t)$ and $Q(\boldsymbol{X}, t)$ obey the following coupled equations:

$$\begin{aligned} P(\boldsymbol{X}, t) &= \int_0^t \mathrm{d}t'\, \phi(t-t') Q(\boldsymbol{X}, t') + \phi(t)\delta_{\boldsymbol{X}\,\boldsymbol{0}} \,, \\ Q(\boldsymbol{X}, t) &= \sum_{\boldsymbol{e}} \int_0^t \mathrm{d}t'\, \psi(\boldsymbol{e}, t-t') Q(\boldsymbol{X}-\boldsymbol{e}, t') + \sum_{\boldsymbol{e}} \psi(\boldsymbol{e}, t)\delta_{\boldsymbol{X}-\boldsymbol{e},\boldsymbol{0}} \,, \end{aligned} \tag{A.26}$$

which are easily solved by Fourier–Laplace transform (on space and time, respectively),

$$\tilde{Q}(\boldsymbol{k}; E) = \frac{\tilde{\psi}(\boldsymbol{k}; E)}{1 - \tilde{\psi}(\boldsymbol{k}; E)} \,, \qquad \tilde{P}(\boldsymbol{k}; E) = \frac{1 - \tilde{\psi}(\boldsymbol{k}=\boldsymbol{0}, E)}{E[1 - \tilde{\psi}(\boldsymbol{k}, E)]} \tag{A 27}$$

Remark When considering other types of initial conditions (e.g., for a stationary initial state) the first jump requires a special treatment and the solution (A 27) is modified For a thorough discussion of this point, see e.g ref. [G13] and references therein.

The CTRWs considered in chapter 1 are *separable*, i.e., such that

$$\psi(\boldsymbol{e}, t) = p(\boldsymbol{e})\psi(t) \,, \tag{A 28}$$

where $p(\boldsymbol{e})$ is the jump length distribution and $\psi(t)$ is the waiting-time distribution (w.t.d.). Then

$$\tilde{P}(\boldsymbol{k}; E) = \frac{1 - \tilde{\psi}(E)}{E[1 - \tilde{\psi}(E)p(\boldsymbol{k})]} \,, \tag{A 29}$$

from which the diffusion behaviour and diffusion front of section 1 2.3.1 are easily derived by expansion for small E and small k. Two particular cases are worth mentioning:

* Poissonian w.t.d $\psi(t) = W \exp(-Wt)$, for which jumps are performed at a constant rate W In this case

$$\tilde{P}(\boldsymbol{k}, E) = \frac{1}{E + W[1 - p(\boldsymbol{k})]} \,. \tag{A.30}$$

* The discrete-time random walk of section A.1 is recovered provided one chooses $\psi(t) = \delta(t - \tau_0)$ Then

$$\tilde{P}(\boldsymbol{k}, E) = \frac{1 - e^{-E\tau_0}}{E} \frac{1}{1 - e^{-E\tau_0} p(\boldsymbol{k})} , \tag{A.31}$$

and one should identify λ in (A 13) with $\exp(-E\tau_0)$ Note, however, that the two expressions (A.15) and (A 31) are different, not surprisingly, the resulting values of $P(\boldsymbol{X}, t)$ only coincide for times $t = n\tau_0$ with integer n

Appendix B. Limit distributions for sums of independent random variables; stable laws

Consider the following sum

$$Z_n = \sum_{k=1}^{n} x_k , \tag{B.1}$$

where the x_k are independent and all distributed according to the same probability distribution $p(x)$ One would like to know the answer to the following questions
- How must one choose the two normalizations A_n, B_n in order to obtain a limit distribution (when $n \to \infty$) for the rescaled variable $u = Z_n/B_n - A_n$?
- What is the distribution $P(u)$ such that

$$\text{Prob}[u_1 \le u \le u_2] \xrightarrow[n\to\infty]{} \int_{u_1}^{u_2} P(u)\, du \text{ ?} \tag{B 2}$$

In this case, one says that p belongs to the attraction basin of P

This problem is a classic in probability theory since the work of Khintchine and Levy Useful references are refs [G2–G4], with a particular mention of the brilliant book by Gnedenko and Kolmogorov [G2], from which we have extracted most of the material presented in this appendix

B 1 Attraction basin of the normal law and convergence towards it

The following theorem characterizes fully the situations where P is the Gaussian (normal) law·

Theorem 1 (Khintchine, Feller, Levy) $p(x)$ belongs to the attraction basin of the normal law if and only if

$$\lim_{X\to\infty} X^2 \frac{\int_{|x|>X} p(x)\, dx}{\int_{|x|>X} x^2 p(x)\, dx} = 0 \tag{B 3}$$

This theorem is the most refined form of the CLT mentioned in section 1 1 It allows one to state that, for example, a distribution $p(x)$ decaying as x^{-3} for large x belongs to the attraction basin of the normal

distribution, even though its variance is infinite. All the distributions decaying faster than the latter also belong to the attraction basin of the Gaussian, which is thus extremely vast. This is of course the reason why the Gaussian law is omnipresent in physical situations, "anomalous" behaviour being comparatively rarer

Concerning the normalisations A_n, B_n one has the following theorem:

Theorem 2 If and only if $p(x)$ has a finite variance $\sigma = \langle x^2 \rangle - \langle x \rangle^2$ will the normalizations read

$$B_n = \sqrt{\sigma n}\,, \tag{B 4}$$

$$A_n B_n = n\langle x \rangle\,. \tag{B.5}$$

$P(x)$ then reads $G(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

The fact that a finite σ is a sufficient condition was already known to Laplace (see, e.g., ref. [G9]).

Convergence towards the normal law [Chebyshev 1887]. It is possible to characterize in a precise manner the convergence of $P(x)$ towards the normal law when $n \to \infty$ by a systematic expansion of the difference in powers of $n^{-1/2}$. One obtains (for $\langle x \rangle = 0$, $\sigma = 1$)

$$\int_{-\infty}^{Z} [P_n(u) - G(u)]\,\mathrm{d}u \simeq (2\pi)^{-1/2} \exp(-Z^2/2)\left(\frac{Q_1(Z)}{n^{1/2}} + \frac{Q_2(Z)}{n} + \cdots + \frac{Q_k(Z)}{n^{k/2}} + \cdot\ \cdot\right), \tag{B 6}$$

where the Q_k are polynomials (see ref. [G2] for their general expression), the first two of which read ($\lambda_k = \langle x^k \rangle_{\mathrm{conn}} / \sigma^{k/2}$)

$$Q_1(x) = \tfrac{1}{6}\lambda_3(1 - x^2)\,, \qquad Q_2(x) = (10/6!)\lambda_3^2 x^5 + \tfrac{1}{8}(\tfrac{1}{3}\lambda_4 - \tfrac{10}{9}\lambda_3^2)x^3 + (\tfrac{5}{24}\lambda_3^2 - \tfrac{1}{8}\lambda_4)x$$

Hence the way P converges towards the normal law G *depends upon the details of the initial distribution* p through higher and higher moments for shorter "times".

B.2. Stable laws; general characterization and attraction basins

The distribution of the sum of two independent variables is the *convolution* of their probability distributions. Therefore the fundamental property allowing the classification of all possible limiting distributions is the invariance of these under convolution. More precisely,

Theorem 3. For $P(x)$ to be a possible limiting distribution for the above reduced variable, there must exist, for all a_1, $a_2(>0)$, b_1, b_2 two quantities $a(>0)$, b such that

$$P(a_1 x + b_1) * P(a_2 x + b_2) = P(ax + b)\,. \tag{B.7}$$

In particular, the normal law satisfies this condition. In Fourier space, convolution is simply multiplication of the Fourier transforms. Hence the classification of stable laws takes a particularly simple form stated in terms of their characteristic functions.

Theorem 4. Canonical representation of stable laws (Levy, Khintchine). $P(x) = (2\pi)^{-1} \times \int_{-\infty}^{\infty} e^{ikx} \hat{P}(k)\, dk$ is a stable law if and only if its characteristic function reads

$$\ln \hat{P}(k) = i\gamma k - C|k|^{\mu}[1 + i\beta \operatorname{sign}(k)\, \omega(k, \mu)], \tag{B.8}$$

where μ, β, γ, C are real numbers such that $-1 \leq \beta \leq +1$, $0 < \mu \leq 2$, $C \geq 0$, and

$$\omega(k, \mu) = \tan(\pi\mu/2) \quad \text{for } \mu \neq 1, \qquad \omega(k, \mu) = (2/\pi) \ln |k| \quad \text{for } \mu = 1 \tag{B.9}$$

γ and C are simply "scale" factors, associated to change in origin ($x \to x + cte$) or dilatation ($x \to rx$) The two really important parameters are thus μ and β, and we shall denote the stable laws by $L_{\mu,\beta}$ μ characterizes their large-x behaviour: $L_{\mu,\beta} \simeq |x|^{-(1+\mu)}$ (for $\mu < 2$). All the positive moments $\langle |x|^k \rangle$ of $L_{\mu,\beta}$ are finite for $k < \mu$ and infinite otherwise In particular, the variance of $L_{\mu,\beta}$ is infinite for $\mu < 2$ The number β characterizes the asymmetry of these laws·

– for $\beta = 0$, one has an even function of x,

– for $\beta = \pm 1$, the law is "maximally" asymmetric, it is, for $0 < \mu < 1$, concentrated in $]-\infty, \gamma]$ for $\beta = -1$ and in $[\gamma, \infty[$ for $\beta = +1$

For $\mu = 2$, β disappears since in that case $\omega = 0$; one recovers the unique normal law centred on γ. Some explicit forms and asymptotic expansions of $L_{\mu,\beta}$ will be given in the next section The initial problem is completely solved by the following theorem, characterizing the *attraction basin* of stable laws

Theorem 5a (Gnedenko, Doeblin). $p(x)$ belongs to the attraction basin of $L_{\mu,\beta}$ if and only if its repartition function $R(X) = \int_{-\infty}^{X} p(x)\, dx$ satisfies the following properties:

$$\text{(i)} \quad \lim_{X \to \infty} \frac{R(-X)}{1 - R(X)} = \frac{1 - \beta}{1 + \beta}, \tag{B 10}$$

(ii) For any r,

$$\lim_{X \to \infty} \frac{1 - R(X) + R(-X)}{1 - R(rX) + R(-rX)} = r^{\mu} \tag{B 11}$$

This theorem echoes theorem 1 for the normal law. It means that $L_{\mu,\beta}$ attracts all the distributions $p(x)$ which essentially behave as $L_{\mu,\beta}$ at infinity. As such, this theorem does not specify the normalizations A_n, B_n to be chosen (see, however, ref. [G2], p 175) A very important practical case is when $p(x)$ decays purely algebraically; in this case, $B_n \simeq n^{1/\mu}$ (apart from logarithms for $\mu = 1, 2$), and the most useful theorem, specifying the behaviour guessed by the simple arguments of section 1 1, is:

Theorem 5b (Gnedenko) $p(x)$ belongs to the attraction basin of $L_{\mu,\beta}$ with $B_n = n^{1/\mu}$, if and only if

$$p(x) \simeq c_- |x|^{-(1+\mu)} \quad \text{if } x \to -\infty, \qquad p(x) \simeq c_+ x^{-(1+\mu)} \quad \text{if } x \to \infty \tag{B.12}$$

Then $\beta = (c_+ - c_-)/(c_+ + c_-)$ and

$$A_n = 0\,, \qquad C = \frac{\pi(c_+ + c_-)}{2\mu \sin(\pi\mu/2)\,\Gamma(\mu)}\,, \quad \text{for } 0<\mu<1\,, \tag{B.13}$$

$$A_n B_n = n\langle x\rangle\,, \qquad C = \frac{\pi(c_+ + c_-)}{2\mu^2 \sin(\pi\mu/2)\,\Gamma(\mu-1)}\,, \quad \text{for } 1<\mu<2\,; \tag{B.14}$$

for $\mu = 1$, $\mu = 2$, see ref. [G2].

Let us sketch the proof that a distribution $p(x)$ satisfying the above conditions indeed belongs to the attraction basin of $L_{\mu,\beta}$. Following the lines of section 1.1, one has to study the limit of $\hat{p}(k/N^{1/\mu})$ for large N. It is easily seen that in this limit one has

$$\hat{p}(k/N^{1/\mu}) \simeq 1 - \frac{|k|^\mu}{N}(c_+ + c_-)\,\mathrm{Re}(I_\mu)\left(1 - \mathrm{i}\,\mathrm{sign}(k)\,\frac{c_+ - c_-}{c_+ + c_-}\,\frac{\mathrm{Im}(I_\mu)}{\mathrm{Re}(I_\mu)}\right), \tag{B.15}$$

$$I_\mu = \int_0^\infty \mathrm{d}u\,(1 - \mathrm{e}^{\mathrm{i}u})u^{-1-\mu} = \mathrm{e}^{-\mathrm{i}\pi\mu/2}\,\frac{\pi}{\mu \sin(\pi\mu)\,\Gamma(\mu)}\,, \quad \text{for } 1>\mu>0\,;$$

$$\hat{p}(k/N^{1/\mu}) \simeq 1 - \frac{\mathrm{i}k\langle x\rangle}{N^{1/\mu}} - \frac{|k|^\mu}{N}(c_+ + c_-)\,\mathrm{Re}(J_\mu)\left(1 - \mathrm{i}\,\mathrm{sign}(k)\,\frac{c_+ - c_-}{c_+ + c_-}\,\frac{\mathrm{Im}(J_\mu)}{\mathrm{Re}(J_\mu)}\right), \tag{B.16}$$

$$J_\mu = \int_0^\infty \mathrm{d}u\,(1 + \mathrm{i}u - \mathrm{e}^{\mathrm{i}u})u^{-1-\mu} = \mathrm{e}^{-\mathrm{i}\pi\mu/2}\,\frac{\pi}{\mu^2 \sin(\pi\mu)\,\Gamma(\mu-1)}\,, \quad \text{for } 2>\mu>1\,.$$

The expressions for C and β directly follow from these expansions.

B.3. Some useful explicit expressions and expansions

In this section we only quote the most useful results on stable laws. More information can be found in ref. [G2–G4].

The symmetrical laws $L_{\mu,0}$. One can always choose $\gamma = 0$, $C = 1$ [since $L^C_{\mu,\beta}(x) = C^{-1/\mu}L^1_{\mu,\beta}(xC^{-1/\mu})$] and hence study the density $L_{\mu,0}$ defined as

$$L_{\mu,0} = (2\pi)^{-1}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx - |k|^\mu}\,\mathrm{d}k\,. \tag{B.17}$$

$L_{2,0}$ of course is identical to the Gaussian G, and $L_{1,0}$ is the Cauchy distribution $L_{1,0} = 1/\pi(1+x^2)$.
– $L_{\mu,0}$ takes a finite value at the origin: $L_{\mu,0}(0) = (\pi\mu)^{-1}\Gamma(1/\mu)$, which becomes very large as $\mu \to 0$.
– The (Cauchy) expansion around $x = 0$ reads

$$L_{\mu,0}(x) = (\pi\mu)^{-1}\sum_{k=0}^{\infty}(-)^k\,\frac{x^{2k}}{2k!}\,\Gamma((2k+1)/\mu)\,, \tag{B.18}$$

the radius of convergence of which is zero for $\mu < 1$ and infinite for $1 < \mu < 2$

– The (Wintner) expansion for large arguments is

$$L_{\mu,0}(x) = (\pi)^{-1} \sum_{k=1}^{\infty} (-)^{k+1} \frac{x^{-(\mu k+1)}}{k!} \Gamma(1+k\mu) \sin(\pi\mu k/2)\,, \tag{B 19}$$

the leading term of which reads

$$L_{\mu,0}(x) = (\pi)^{-1} x^{-(\mu+1)} \Gamma(1+\mu) \sin(\pi\mu/2) \tag{B 20}$$

A detailed study of the behaviour of those properties in the (singular) limit $\mu = 0$ can be found in ref [G4]

The negative moments of $L_{\mu\,0}$ read

$$\langle x^{-\nu} \rangle = \frac{\Gamma(\nu/\mu)}{\mu\Gamma(\nu)\cos(\pi\nu/2)} \tag{B.21}$$

The asymmetrical laws $L_{\mu\,\pm 1}$ Still choosing $\gamma = 0$, one can show that $L_{\mu\,1}$ (which we shall denote simply by L_μ) can be expressed as an inverse Laplace transform,

$$L_\mu(x) = \frac{1}{2\pi \mathrm{i}} \int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \mathrm{d}s\, \mathrm{e}^{sx - C' s^\mu}\,, \tag{B 22}$$

with $C' = C/\cos(\pi\mu/2)$ This law has the support $[0, +\infty[$ for $\mu < 1$ and $]-\infty, +\infty[$ for $1 < \mu < 2$

N B A stretched exponential decay (Kohlrausch law) is thus obtained if the relaxation time density is an L_μ stable law,

$$\mathrm{e}^{-C' t^\mu} = \int_0^\infty L_\mu(\tau^{-1})\, \mathrm{e}^{-t/\tau}\, \mathrm{d}(\tau^{-1}) \tag{B 23}$$

L_μ takes a particularly simple form for $\mu = 1/2$ or $\mu = 1/3$·

– For $\mu = 1/2$,

$$L_{1/2}(x) = \Theta(x) \frac{C}{\sqrt{2\pi}\, x^{3/2}} \mathrm{e}^{-C^2/2x} \tag{B 24}$$

(Θ is the step function). This distribution has a simple physical interpretation: it is the limiting law of return times to the origin for a one-dimensional symmetrical random walk The nth return time τ_n to the origin is typically of order n^2, and

$$\mathrm{Prob}[z \le 2\tau_n/\pi n^2 \le z + \mathrm{d}z] \xrightarrow[n\to\infty]{} L_{1/2}(z)\, \mathrm{d}z \tag{B.25}$$

– For $\mu = 1/3$, $L_{1/3}$ is a modified Bessel function of order 1/3 (see section 1 2 4 for a physical application, for $x > 0$,

$$xL_{1/3}(x) = (u/\pi) \sin(\pi/3)\, K_{1/3}(u)\,, \quad u = 2(2C/3^{3/2}x^{1/3})^{3/2} \tag{B 26}$$

For large x, the expansion of $L_\mu(x)$ reads, for $\mu < 1$,

$$L_\mu(x) = -(\pi x)^{-1} \sum_{k=1}^{\infty} \left(\frac{-C'}{x^\mu}\right)^k \frac{\Gamma(1+k\mu)}{\Gamma(1+k)} \sin(\pi\mu k) \tag{B.27}$$

Hence the leading behaviour for x large is, for $\mu < 1$,

$$L_\mu(x) \simeq C \frac{2\mu\Gamma(\mu)}{\pi x^{(1+\mu)}} \sin(\pi\mu/2) . \tag{B 28}$$

For small x, L_μ exhibits an *essential singularity*,

$$xL_\mu(x) \simeq [2\pi(1-\mu)\zeta]^{-1/2} \exp\left(-\frac{1-\mu}{\mu\zeta}\right), \quad \zeta = \left(\frac{x^\mu}{\mu C'}\right)^{1/(1-\mu)} \tag{B 29}$$

This behaviour controls the shape of the diffusion front in the random walk problems encountered in chs. 1 and 3.

The negative moments of L_μ read

$$\langle x^{-\nu} \rangle = C'^{-\nu/\mu} \Gamma(\nu/\mu)/\mu\Gamma(\nu) . \tag{B.30}$$

Appendix C. Diffusion behaviour and diffusion front in the Matheron–de Marsily layered model

In this appendix, some analytical results are derived for the diffusion behaviour and diffusion fronts of the layered model [Mat80] described in section 1.3.2, with particular emphasis on the different possible ensemble averages. This question has been the subject of recent investigations [Dou89a, Red89, Bou89f] (we are grateful to Redner for having drawn our attention to some of the points below).

The velocity distribution will be taken to be white noise,

$$\langle V \rangle = 0 , \qquad \langle V(Z)V(Z') \rangle = \sigma_V \delta(Z - Z') ,$$

and the thermal noise along Z will be neglected (it is a non-leading contribution to diffusion at large time, so we can set $D_\parallel = 0$).

The motion along Z is a standard Brownian motion,

$$Z_t = \int^t \eta(t')\, dt' , \quad \overline{\eta_t \eta_{t'}} = 2D_\perp \delta(t - t') , \qquad \overline{Z_t^2} = 2D_\perp t , \tag{C.1}$$

and the position X_t depends on both the "thermal history" $(Z_{t'}; t' \le t)$ and the environment $(V(Z))$. The beauty and simplicity of this model is that one has an explicit form for this dependence,

$$X_t = \int^t V[Z_{t'}]\, dt' . \tag{C.2}$$

In taking averages, it will be necessary to separate independent variables in this expression. Defining $n_t(Z)$ as the number of times the Brownian motion $(Z_{t'}; t' < t)$ has visited position Z,

$$n_t(Z) = \int^{t} \delta(Z - Z_{t'})\, \mathrm{d}t' , \tag{C.3}$$

(C 2) can be cast in the convenient form

$$X_t[(Z_{t'}); (V(Z))] = \int_{-\infty}^{+\infty} \mathrm{d}Z\, n_t(Z) V(Z) \tag{C.4}$$

The *thermal average* of the position thus reads

$$\overline{X_t} = \int \mathrm{d}Z\, \overline{n_t(Z)} V(Z) \tag{C 5}$$

$\overline{n_t(Z)}$ is easily computed from the knowledge of the (Gaussian) probability distribution

$$P_\perp(Z, t) = (4\pi D_\perp t)^{-1/2} \exp(-Z^2/4D_\perp t)$$

of the transverse Brownian motion One obtains

$$\overline{n_t(Z)} = \int_0^t \mathrm{d}t'\, P_\perp(Z, t') = \frac{|Z|}{4\sqrt{\pi} D_\perp}\, \Gamma(-\tfrac{1}{2}, Z^2/4D_\perp t) , \tag{C 6}$$

where $\Gamma(-\frac{1}{2}; x)$ denotes the incomplete gamma function (note that it is simply related to the error function) Thus

$$\overline{X_t} = \frac{1}{4\sqrt{\pi} D_\perp} \int_{-\infty}^{+\infty} \mathrm{d}Z\, |Z| \Gamma(-\tfrac{1}{2}, Z^2/4D_\perp t) V(Z) \tag{C 7}$$

$\overline{X_t}$ still depends on the environment $(V(Z))$ In particular, *it does not converge to zero at large time* for a given $(V(Z))$, only its average over environments $\langle \overline{X_t} \rangle$ is zero (for all times, since no global bias is present) $\overline{X_t}$ has a distribution over environments which is obviously a Gaussian of variance $\langle (\overline{X_t})^2 \rangle$ The latter can be calculated in closed form from (C 7),

$$\begin{aligned} &\langle (\overline{X_t})^2 \rangle = \sigma_V \int \mathrm{d}Z\, [\overline{n_t(Z)}]^2 = \sigma_V D_\perp^{-1/2} I t^{3/2} , \\ &I = \frac{1}{\pi} \int_0^\infty u^2 \Gamma(-\tfrac{1}{2}, u^2)^2\, \mathrm{d}u = \frac{4}{3\sqrt{\pi}} (\sqrt{2} - 1) \end{aligned} \tag{C 8}$$

The thermal average of the squared position is related to the correlation function $\overline{n_t(Z) n_t(Z')}$ as

$$\overline{X_t^2} = \int \mathrm{d}Z\, \mathrm{d}Z'\, V(Z) V(Z') \overline{n_t(Z) n_t(Z')} , \tag{C 9}$$

and its average is also easily computed,

$$\langle \overline{X_t^2} \rangle = \sigma_V \int \mathrm{d}Z\, \overline{n_t(Z)^2}$$

$$= 2\sigma_V \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, P_\perp(Z, t'|Z, t'') P_\perp(Z, t''|0, 0)$$

$$= 2\sigma_V \int_0^t \mathrm{d}t' \int_0^{t'} \frac{\mathrm{d}t''}{\sqrt{4\pi D_\perp (t' - t'')}} = \frac{4}{3} \frac{\sigma_V}{\sqrt{\pi D_\perp}} t^{3/2} . \tag{C.10}$$

One concludes from these results that both $\langle \overline{X_t^2} \rangle - (\langle \overline{X_t} \rangle)^2 = \langle \overline{X_t^2} \rangle$ and $\langle \overline{X_t^2} \rangle - \langle (\overline{X_t})^2 \rangle$ behave anomalously as $D_\perp^{-1/2} t^{3/2}$ at large time, but with *different prefactors*,

$$\langle \overline{X_t^2} \rangle = \frac{4}{3} \frac{\sigma_V}{\sqrt{\pi D_\perp}} t^{3/2} , \tag{C 11a}$$

$$\langle \overline{X_t^2} - (\overline{X_t})^2 \rangle = \tfrac{4}{3}(2 - \sqrt{2}) \frac{\sigma_V}{\sqrt{\pi D_\perp}} t^{3/2} . \tag{C 11b}$$

(Note that only the average one, $\langle \overline{X_t^2} \rangle$, was obtained in the original paper [Mat80].) It is expected (as would be confirmed by a calculation of its fluctuation) that the behaviour of $\overline{X_t^2} - (\overline{X_t})^2$ is indeed *self-averaging* and behaves as its average (C.11b) for a *given* environment ($V(Z)$) However, it follows from (C.11) that the distribution of the rescaled position $X_t/t^{3/4}$ over thermal histories [i.e., $P(X, t)$] for a given sample *does not obey a "generalized CLT" at large time* (*i.e., does not reach a limit form*) Only the average front $\langle P(X, t) \rangle$ does, on which we now comment.

* It is easily shown from (C.4) that X_t, being an integral of $V(Z)$, has a distribution over samples for a fixed thermal history which is a Gaussian of variance

$$Q = \sigma_V \int \mathrm{d}Z\, n_t(Z)^2 \tag{C.12}$$

This quantity has still a distribution over thermal histories. Computing its average (that is, $\langle \overline{P(X, t)} \rangle$) requires the knowledge of the distribution of Q (or alternatively of the characteristic function $\mathrm{e}^{\overline{\alpha Q}}$). We shall not attempt to calculate fully this distribution here, but rather make a conjecture on the *tail* of $\langle \overline{P(X, t)} \rangle$, in the regime $X \gg t^{3/4}$ Physically, the main contributions to this tail is from walks which have visited a small number of layers, and thus for which Q is *much larger than its average* (since this quantity counts the number of at least doubly visited layers). Let us assume that the distribution of Q (over histories) behaves for $Q \gg \bar{Q}$ as $\exp[-(Q/\bar{Q})^\beta]$, where β is some exponent The tail of $\langle \overline{P(X, t)} \rangle$ is deduced from (C.12) by a saddle point estimate,

$$\int \mathrm{d}Q \exp[-(Q/\bar{Q})^\beta - X^2/Q] \sim \exp[-(X/\langle \overline{X^2} \rangle^{1/2})^\delta] , \quad \delta = \frac{2\beta}{1 + \beta}$$

It can be argued [Bou89f] (by estimating the weight of "confined" walks) that the distribution of Q is in fact *Gaussian* at large Q, i.e , $\beta = 2$. This suggests that the tail of $\langle \overline{P(X, t)} \rangle$ for $X \gg t^{3/4}$ has the

markedly *non-Gaussian* behaviour $\exp[-(X/t^{3/4})^{4/3}]$. One observes that the value $\delta = 1/(1-\nu) = 4$ discussed through linear response arguments in section 5 3.1 and 5.3 2 is *not obeyed*; as pointed out at the end of section 5.3.1, this is not surprising in view of the existence of correlations up to the scale $\sigma_V/DF^3 \gg kT/F$. The shape of the full scaling function associated with $\langle P(X,t)\rangle$,

$$\langle P(X,t)\rangle \to t^{-3/4} f(X/t^{3/4}) ,$$

has been very recently investigated numerically [Bou89f] and analytically [Dou90, Zum90]. Another open question on this model is whether the histogram of the position of a single walker for a *given* history *and* environment indeed coincide (when rescaled by $t^{3/4}$) with $\langle P(X,t)\rangle$, as ergodicity would suggest, or whether the model displays some kind of "ergodicity breaking"

Appendix D. Two theorems on electrical networks

D 1 Derrida's proof of relation (2.15) [G14]

If one considers a random network (σ_{ij} is the conductance between site i and j) and if each site is connected to the mass by a capacitance C, the time dependence of the potential V_i on site i is given by

$$C \frac{dV_i}{dt} = \sum_j \sigma_{ij}(V_j - V_i) \tag{D.1}$$

We see that this equation is the same as the master equation (2 1) for P_j So the properties of the two problems should be related. We are now going to see that the diffusion constant for the diffusion problem is related to the conductivity of the random resistor network.

(*a*) *The diffusion constant.* One can define a diffusion constant D_α corresponding to the direction $\boldsymbol{\alpha}$ by

$$\langle [\overline{\boldsymbol{x}(t)\cdot\boldsymbol{\alpha}}]^2 \rangle \sim 2D_\alpha t . \tag{D.2}$$

$\langle\ \rangle$ means average over the starting points.

We are now going to calculate D_α for a *periodic lattice* in d dimensions with an elementary cell Ω containing l^d sites with arbitrary hopping rates $W_{xx'}$ in this cell. The master equation is

$$dP_x/dt = \sum_{x'} W_{xx'}(P_{x'} - P_x) , \tag{D.3}$$

and the periodicity of the lattice means that $d \cdot l^d$ arbitrary values of $W_{xx'}$ are given and that

$$W_{xx'} = W_{x+nl,x'+nl} \quad \text{for any } \boldsymbol{n} \in \mathbb{Z}^d$$

Because the lattice is periodic, it is convenient to introduce two quantities for each site x of the cell,

$$Q_x = \sum_{\boldsymbol{n}} P_{x+nl} , \qquad \boldsymbol{R}_x = \sum_{\boldsymbol{n}} (\boldsymbol{x} + \boldsymbol{n}l) P_{x+nl}$$

These quantities satisfy the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_x = \sum_{x'} W_{xx'}(Q_{x'} - Q_x), \qquad \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{R}_x = \sum_{x'} W_{xx'}(\boldsymbol{R}_{x'} - \boldsymbol{R}_x) + \sum_{x'} (\boldsymbol{x} - \boldsymbol{x}') W_{xx'} Q_{x'} \tag{D 4}$$

Let us see how $(\mathrm{d}/\mathrm{d}t)\langle(\boldsymbol{x}\cdot\boldsymbol{\alpha})^2\rangle$ can be expressed in terms of Q_x and $\boldsymbol{R}_x$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle(\boldsymbol{x}\cdot\boldsymbol{\alpha})^2\rangle &= \sum_x (\boldsymbol{x}\cdot\boldsymbol{\alpha})^2 \frac{\mathrm{d}P_x}{\mathrm{d}t} = \sum_x (\boldsymbol{x}\cdot\boldsymbol{\alpha})^2 \sum_{x'} W_{xx'}(P_{x'} - P_x) \\ &= \sum_{x,x'} [2(\boldsymbol{x}\cdot\boldsymbol{\alpha})(\boldsymbol{x}'\cdot\boldsymbol{\alpha} - \boldsymbol{x}\cdot\boldsymbol{\alpha}) W_{x'x} P_x + (\boldsymbol{x}'\cdot\boldsymbol{\alpha} - \boldsymbol{x}\cdot\boldsymbol{\alpha})^2 W_{x'x}] \\ &= \sum_{x\in\Omega} \sum_{x'} [2(\boldsymbol{R}_x\cdot\boldsymbol{\alpha})(\boldsymbol{x}'\cdot\boldsymbol{\alpha} - \boldsymbol{x}\cdot\boldsymbol{\alpha}) W_{x'x} + (\boldsymbol{x}'\cdot\boldsymbol{\alpha} - \boldsymbol{x}\cdot\boldsymbol{\alpha})^2 Q_x W_{x'x}]\end{aligned} \tag{D.5}$$

For long times, Q_x and $\boldsymbol{R}_x$ have a finite limit,

$$Q_x \to 1/l^d, \qquad \boldsymbol{R}_x \to (1/l^d)\boldsymbol{r}_x, \tag{D 6}$$

where $\boldsymbol{r}_x$ are solutions of

$$\sum_{x'} [W_{xx'}(\boldsymbol{r}_{x'} - \boldsymbol{r}_x) + (\boldsymbol{x} - \boldsymbol{x}')W_{xx'}] = 0. \tag{D.7}$$

So the diffusion constant D_α in the direction $\boldsymbol{\alpha}$ is given by

$$\begin{aligned}2D_\alpha &= \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \langle \overline{[\boldsymbol{x}(t)\cdot\boldsymbol{\alpha}]^2} \rangle \\ &= \frac{1}{l^d} \sum_{x\in\Omega} \sum_{x'} [2(\boldsymbol{r}_x\cdot\boldsymbol{\alpha})(\boldsymbol{x}'\cdot\boldsymbol{\alpha} - \boldsymbol{x}\cdot\boldsymbol{\alpha}) W_{x'x} + (\boldsymbol{x}'\cdot\boldsymbol{\alpha} - \boldsymbol{x}\cdot\boldsymbol{\alpha})^2 W_{x'x}].\end{aligned} \tag{D.8}$$

We are now going to see that the conductivity is given by a very similar expression.

(*b*) *The conductivity of a random resistor network.* Consider again a periodic lattice with a unit cell Ω of sites. All the conductances $\sigma_{xx'}$ in the unit cell are arbitrary and one has

$$\sigma_{xx'} = \sigma_{x+nl,x'+nl}, \quad \boldsymbol{n} \in \mathbb{Z}^d.$$

Let us put two electrodes perpendicular to a direction $\boldsymbol{\alpha}$. Since the medium is periodic, one expects a potential V_x which has the following form:

$$V_x = -\boldsymbol{\alpha}\cdot\boldsymbol{x}E + \psi_x, \quad \psi_x = \psi_{x+nl}. \tag{D.9}$$

One can define $\boldsymbol{r}_x$ by $\psi_x = \boldsymbol{r}_x\cdot\boldsymbol{\alpha}E$. Then the conservation of current reads

$$\sum_{x'} \sigma_{xx'}(V_{x'} - V_x) = 0,$$

which gives

$$-\sum_{i} \sigma_{ii'}(\boldsymbol{x}' - \boldsymbol{x}) + \sum_{i'} \sigma_{ii'}(\boldsymbol{r}_i - \boldsymbol{r}_{i'}) = 0 \tag{D 10}$$

We see that the equation which determines $\boldsymbol{r}_i$ is the same as in the diffusion problem

In a cube of l^d sites, the current in direction $\boldsymbol{\alpha}$ is

$$\frac{1}{2} \sum_{i \in \Omega} \sum_{i'} \boldsymbol{\alpha} \cdot (\boldsymbol{x}' - \boldsymbol{x})(V_i - V_{i'})\sigma_{ii'}$$

(the factor 1/2 is due to the fact that each bond is counted twice) The current j_α per site is

$$j_\alpha = E\sigma_\alpha = \frac{E}{2l^d} \sum_{i \in \Omega} \sum_{i'} [(\boldsymbol{x}' \cdot \boldsymbol{\alpha} - \boldsymbol{x} \cdot \boldsymbol{\alpha})^2 \sigma_{ii'} + 2(\boldsymbol{r}_i \cdot \boldsymbol{\alpha})(\boldsymbol{x}' \cdot \boldsymbol{\alpha} - \boldsymbol{x} \cdot \boldsymbol{\alpha})\sigma_{i'i}] , \tag{D 11}$$

by definition of the conductivity σ_α We see that the expressions of D_α and σ_α are identical (since $\boldsymbol{r}_x$ are given by the same equations)

We have therefore shown that for any periodic lattice

$$D_\alpha = \sigma_\alpha \quad \text{if } W_{ii'} = \sigma_{ii} \tag{D 12}$$

Let us make two remarks concerning the validity of this result.

(i) It was obtained by assuming that $Q_i \to 1/l^d$ and $\boldsymbol{R}_x$ has a limit as $t \to \infty$ This is true only if there is no isolated cluster or site Therefore one should slightly modify this result in the case of percolation networks (see section 6 3 2 1)

(ii) The relation $D_\alpha = \sigma_\alpha$ has been derived for a periodic lattice of period l It is not obvious that the diffusion coefficients of the disordered medium can be obtained by taking D_α in the limit $l \to \infty$ because the two limits $t \to \infty$ and $l \to \infty$ may not commute, see, however, the discussion in chapters 2 and 3

D.2. The "path integral" representation of the conductance [Doy84, Gef87]

Take a disordered lattice characterized by random hopping rates $W_{ij} = W_{ji}$ If one imposes a constant number of particles at sites A and B (N_A and N_B), a non-zero current I of particles has to be supplied (or taken away) at A if $N_A \neq N_B$,

$$I = -N_A \sum_i W_{Ai} + \sum_i N_i W_{iA} = \sum_i W_{iA}(N_i - N_A) \tag{D 13}$$

Solving the master equation with N_A, N_B fixed corresponds, in the electrical language, to solving Kirchoff's rules with fixed *potentials* V_A, V_B I then corresponds to the current injected at A Now, if one calls T_{AB} the total probability for a particle to leave site A and reach site B *without* returning to A at intermediate times and Q_A the total probability for a particle to leave A and come back to A *without* hitting B at intermediate times, one has

$$T_{AB} + Q_A = 1 \tag{D 14}$$

The probability to leave A *per unit time* is clearly $\Sigma_i W_{Ai}$. The current I may thus also be computed as

$$I = \left(N_{Ai} \sum_i W_{Ai}\right)(-1 + Q_A) + \left(N_B \sum_i W_{Bi}\right) T_{BA} . \tag{D 15}$$

Since the hopping rates are symmetrical, the total probability to hop from A to B is equal to that of hopping from B to A; thus

$$\sum_i W_{Ai} T_{AB} = \sum_i W_{Bi} T_{BA} . \tag{D 16}$$

Otherwise stated, if $N_A = N_B$, an equilibrium state with $N_i = N_A = N_B$ for all i can be reached and $I = 0$ From (D.14) and (D.15), (D.16) one derives

$$I = (N_A - N_B) T_{AB} \left(-\sum_i W_{Ai}\right) , \tag{D 17}$$

which, upon the identification $W_{ij} = \sigma_{ij}$, $V_i = N_i$, yields the conductance Σ_{AB} between A and B as

$$\Sigma_{AB} = \left(\sum_i W_{Ai}\right) T_{AB} = \left(\sum_i W_{Bi}\right) T_{BA} \tag{D.18}$$

Note that, in other words, $\Sigma_{AB} = G_{BA} - G_{AA} + G_{AB} - G_{BB}$, where $G_{xy} = P(x, y, E = 0)$ is the Green function of the master equation (D.3).

Hilfer and Blumen [Hil88] have furthermore shown that a simple relation exists between T_{AB} and some *time constants* in the diffusion problem. Namely, if one calls $\overline{t_{AA}}$ the mean first return time to A and $\overline{t_{AB}}$ the mean first passage time at B starting at A, they have shown that

$$T_{AB} = \overline{t_{AA}} / (\overline{t_{AB}} + \overline{t_{BA}})$$

Appendix E. Fluctuation–dissipation theorem in the potential case

If the Brownian particle follows an overdamped Langevin equation in which the force $\boldsymbol{F}$ is the gradient of some potential, then one can prove a fluctuation–dissipation theorem Its formulation depends on the nature of the potential: either it grows sufficiently fast for large distances to localize the particle [i e., $\lim_{t\to\infty} \overline{x^2}(t)$ is finite] or the particle can escape and $\overline{x^2}(t)$ grows without bounds, usually as Dt. The problem dealt with in this appendix is to obtain the response of the particle to a weak oscillating external field $\varepsilon f e^{i\omega t}$. Let us illustrate how this can be reached on the example of a one-dimensional "confining" potential; the calculations are easily transposed to higher dimensions or non-confining potentials (see below).

We thus consider a particle in equilibrium evolving according to $\gamma \dot{x} = -U'(x) + \eta(t) + \varepsilon f e^{i\omega t}$. The associated Fokker–Planck equation for P can be transformed into a Schrodinger equation (see, e.g ,

ref. [G5] and section 3 3 5), which is expanded in powers of ε, $\Psi = \Psi_0 + \varepsilon \Psi_1 e^{i\omega t}$, with

$$\Psi_0(x) = \sqrt{P_0(x)} = Z^{-1} e^{-U/2kT},$$

which is the ground state of

$$H = -D_0 \partial_x^2 + (4D_0 \gamma^2)^{-1} U'(x)^2 - \frac{1}{2\gamma} U''(x), \quad D_0 = \frac{kT}{\gamma}$$

The equation for Ψ_1 reads

$$i\omega \Psi_1 = -H\Psi_1 - (2f/\gamma)\partial_x \Psi_0$$

Introducing $H = \Sigma_\alpha E_\alpha |\alpha\rangle\langle\alpha|$, $|0\rangle \equiv \Psi_0$, $E_0 = 0$ and $\Psi_1 = \Sigma_\beta c_\beta |\beta\rangle$, one obtains

$$c_\beta = -2f\langle\beta|\partial_x|0\rangle / \gamma(E_\beta + i\omega).$$

Noticing that $2D_0\partial_x = [x, H]$, this transforms into

$$\langle \delta x(\omega)\rangle = \int dx\, x P_1(x) = (D_0\gamma)^{-1} \sum_\alpha \frac{E_\alpha}{E_\alpha + i\omega} |\langle\alpha|x|0\rangle|^2 f e^{i\omega t},$$

or, defining the susceptibility $\chi(\omega)$ as $\langle \delta x(\omega)\rangle = \chi(\omega) f e^{i\omega t}$, one has, using $D_0\gamma = kT$,

$$kT\chi(\omega) = \sum_\alpha \frac{E_\alpha}{E_\alpha + i\omega} |\langle\alpha|x|0\rangle|^2$$

The zero-frequency limit takes a very simple form (using $\Sigma_\alpha |\alpha\rangle\langle\alpha| = 1$),

$$\chi(0) = \frac{\int dx\, x^2 P_0(x)}{kT},$$

which is the standard fluctuation–dissipation theorem

Remark. If $U(x)$ is a harmonic potential, $2U(x) = kT_0(x/l)^2$, then one has an explicit formula for $\chi(\omega)$ at all frequencies,

$$\chi(\omega) = \frac{l^2}{kT} \frac{1}{(T_0/T)^2 + i\omega\tau}, \quad \tau = \frac{l^2}{D_0}$$

Following the same lines, one may prove that in the case of a non-confining potential, the mobility, defined as $m(\omega) = \omega\langle\delta x(\omega)\rangle / f e^{i\omega t}$, has a zero-frequency limit given by the Einstein relation,

$$m(\omega = 0) = D/kT,$$

with $D \neq D_0$ the diffusion constant modified by the force field.

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