Approximate Distribution of Hitting Probabilities for a Regular Surface with Compact Support in 2D

D.S.Grebenkov[1]

Laboratoire de Physique de la Matière Condensée, Ecole Polytechnique
91128 Palaiseau Cedex France

Department of Statistical Physics, Saint Petersburg State University
ul. Ulyanovskaya 1, Petrodvorets, 198904, Saint Petersburg, Russia

Abstract

Generalizing the well-known relations on characteristic functions on a plane to the case of a one-dimensional regular surface (curve) with compact support, we establish implicit equations for these functions. Introducing an approximation, we solve the combinatorial problems and reduce these equations to a set of linear equations for a finite number of unknown functions. Imposing natural conditions, we obtain a closed system of linear equations which can be solved for a given surface. Its solutions can be used to calculate the distribution of hitting probabilities for a regular surface with compact support.

In order to verify the accuracy of the approximate distribution of hitting probabilities, numerical analysis is being made for a chosen surface.

Introduction

Search for the distribution of hitting probabilities is an old and a well-known problem. Consider random walk on d-dimensional lattice (in continuous case consider Brownian motion). Then fix a surface of interest $\mathcal{S}$. Suppose that any random walk starts from a given point z which does not lie on $\mathcal{S}$. The problem is to calculate the distribution $P_z(x)$ of probabilities of first contact with points x of the surface $\mathcal{S}$. In other words, we are looking for the probability that random walks from $z \notin \mathcal{S}$ to $x \in S$ do not touch other points $y \in \mathcal{S}\backslash\{x\}$. Of course, the distribution $P_z(x)$ depends on z and $\mathcal{S}$.

This problem has been solved exactly for some particular surfaces. For instance, the case of a planar surface in 2D (an ordinary straight line) is described in any book on probability theory (see [1], [2]). Its generalization for d-dimensional hyperplane is also simple (for example, see the end of Section 2). Note that exact solutions have been found only for some particular surfaces but not in the general case. In the general case, the asymptotic behavior is widely studied, [2].

Problems of the hitting probabilities do not only have a purely mathematical interest. They are important for a wide class of physical problems, in particular, for the problems of Laplacian transfer across an interface, for instance,

[1]E-mail: Denis.Grebenkov@polytechnique.fr

diffusion through a membrane, electrod problems, heterogeneous catalysis, etc. (see [3], [4], [5] for details). Indeed, if we are interested in diffusion through a semi-permeable membrane (points of this membrane can absorb or reflect touching particle with certain probabilities), we can write the total probability of absorption by a chosen point of the membrane as a sum of probabilities to be absorbed after 0, 1, 2, etc. reflections (rigorous formalism is described in [5], [6]). Here we face the task to calculate the distribution of hitting probabilities. Note that using this distribution solely for a planar membrane, we have recently obtained some important results about general characteristics of the Laplacian transfer across an interface, [7]. To solve these problems one needs to know the distribution of hitting probabilities for a general surface. Here we propose a method to approximate the distribution of hitting probabilities for a rather general case in 2D.

In the first section we introduce definitions and conditions which are required in what follows. In the second section we briefly describe a well-known case of the hitting probabilities on a horizontal axis. Main results are contained in the third section. Section 4 is devoted to some numerical results. In the last section we make conclusions and discuss possible generalizations.

1 Definitions

Consider a square lattice on a plane. Let us define a *regular surface*[2] *with compact support* $\mathcal{S} = \{ (x, S(x)) \}$ by a function $S(x)$ with integer x subject to the following conditions:

1. *Bijection*: The function $S(x)$ is a bijection between the set of integer numbers (absiccae x) and the set of surface points ;

2. *Regularity*: For any x, $|S(x+1) - S(x)| \leq 1$;

3. *Compactness*: $\exists M : S(x) = 0$ for $|x| \geq M$, i.e. the non-plane part of the surface has a finite size. In other words, function $S(x)$ has a compact support. Moreover, we suppose that the surface is centered : $S(\pm(M-1)) \neq 0$.

Let us briefly discuss this definition. The second condition allows to simplify all calculations and formulae, but it does not seem to be essential (see Section 5). Note that this assumption can be viewed as *a regularity condition* for the surface in continuous case : $S'(x) \leq 1$.

On the contrary, the third condition is important. It tells us that the surface in question is a finite "perturbation" of a planar surface (line). In other words, this surface is composed of two parts: a complex but compact part in the center with two plane "tails". Moreover, it is important that both tails lie on the same height (which is chosen as 0). This feature will allow to obtain an approximate distribution of hitting probabilities by using the same ideas as for a planar surface (see Section 2).

We call all the points $\{(x, y) \ : \ y = n\}$ the nth *level*. Denote

$$N = \max\{S(x)\}, \qquad N^* = -\min\{S(x)\},$$

i.e. the surface lies between $(-N^*)$th and Nth levels.

[2]Even for two-dimensional case we prefer to use the word "surface" instead of "curve" or something else.

All points $\mathcal{M} = \{(x,y) \ : \ \forall x \quad y < S(x)\}$ are called *internal*. All points $\mathcal{E} = \{(x,y) \ : \ \forall x \quad y > S(x)\}$ are called *external*. The external points near the surface, $\{(x, S(x)+1)\}$ are called *near-boundary points*. The functions defined on these points, are called *near-boundary functions* (see below). Often we shall use the words "surface", "near-boundary functions", etc. thinking only about the non-trivial part, i.e. for $|x| < M$.

The external points with $y = 0$ are called *ground points*. The functions defined on these points, are called *ground functions*. Let $J = \{k \ : \ (k,0) \in \mathcal{E}\}$ the set of abscissae of ground points. Let also $J_0 = \{k \in (-M, M) \ : \ (k,0) \in S\}$ the set of abscissae of boundary points on zeroth level (only non-plane part!).

We introduce the hitting probabilities $P_{k,n}(x)$, i.e. the probability of the first contact with the surface at point $(x, S(x))$ if started from (k,n). Their characteristic functions $\phi_{m,n}(\theta)$ are

$$\phi_{k,n}(\theta) = \sum_{x=-\infty}^{\infty} P_{k,n}(x) e^{ix\theta}. \tag{1}$$

The inverse Fourier transform allows to obtain $P_{k,n}(x)$,

$$P_{k,n}(x) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-ix\theta} \phi_{k,n}(\theta). \tag{2}$$

2 Planar surface

At the beginning, we consider the trivial and well-known case of a planar surface (horizontal axis): $S(x) = 0$. This case is useful to remind the technique of manipulation with characteristic functions.

Suppose that $n > 0$. The probability $P_{k,n}(x)$ satisfies a simple identity

$$P_{k,n}(x) = \frac{1}{4}\Big[P_{k+1,n}(x) + P_{k-1,n}(x) + P_{k,n+1}(x) + P_{k,n-1}(x)\Big], \tag{3}$$

which can be also written for characteristic functions,

$$\phi_{k,n}(\theta) = \frac{1}{4}\Big[\phi_{k+1,n}(\theta) + \phi_{k-1,n}(\theta) + \phi_{k,n+1}(\theta) + \phi_{k,n-1}(\theta)\Big]. \tag{4}$$

Translational invariance along the horizontal axis gives

$$\phi_{k,n}(\theta) = e^{ik\theta} \phi_{0,n}(\theta). \tag{5}$$

Using the obvious condition $P_{k,0}(x) = \delta_{k,x}$, we obtain

$$\phi_{k,0}(\theta) = e^{ik\theta}. \tag{6}$$

The last trick is the following. If the starting point is placed in the n-th level, the random walk must cross the $(n-1)$-th level at some point $(m, n-1)$ to reach zeroth level. The probability to pass from (k,n) to $(m, n-1)$ without

touching other points in the $(n-1)$-th level is exactly $P_{k,1}(m)$. Therefore we can write

$$P_{k,n}(x) = \sum_m P_{k,1}(m) P_{m,n-1}(x).$$

In terms of characteristic functions this convolution is just a product of the two corresponding characteristic functions,

$$\phi_{k,n}(\theta) = \phi_{0,1}(\theta)\phi_{k,n-1}(\theta).$$

Using the translational invariance (5), we obtain

$$\phi_{k,n}(\theta) = e^{ik\theta}[\phi_{0,1}(\theta)]^n. \tag{7}$$

Substitution of expressions (6) and (7) into relation (4) for $n = 1$ and $k = 0$ leads to

$$\phi_{0,1}(\theta) = \frac{1}{4}\left(e^{-i\theta}\phi_{0,1}(\theta) + e^{i\theta}\phi_{0,1}(\theta) + 1 + [\phi_{0,1}(\theta)]^2\right), \qquad \text{or}$$

$$\phi_{0,1}^2 - (4 - 2\cos\theta)\phi_{0,1} + 1 = 0. \tag{8}$$

This quadratic equation has two solutions, and we should choose the one for which $\phi_{0,1}(\theta) \leq 1$ (property of characteristic function). It is denoted $\varphi(\theta)$,

$$\varphi(\theta) = 2 - \cos\theta - \sqrt{(2-\cos\theta)^2 - 1}. \tag{9}$$

So, we obtain for the planar surface

$$\phi_{k,n}(\theta) = e^{ik\theta}\varphi^n(\theta). \tag{10}$$

Inverting this relation with the help of (2), we obtain the distribution of hitting probabilities for the planar surface,

$$P_{k,n}^{planar}(x) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i(k-x)\theta}\varphi^n(\theta) = H_{k-x}^n. \tag{11}$$

This well-known result will be used for a general case (some properties of coefficients H_k^n are described in Appendix 1). The formulae (9) and (10) can be generalized for d-dimentional hypercubic lattice,

$$\varphi(\theta_1 ... \theta_{d-1}) = d - \sum_{i=1}^{d-1}\cos(\theta_i) - \sqrt{\left(d - \sum_{i=1}^{d-1}\cos(\theta_i)\right)^2 - 1},$$

$$\phi_{\mathbf{x},n}(\theta_1 ... \theta_{d-1}) = \exp\left[i\sum_{i=1}^{d-1} x_i\theta_i]\right]\varphi^n(\theta_1 ... \theta_{d-1}).$$

3 Regular surface with compact support

We shall consider the characteristic functions $\phi_{k,n}(\theta)$ as a vector

$$\Phi^{(n)}(\theta) = \begin{pmatrix} \phi_{-L,n}(\theta) \\ \phi_{-L+1,n}(\theta) \\ \cdots \\ \phi_{L,n}(\theta) \end{pmatrix}$$

of $(2L+1)$ components where parameter L is supposed large, and it will tend to infinity at the end of calculation.

For the planar surface we had relation (4) which can be written in matrix form

$$A\Phi^{(n)} = \Phi^{(n-1)} + \Phi^{(n+1)}, \tag{12}$$

where the matrix A is

$$A_{i,i} = 4, \qquad A_{i,i+1} = A_{i+1,i} = -1, \qquad A_{2L+1,1} = A_{1,2L+1} = -1.$$

The last equalities are artificial : we added them to obtain a cyclic structure of A. But at the limit $L \to \infty$ this little modification vanishes. The eigenvalues of A are

$$\lambda_h = 4 - 2\cos(\theta_h), \qquad \text{with } \theta_h = \frac{2\pi h}{2L+1},$$

and the eigenvectors are given as

$$V_h = \begin{pmatrix} e^{-iL\theta_h} \\ e^{-i(L-1)\theta_h} \\ \cdots \\ e^{iL\theta_h} \end{pmatrix}.$$

Now we generalize the relation (12) to the case of a regular surface with compact support by introducing vector $\Delta\Phi^{(n)}$,

$$A\Phi^{(n)} = \Phi^{(n-1)} + \Phi^{(n+1)} + \Delta\Phi^{(n)} \tag{13}$$

(this relation can be regarded as the definition of $\Delta\Phi^{(n)}$). Let us introduce

$$c_n(\theta,\theta_h) = (\Phi^{(n)}, V_h^*), \qquad \Delta c_n(\theta,\theta_h) = (\Delta\Phi^{(n)}, V_h^*). \tag{14}$$

We can rewrite (13) in terms of c_n and Δc_n,

$$\lambda_h c_n = c_{n-1} + c_{n+1} + \Delta c_n. \tag{15}$$

If we can express c_n in terms of λ_h, $\varphi(\theta)$, c_0 and $\{\Delta c_k\}$, we find $\Phi^{(n)}$ as a decomposition in the eigenbasis V_h,

$$\Phi^{(n)} = \frac{1}{2L+1}\sum_h c_n(\theta,\theta_h)V_h \tag{16}$$

(factor $(2L+1)^{-1}$ is due to normalization (V_h, V_h^*)). In order to solve the recurrence relations (15), we should close them by certain conditions. For the lower half plane we take a sufficiently large number $N_l > N^*$, and

$$\Phi_{-N_l} = 0, \qquad \text{or} \qquad c_{-N_l} = 0, \tag{17}$$

because it is impossible to penetrate through the surface. More generally, according to the definition of hitting probabilities, we should maintain

$$\phi_{m,n}(\theta) = \begin{cases} e^{im\theta}, & \text{if } (m,n) \in \mathcal{S}, \\ 0, & \text{if } (m,n) \in \mathcal{M}. \end{cases} \tag{18}$$

For the upper half plane we shall use the following trick. We take a large number $N_u \gg N$ and consider $\phi_{x,N_u+1}(\theta)$. As for a planar surface, we can write $P_{x,N_u+1}(n)$ in convolution form,

$$P_{x,N_u+1}(n) = \sum_m H^1_{x-m} P_{m,N_u}(n),$$

where H^1_{x-m} is the probability to hit point (m, N_u) if started from (x, N_u+1). For characteristic functions it is simply

$$\phi_{x,N_u+1}(\theta) = \sum_m H^1_{x-m} \phi_{m,N_u}(\theta).$$

In the plane case we used the translational invariance along the horizontal axis to simplify this sum. Evidently, such a symmetry breaks down in the general case. But if we take N_u sufficiently large, i.e. we "look" on the membrane from a remoted point, we can suppose that translational invariance is approximately valid,

$$\phi_{m,N_u}(\theta) \approx e^{i(m-x)\theta} \phi_{x,N_u}(\theta). \tag{19}$$

Using this approximation, we immediately obtain

$$\phi_{x,N_u+1}(\theta) = \varphi(\theta) \phi_{x,N_u}(\theta),$$

where function φ is defined by (9). This is our approximation in the upper half plane which allows to close the recurrence relations (15),

$$c_{N_u+1}(\theta, \theta_h) = \varphi(\theta) c_{N_u}(\theta, \theta_h). \tag{20}$$

The main idea is to step down from N_u-th and $(-N_l)$-th levels to zeroth level. We shall consider the upper and lower half planes separately because the relations (17) and (20) are different. Note the essential complication of the general case with respect to the planar surface. For the planar surface we had $\Delta c_n = 0$ for any n, and the system of equations (15) was closed. It was sufficient to solve these recurrence relations by substitution $c_n = c_0 c^n$ in (15), and we obtained the final form of $\Phi^{(n)}$. On the contrary, for the general surface $\Delta c_n \neq 0$, and they depend on the near-boundary functions $\phi_{m,n}$ (see below). Consequently, the decomposition (16) itself becomes a system of implicit equations for $\phi_{m,n}$. For the moment, the problem is complex. It will be solved by two steps. First, we obviate the combinatorial problems, i.e. we find the explicit solution of reccurence relations (15). Second, we solve the equations for $\phi_{m,n}$.

3.1 Solution of reccurence relations.

A direct verification shows that

$$c_n = \beta_{N_u-n} c_{N_u} - \sum_{l=1}^{N_u-n} \alpha_l \Delta c_{n+l}, \tag{21}$$

$$c_{-n} = \alpha_{N_l-n} c_{-N_l+1} - \sum_{l=1}^{N_l-n-1} \alpha_l \Delta c_{-n-l} \tag{22}$$

is a general solution of (15) (we omitted the index h which does not change the structure of the solution), where

$$\alpha_0 = 0, \quad \alpha_1 = 1, \qquad \alpha_{n+2}(\theta_h) = \lambda_h \alpha_{n+1}(\theta_h) - \alpha_n(\theta_h), \tag{23}$$

$$\beta_n(\theta, \theta_h) = \alpha_n(\theta_h)\big[\lambda_h - \varphi(\theta)\big] - \alpha_{n-1}(\theta_h), \qquad \beta_0 = 1. \tag{24}$$

We are looking for the explicit representation for α_n in the form $\alpha_n = x^{n-1}$. Substituting this into (23), we obtain equation

$$x^2 - \lambda_h x + 1 = 0,$$

which has two well-known solutions: φ and φ^{-1} (compare this equation with (8)). As the expression (23) is linear, we find a general solution as linear combination of φ^{n-1} and φ^{1-n} such that $\alpha_0 = 0$. We obtain

$$\alpha_n(\theta_h) = \frac{1 - \varphi^{2n}(\theta_h)}{1 - \varphi^2(\theta_h)} \varphi^{1-n}(\theta_h), \tag{25}$$

or as a geometrical sequence

$$\alpha_n(\theta_h) = \sum_{j=0}^{n-1} \varphi^{2j+1-n}(\theta_h). \tag{26}$$

Formulae (21) and (22) are valid for any $n \geq 0$, in particular, for $n = 0$, and we can express c_{N_u} and c_{-N_l+1} in terms of c_0 and $\{\Delta c_l\}$,

$$c_{N_u} = \frac{1}{\beta_{N_u}} \left(c_0 + \sum_{l=1}^{N_u} \alpha_l \Delta c_l \right), \qquad c_{-N_l+1} = \frac{1}{\alpha_{N_l}} \left(c_0 + \sum_{l=1}^{N_l-1} \alpha_l \Delta c_{-l} \right),$$

hence

$$c_n = \frac{\beta_{N_u-n}}{\beta_{N_u}} \left(c_0 + \sum_{l=1}^{N_u} \alpha_l \Delta c_l \right) - \sum_{l=1}^{N_u-n} \alpha_l \Delta c_{n+l}, \tag{27}$$

$$c_{-n} = \frac{\alpha_{N_l-n}}{\alpha_{N_l}} \left(c_0 + \sum_{l=1}^{N_l-1} \alpha_l \Delta c_{-l} \right) - \sum_{l=1}^{N_l-n-1} \alpha_l \Delta c_{-n-l}. \tag{28}$$

Let us introduce

$$f_n^{(N_u)}(\theta, \theta_h) = \frac{\beta_{N_u-n}}{\beta_{N_u}}, \qquad \tilde{f}_n^{(N_l)}(\theta_h) = \frac{\alpha_{N_l-n}}{\alpha_{N_l}}$$

(the right-hand side of $f_n^{(N_u)}$ depends on θ through factor φ; the dependence on θ_h is due to α_n which contains λ_h).

Using the reccurence properties of α_n (see Appendix 2), we simplify relations (27) and (28),

$$c_n = f_n^{(N_u)} \left(c_0 + \sum_{l=1}^{n} \alpha_l \Delta c_l \right) + \alpha_n \sum_{l=n+1}^{N_u} f_l^{(N_u)} \Delta c_l, \tag{29}$$

$$c_{-n} = \tilde{f}_n^{(N_l)} \left(c_0 + \sum_{l=1}^{n} \alpha_l \Delta c_{-l} \right) + \alpha_n \sum_{l=n+1}^{N_l-1} \tilde{f}_l^{(N_l)} \Delta c_{-l}. \tag{30}$$

Rewriting the definition (24) of β_n as

$$\beta_n(\theta,\theta_h) = \alpha_{n+1}(\theta_h) - \varphi(\theta)\alpha_n(\theta_h),$$

after simplifications we obtain

$$f_n^{(N_u)}(\theta,\theta_h) = \frac{\varphi^n(\theta_h)[1-\varphi(\theta)\varphi(\theta_h)] - \varphi^{2N_u-n+1}(\theta_h)[\varphi(\theta_h)-\varphi(\theta)]}{[1-\varphi(\theta)\varphi(\theta_h)] - \varphi^{2N_u+1}(\theta_h)[\varphi(\theta_h)-\varphi(\theta)]}.$$

We remind that N_u is an *arbitrary* sufficiently large number. Therefore we can take the limit $N_u \to \infty$. Knowing that $\varphi(\theta_h) < 1$ for $\theta_h \neq 0$, we obtain that in this limit φ^{2N_u-n+1} and φ^{2N_u+1} vanish, i.e.

$$f_n^{(\infty)}(\theta,\theta_h) = \varphi^n(\theta_h). \tag{31}$$

Using formula (25), we can also write the explicit representation for $\tilde{f}_n^{(N_l)}$,

$$\tilde{f}_n^{(N_l)}(\theta_h) = \frac{1-\varphi^{2(N_l-n)}(\theta_h)}{1-\varphi^{2N_l}(\theta_h)}\varphi^n(\theta_h).$$

As above, we take the limit $N_l \to \infty$ to obtain

$$\tilde{f}_n^{(\infty)}(\theta_h) = \varphi^n(\theta_h). \tag{32}$$

Note that we obtained the same limits $f_n^{(\infty)}$ and $\tilde{f}_n^{(\infty)}$ imposing the different conditions (17) and (20) in the lower and upper half planes respectively. Using formulae (31) and (32), we can write

$$\begin{aligned} c_n &= \varphi^n(\theta_h)\left(c_0(\theta,\theta_h) + \sum_{l=1}^{n}\alpha_l(\theta_h)\Delta c_l(\theta,\theta_h)\right) + \alpha_n(\theta_h)\sum_{l=n+1}^{\infty}\varphi^l(\theta_h)\Delta c_l(\theta,\theta_h), \\ c_{-n} &= \varphi^n(\theta_h)\left(c_0(\theta,\theta_h) + \sum_{l=1}^{n}\alpha_l(\theta_h)\Delta c_{-l}(\theta,\theta_h)\right) + \alpha_n(\theta_h)\sum_{l=n+1}^{\infty}\varphi^l(\theta_h)\Delta c_{-l}(\theta,\theta_h). \end{aligned}$$

Introducing functions

$$\gamma_l^{(n)}(\theta) = \begin{cases} \varphi^n(\theta)\alpha_l(\theta), & \text{if } l \le n, \\ \varphi^l(\theta)\alpha_n(\theta), & \text{if } l > n, \\ 0, & \text{if } l \le 0 \text{ or } n \le 0 \end{cases} \tag{33}$$

(the last convention will be used in the following), we can write c_n in the unique form (for $n > 0$ and $n < 0$),

$$c_n = \varphi^{|n|}(\theta_h)c_0(\theta,\theta_h) + \sum_{l=-N^*}^{N}\left[\gamma_l^{(n)}(\theta_h) + \gamma_{-l}^{(-n)}(\theta_h)\right]\Delta c_l(\theta,\theta_h), \tag{34}$$

where summation over l is just from $-N^*$ to N, because corrections Δc_{N+l} and Δc_{-N^*-l} are equal to 0 for $l > 0$ (see Section 3.4 for details).

3.2 Coefficient c_0

Let us calculate the coefficient c_0,

$$c_0(\theta,\theta_h) = (\Phi_0, V_h^*) = \sum_{k=-L}^{L} e^{-ik\theta_h}\phi_{k,0} = c_0^{(0)} + c_0^{(1)}.$$

As it was mentioned above, the plane "tails" of the surface lie on the zeroth level, thus

$$(\Phi_0)_k = e^{ik\theta}, \text{ if } |k| \geq M.$$

Using this explicit form, we are going to compute the contribution $c_0^{(0)}$ of plane "tails"

$$c_0^{(0)}(\theta,\theta_h) = 2\sum_{k=M}^{L}\cos(\theta-\theta_h) = \frac{\sin(L+0.5)(\theta-\theta_h)}{\sin(\theta-\theta_h)/2} - \frac{\sin(M-0.5)(\theta-\theta_h)}{\sin(\theta-\theta_h)/2} \tag{35}$$

(in the case $\theta = \theta_h$ one should consider this relation in the limit sense when $\theta \to \theta_h$). Note the simple relation

$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\pi}\cos(2k\theta)\frac{\sin(2M-1)\theta}{\sin\theta} = \chi_{(-M,M)}(k), \qquad \chi_A(k) = \begin{cases} 1, & \text{if } k \in A, \\ 0, & \text{if } k \notin A. \end{cases} \tag{36}$$

The contribution $c_0^{(1)}$ of intermediate part of the surface (with $|x| < M$) can contain some non-trivial functions $\phi_{m,0}(\theta)$, with $k \in J$ and $k \in J_0$

$$c_0^{(1)}(\theta,\theta_h) = \sum_{k\in J}\phi_{k,0}(\theta)e^{-ik\theta_h} + \sum_{k\in J_0} e^{ik(\theta-\theta_h)}. \tag{37}$$

3.3 Limit $L \to \infty$

Expression (34) transforms to the integral expression for $\phi_{k,n}$ by taking the limit $L \to \infty$. Here we write only the expression for $n \geq 0$, the opposite case will be easily obtained later. Note that the first term of (35) tends to δ-function in the limit $L \to \infty$. It removes the integration in the first term, i.e. we obtain

$$\phi_{k,n}(\theta) = \varphi^n(\theta)e^{ik\theta} - \phi_{k,n}^{(0)}(\theta) + \tag{38}$$

$$+\int_{-\pi}^{\pi}\frac{d\theta'}{2\pi}e^{ik\theta'}\left(\varphi^n(\theta')c_0^{(1)}(\theta,\theta') + \sum_{l=1}^{N}\gamma_l^{(n)}(\theta')\Delta c_l(\theta,\theta')\right).$$

The first term is the contribution of plane "tails". The second term corresponds to the perturbation on zeroth level due to $c_0^{(0)}$,

$$\phi_{k,n}^{(0)}(\theta) = \int_{-\pi}^{\pi}\frac{d\theta'}{2\pi}e^{ik\theta'}\varphi^n(\theta')\frac{\sin(M-0.5)(\theta-\theta')}{\sin(\theta-\theta')/2}. \tag{39}$$

The last term of (38) contains some unknown functions $\phi_{m,l}(\theta)$ through the coefficients $\Delta c_l(\theta,\theta')$. It is denoted as $T[\phi]$, and we are going to calculate it.

3.4 Coefficients Δc_l

To get ahead with the expression (38), we should write explicitly the coefficients $\Delta c_l(\theta,\theta')$. It is not so easy for the general case. Indeed, for these purposes one calculates the contributions of each point on l-th level. The problem is that there are many conditions, and they lead to complex formulae difficult to manipulate. We are going to present the other way.

What is the origin of the vector $\Delta\Phi^{(n)}$? Let us recall the definition (13) where vectors $\Delta\Phi^{(n)}$ were introduced to generalize the expression (12). A brief reflection shows that

$$(\Delta\Phi^{(n)})_m = \begin{cases} 0, & \text{if } (m,n)\in\mathcal{E}, \\ 4\phi_{m,n}-\phi_{m-1,n}-\phi_{m+1,n}-\phi_{m,n-1}-\phi_{m,n+1}, & \text{if } (m,n)\notin\mathcal{E} \end{cases}.$$

In other words, the relation (12) is satisfied automatically for any external point, but it should be imposed artificially for each surface and internal points.

Now it is the moment to remind the formula (18) which tells that functions $\phi_{m,n}(\theta)$ are equal to zero on the internal points. Therefore we can consider only the points near the surface $\mathcal{S}$. A direct verification shows that

$$(\Delta\Phi^{(S(m))})_m = 4e^{im\theta}-\phi_{m,S(m)+1}-\begin{cases} e^{i(m+1)\theta}, & \text{if } S(m+1)-S(m)=0 \\ \phi_{m+1,S(m)}, & \text{if } S(m+1)-S(m)=-1 \\ 0, & \text{if } S(m+1)-S(m)=1 \end{cases}$$

$$-\begin{cases} e^{i(m-1)\theta}, & \text{if } S(m-1)-S(m)=0 \\ \phi_{m-1,S(m)}, & \text{if } S(m-1)-S(m)=-1 \\ 0, & \text{if } S(m-1)-S(m)=1 \end{cases}, \tag{40}$$

$$(\Delta\Phi^{(S(m)-1)})_m = -e^{im\theta}-e^{i(m+1)\theta}\delta_{S(m),S(m+1)+1}-e^{i(m-1)\theta}\delta_{S(m),S(m-1)+1},$$

$$(\Delta\Phi^{(n)})_m = 0, \quad \text{if } n\neq S(m) \text{ and } n\neq S(m)-1$$

(here we use Kronecker δ-symbols, $\delta_{ii}=1$ and $\delta_{ij}=0$ if $i\neq j$). Usually there are several nonzero components of $\Delta\Phi^{(n)}$ for each n, because each level contains several surface points. But there exists one exception – zeroth level, where there is infinity of surface points due to the plane "tails". Thus, the vector $\Delta\Phi^{(-1)}$ has exceptional structure[3]. It contains the usual terms due to the non-trivial part of the surface, and the contribution of plane "tails". Note that the last one is equal to $-c_0^{(0)}$ which was calculated in Section 3.2. Later we shall use this result for the lower half plane.

3.5 Approximate distribution of hitting probabilities

According to the definition (14) of Δc_l, we can rewrite $T[\phi]$ as

$$T[\phi] = \int_{-\pi}^{\pi}\frac{d\theta'}{2\pi}e^{ik\theta'}\sum_{m=-M+1}^{M-1}e^{-im\theta'}\sum_{l=1}^{N}\gamma_l^{(n)}(\theta')(\Delta\Phi^{(l)})_m,$$

[3]There exists another vector with exceptional structure, $\Delta\Phi^{(0)}$. However, we construct our treatment by the way when this vector does not appear in expressions (see (29), (30)). In other words, we step down from N_u-th and $(-N_l)$-th levels to zeroth level in order to do not pass through the zeroth level.

where we changed the order of summation over m and l. However, in the last sum there are only two terms corresponding to $(\Delta\Phi^{(S(m))})_m$ and $(\Delta\Phi^{(S(m)-1)})_m$, if $S(m) \geq 1$ (otherwise, this sum is equal to 0). Using expression (40), we obtain explicitly

$$T[\phi] = \sum_{m=-M+1}^{M-1} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{i(k-m)\theta'} \Bigg(\gamma^{(n)}_{S(m)}(\theta') \Big[4e^{im\theta} - \phi_{m,S(m)+1} - e^{i(m+1)\theta} \delta_{S(m),S(m+1)}$$

$$- \phi_{m+1,S(m)} \delta_{S(m),S(m+1)+1} - e^{i(m-1)\theta} \delta_{S(m),S(m-1)} - \phi_{m-1,S(m)} \delta_{S(m),S(m-1)+1} \Big]$$

$$- \gamma^{(n)}_{S(m)-1}(\theta') \Big[e^{im\theta} + e^{i(m-1)\theta} \delta_{S(m),S(m-1)+1} + e^{i(m+1)\theta} \delta_{S(m),S(m+1)+1} \Big] \Bigg).$$

(here we have used the last convention in the definition (33) of $\gamma^{(n)}_l$ to avoid any terms with $S(m) \leq 0$).

The last step is to transform this huge expression for characteristic functions into hitting probabilities using the formula (2). Note that all functions $e^{im\theta}$ after integration over θ with $e^{-ix\theta}$ give δ-symbols that remove the summation over m in corresponding terms (but we should write factor $\chi_{(-M,M)}(x)$),

$$T[P] = - \sum_{m=-M+1}^{M-1} D^{(k,n)}_{m,S(m)} \Big(P_{m,S(m)+1}(x) + \tag{41}$$

$$P_{m+1,S(m)}(x) \delta_{S(m),S(m+1)+1} + P_{m-1,S(m)}(x) \delta_{S(m),S(m-1)+1} \Big) +$$

$$\chi_{(-M,M)}(x) \Big(4D^{(k,n)}_{x,S(x)} - D^{(k,n)}_{x-1,S(x-1)} \delta_{S(x),S(x-1)} - D^{(k,n)}_{x+1,S(x+1)} \delta_{S(x),S(x+1)} -$$

$$D^{(k,n)}_{x,S(x)-1} - D^{(k,n)}_{x+1,S(x+1)-1} \delta_{S(x),S(x+1)-1} - D^{(k,n)}_{x-1,S(x-1)-1} \delta_{S(x),S(x-1)-1} \Big),$$

where

$$D^{(k,n)}_{m,l} = \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{i(k-m)\theta'} \gamma^{(n)}_l(\theta'). \tag{42}$$

Note that coefficients $D^{(k,n)}_{m,l}$ are universal, they do not depend on a given surface. It means that once calculated, these coefficients can be used for any hitting problem in 2D. They can be also expressed in terms of H^n_k,

$$D^{(k,n)}_{m,l} = \sum_{j=1}^{\min\{n,l\}} H^{2j-1+|l-n|}_{k-m}$$

(if n or l is equal to 0, the sum is also equal to 0). We just indicate several useful properties of these coefficients,

$$D^{(k,n)}_{m,l} = D^{(0,n)}_{m-k,l} = D^{(k-m,n)}_{0,l} = D^{(m-k,n)}_{0,l}, \qquad D^{(k,0)}_{m,l} = D^{(k,n)}_{m,0} = 0,$$

$$D^{(k,n)}_{m+1,l} + D^{(k,n)}_{m-1,l} + D^{(k,n)}_{m,l+1} + D^{(k,n)}_{m,l-1} = 4D^{(k,n)}_{m,l} - \delta_{k,m}\delta_{n,l}.$$

The first part of (41), containing $P_{m,S(m)+1}$, can be represented as

$$T_1[P] = -\sum_{m=-M}^{M} G_m^{(k,n)} P_{m,S(m)+1}(x)$$

with coefficients

$$G_m^{(k,n)} = D_{m,S(m)}^{(k,n)} + \delta_{S(m),S(m+1)-1} D_{m+1,S(m+1)}^{(k,n)} + \delta_{S(m),S(m-1)-1} D_{m-1,S(m-1)}^{(k,n)}.$$

The second part of (41) can be simplified. Indeed, using the properties of δ-symbols, we have

$$T_2 = \chi_{(-M,M)}(x)\Big(4D_{x,S(x)}^{(k,n)} - D_{x-1,S(x)}^{(k,n)} - D_{x+1,S(x)}^{(k,n)} - D_{x,S(x)-1}^{(k,n)}$$

$$+D_{x-1,S(x)}^{(k,n)}\delta_{S(x),S(x-1)+1} + D_{x+1,S(x)}^{(k,n)}\delta_{S(x),S(x+1)+1}\Big).$$

Using the properties of $D_{m,l}^{(k,n)}$, we finally obtain

$$T_2 = \chi_{(-M,M)}(x)\Big(D_{x,S(x)+1}^{(k,n)} + D_{x-1,S(x)}^{(k,n)}\delta_{S(x),S(x-1)+1} + D_{x+1,S(x)}^{(k,n)}\delta_{S(x),S(x+1)+1}\Big). \tag{43}$$

Here we omitted term $\delta_{k,x}\delta_{n,S(x)}$ supposing that starting point (k,n) does not lie on the surface.

Let us get back to the formula (38). Using the inverse Fourier transform (2), we write

$$P_{k,n}(x) = H_{k-x}^n - P_{k,n}^{(0)}(x) + \sum_{m\in J_0} H_{k-m}^n \delta_{m,x} + \sum_{m\in J} H_{k-m}^n P_{m,0}(x) + T_1[P] + T_2. \tag{44}$$

The second term is

$$P_{k,n}^{(0)}(x) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-ix\theta} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{ik\theta'} \varphi^n(\theta') \frac{\sin(M-0.5)(\theta-\theta')}{\sin(\theta-\theta')/2}.$$

Replacing in the first integral $\theta_1 = \theta - \theta'$, we factorize these integrals. The first factor is exactly H_{k-x}^n. The second one was calculated explicitly, see (36), and it equals to $\chi_{[-M+1,M-1]}(x)$. Consequently, the first two terms in (44) can be grouped into $H_{k-x}^n \chi_{(-\infty,-M]\cap[M,+\infty)}(x)$. It means that the solution H_{k-x}^n of the plane case is valid only for the plane "tails", whereas on the non-trivial surface (for $|x| < M$) the main contribution is due to other terms. So, we have obtained an important result,

$$P_{k,n}(x) = \tilde{P}_{k,n}(x) + \sum_{m\in J} H_{k-m}^n P_{m,0}(x) - \sum_{m=-M}^{M} G_m^{(k,n)} P_{m,S(m)+1}(x), \tag{45}$$

where

$$\tilde{P}_{k,n}(x) = H_{k-x}^n \chi_{(-\infty,-M]\cap[M,+\infty)}(x) + T_2 + H_{k-x}^n \chi_{J_0}(x)$$

(the third term is due to the second sum in (37)). Using (43), we can combine first two terms to obtain

for $n > 0$**:**

$$\tilde{P}_{k,n}(x) = D^{(k,n)}_{x,S(x)+1} + D^{(k,n)}_{x-1,S(x)}\delta_{S(x),S(x-1)+1} + D^{(k,n)}_{x+1,S(x)}\delta_{S(x),S(x+1)+1} + H^{n}_{k-x}\chi_{J_0}(x). \tag{46}$$

In order to obtain analogous results for the case $n < 0$, we remind that expression (34) contains two terms: $\gamma^{(n)}_{l}$ and $\gamma^{(-n)}_{-l}$. In the previous treatment we have used only the first term. It means that analogous results for $n < 0$ can be easily obtained by "reflection" of all "ordinates" with respect to horizontal axis,

for $n < 0$**:**

$$P_{k,n}(x) = \tilde{P}^{*}_{k,n}(x) + \sum_{m\in J} H^{-n}_{k-m} P_{m,0}(x) - \sum_{m=-M}^{M} G^{(k,n)*}_{m} P_{m,S(m)+1}(x), \tag{47}$$

with

$$\tilde{P}^{*}_{k,n}(x) = \chi_{(-M,M)}(x)\Big(D^{(k,-n)}_{x,-S(x)-1} + D^{(k,-n)}_{x-1,-S(x)}\delta_{S(x),S(x-1)-1} + \tag{48}$$

$$D^{(k,-n)}_{x+1,-S(x)}\delta_{S(x),S(x+1)-1} + H^{-n}_{k-x}\chi_{J_0}(x)\Big),$$

$$G^{(k,n)*}_{m} = D^{(k,-n)}_{m,-S(m)} + \delta_{S(m),S(m+1)+1} D^{(k,-n)}_{m+1,-S(m+1)} + \delta_{S(m),S(m-1)+1} D^{(k,-n)}_{m-1,-S(m-1)}.$$

In (48) there appears function $\chi_{(-M,M)}(x)$, because for $n < 0$ there is no contribution $P^{(0)}_{k,n}(x)$ due to plane "tails" (see remark at the end of Section 3.4).

Note that we cannot represent analogous expressions (46) and (48) uniquely by writing $|n|$ and $|S(x)|$. It is due to the fact that functions $P_{m,n}$ in the upper half plane ($n > 0$) have no influence on functions $P_{m,n}$ in the lower half plane ($n < 0$) (except through the ground functions), and vice versa. For example, in the sum of near-boundary functions (the last term in (45) and (47)) coefficients $G^{(k,n)}_{m}$ should be equal to 0 if $n > 0$ and $S(x) \leq 0$ or if $n < 0$ and $S(x) \geq 0$.

$\tilde{P}_{k,n}(x)$ can be considered as first approximation to $P_{k,n}(x)$. Note that a priori there is no reason to neglect the second and the third terms in (45). Normally, we should take these terms into account, thus the relation (45) is considered as a system of linear equations on the near-boundary and ground functions. In order to find these functions, we should close the system of linear equation. Note that equations (45) must be satisfied for any k, n and x, and we can choose appropriate values of k and n. To close the system for near-boundary functions, we take $\{(k,n) : k \in [-M,M], n = S(k)+1\}$, i.e. for any $k \in [-M,M]$

$$P_{k,S(k)+1}(x) = \tilde{P}_{k,S(k)+1}(x) + \sum_{m\in J} H^{(S(k)+1)}_{k-m} P_{m,0}(x) - \sum_{m=-M}^{M} G^{(k,S(k)+1)}_{m} P_{m,S(m)+1}(x). \tag{49}$$

To close the system for ground functions, we can choose different conditions. For example, if we consider surfaces with $S(x) > 0$ on $x \in (-M,M)$, there are

no ground functions, thus there is no additional condition other than (49). For the general case (where $J \neq \emptyset$), we propose the following condition

$$P_{k,0}(x) = \frac{1}{4}\Big(P_{k+1,0}(x) + P_{k-1,0}(x) + P_{k,1}(x) + P_{k,-1}(x)\Big), \qquad k \in J. \tag{50}$$

We started from this relation for all the external points (see (3)). Here we just demand that this relation remains valid if we substitute our approximations for $P_{k,1}(x)$ and $P_{k,-1}(x)$.

4 Some numerical verifications

In this section we briefly present some numerical results to check the validity of the approximation.

First of all, in Fig.1 we depict the function $\varphi(\theta)$ which plays a central role in this work.

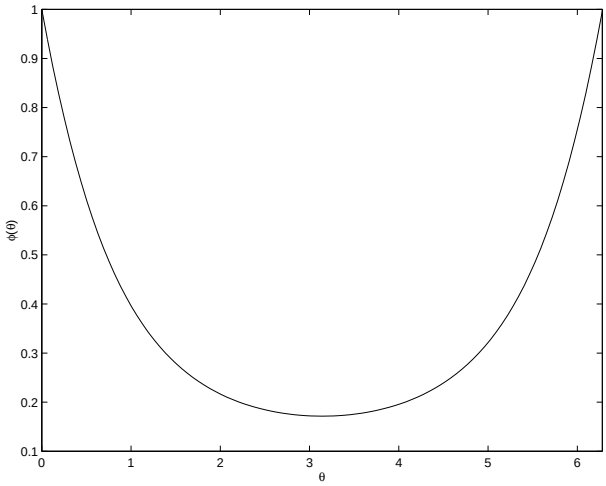

Figure 1. Function $\varphi(\theta)$. It changes in range from $\varphi(0) = 1$ to $\varphi(\pi) = 3 - \sqrt{8}$.

We can see that $\varphi(\theta) \ll 1$ if θ is not in vicinity of $2\pi m$ ($m \in Z$). This property was used in the limits (31) and (32).

Let us consider again the planar surface, $S(x) = 0$. Without simulations, we easily obtain that

$$P_{k,n}(x) = \tilde{P}_{k,n}(x) = H^n_{k-x}.$$

So, in this trivial case our approximation gives the exact result (cf. (11)).

For a particular non-trivial surface the accuracy of the formula (45) can be obtained by comparing its values with numerical simulations of random walks. We have taken a simple surface represented in Fig.2.

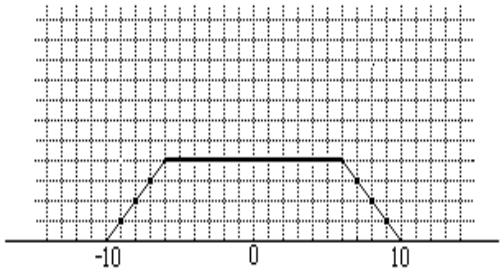

Figure 2. A simple surface with : $N = 4$, $N^* = 0$, $M = 10$.

In this case there are no ground functions. On the contrary, there are 21 near-boundary functions which can be calculated with the help of (49). We present two distributions of hitting probabilities obtained numerically and through formula (45) (see Fig.3).

We can conclude that our approximation is quite good.

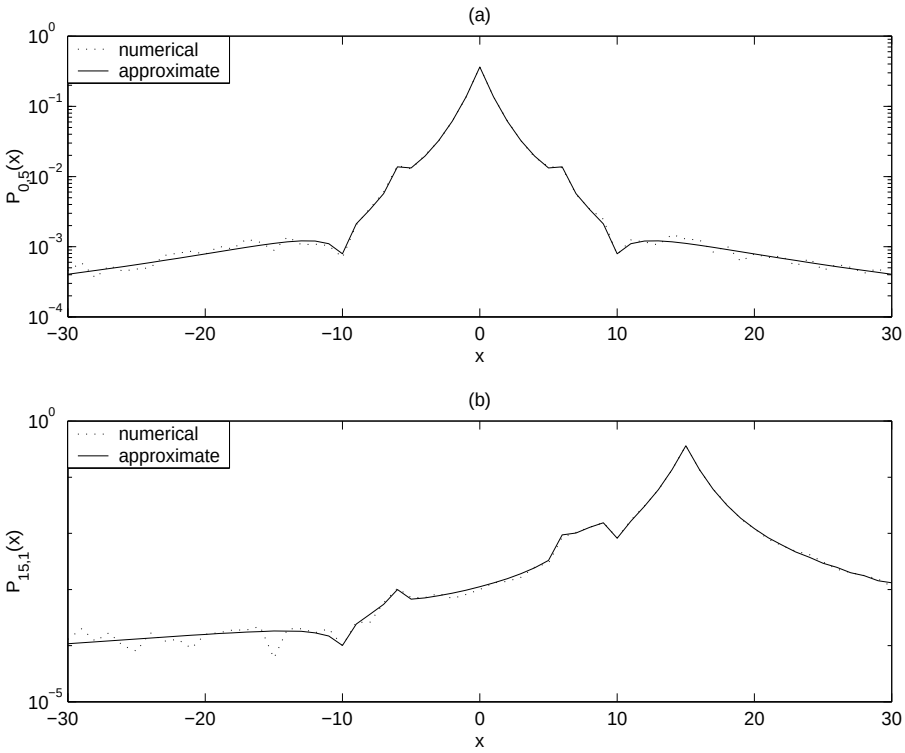

Figure 3. Distributions of hitting probabilities (in log-scale): the probability of the first contact with point $(x, S(x))$ *of the surface if started from the point* $(0, 5)$ *(Fig. 3a) or* $(15, 1)$ *(Fig. 3b).*

5 Conclusions and possible generalizations

Let us sum up what has been done. Using the same technique as for a planar surface, we obtain the reccurence relations for coefficients c_n. For the lower half plane we impose $c_{-N_l} = 0$ for a sufficiently large N_l. This condition tells that random walks cannot penetrate through the surface. For the upper half plane there is no such condition. However, for a sufficiently large N_u we can use an *approximate* condition $c_{N_u+1} = \varphi c_{N_u}$ supposing that from a remoted point the regular surface with compact support looks like a translationally invariant object. Then we find the explicit solution for reccurence relations under these conditions. The influence of our approximation becomes more and more negligible with increasing of N_u. Taking the limits $N_u \to \infty$ and $N_l \to \infty$, we express c_n in terms of explicit functions φ and $\gamma_l^{(n)}$ and coefficients $\{\Delta c_l\}$. Note that two limits $f_n^{(\infty)}$ and $\tilde{f}_n^{(\infty)}$ for the upper and lower half planes are identical. It means that in the upper half plane we could use the condition $c_{N_u+1} = 0$ instead of $c_{N_u+1} = \varphi c_{N_u}$. In other words, one could imagine an absorbing line $y = N_u + 1$, and then send it to infinity (limit $N_u \to \infty$). These two possible approximations give the same final result (31).

Having made these calculations, we express Δc_n in terms of near-boundary functions and combinations of exponential functions. Finally, we obtain a system of linear equations (49) for near-boundary functions $P_{m,S(m)+1}(x)$ and ground functions $P_{m,0}(x)$. It can be solved, and after that one can use the approximation (45) for any point (k, n).

Numerical analysis shows that this approximation works quite good.

The main conclusion is that we have found an approximate distribution of hitting probabilities for a rather general surface, pending certain conditions. In particular, one can make use of these results for a further study of the Laplacian transfer problems.

Now one needs to study the role of conditions which were imposed in the first section. As we said above, the compactness condition is the most important. It tells that

– the surface has a compact support, i.e. there is only *finite* "perturbation" of the planar surface ;

– the plane "tails" have the same height (zero of the vectical axis).

If we want to consider a surface with *infinite* support, we can obtain the same results but with an infinity of near-boundary functions. Thus the system (49) has infinitely many equations, and we cannot proceed any further. The same difficulty appears if the plane "tails" have different height: while we step down from the N_u-th level to the level of a lower "tail", we must pass through the level of a higher "tail". It means that there appears again an infinity of near-boundary functions. Only if we step down from the N_u-th level to the zeroth level (and from the $(-N_l)$-th level to the zeroth level), we can avoid the appearance of an infinity of near-boundary functions.

The regularity condition is used to simplify certain expressions. Nevertheless, it does not seem to be restrictive. Normally, to apply the technique of characteristic functions, we enumerate all sites (points) of the surface. To simplify the problem, one can make one of the following assumptions:

– either suppose that the surface obeys the regularity condition;

– or be interested in the total hitting probability $P_{k,n}^{(total)}(x)$ of points $(x, S(x))$,

$(x, S(x)-1)$, ..., $(x, S(x-1)+1)$ (if we authorize changes of $S(x)$ by more than one unit). In other words, we could identify the points of the surface which have the same x-coordinate. Any of these assumptions allows to enumerate the points of the surface with their x coordinate using function $S(x)$. In the first assumption we consider the surfaces having only one point for each x; in the second assumption the surfaces can have some points with the same x, but we are interested in the total probability for each x.

In order to generalize the method, one can introduce another *parametrization* of the surface. One possible generalization will be presented in our forthcoming paper.

6 Appendices

6.1 Coefficients H_k^n

As we have seen, coefficients H_k^n play a central role in all calculations of hitting probabilities in 2D. Here we briefly present some useful properties of H_k^n. In real form (11) becomes,

$$H_k^n = \int_0^\pi \frac{d\theta}{\pi} \cos(k\theta)\varphi^n(\theta). \tag{51}$$

We can write two inequalities for $\theta \in (0, \pi)$:

$$e^{-\theta} \leq \varphi(\theta) \leq e^{-\theta} \frac{1}{\cos(\theta/2)},$$

which can be useful for estimations.

Now we are going to calculate the asymptotics of H_k^n for large k. Integrating the expression (51) by parts four times and using the values of the derivatives $\varphi^k(s)$ at the points 0 and π (see Table 1), we obtain the asymptotic behaviour

$$H_k^n = \frac{n}{\pi k^2} - \frac{n(n^2-0.5)}{\pi k^4} + O(k^{-6}), \qquad k \gg 1. \tag{52}$$

θ	φ	φ'	φ''	φ'''
0	1	-1	1	$-1/2$
π	$3-\sqrt{8}$	0	$3\sqrt{2}/4-1$	0

Table 1. The values of the derivatives $\varphi^{(k)}(s)$ at the points 0 and π.

The formula (52) works rather well for $k \geq 10$.

$n \backslash k$	0	1	2	3	4	5
1	0.3633	0.1366	0.0609	0.0319	0.0189	0.0124
2	0.1803	0.1221	0.0756	0.0477	0.0315	0.0219
3	0.1136	0.0958	0.0715	0.0517	0.0376	0.0278
4	0.0826	0.0759	0.0631	0.0501	0.0392	0.0307
5	0.0651	0.0620	0.0548	0.0464	0.0384	0.0315

Table 2. The values of coefficients H_k^n for small k (n in range from 1 to 5).

Table 2 shows values of H_k^n for small k. The asymptotics of H_k^n for large n is

$$H_k^n = \frac{n}{\pi(n^2+k^2)} + O(n^{-3}),$$

i.e. we obtained the same behaviour as for the brownian motion. It is quite a reasonable result : if we look on the surface from a remoted point, there is no difference between continuous and discrete cases.

6.2 Manipulation with coefficients α_n and β_n

Here we present some properties of coefficients α_n and β_n. Also we prove the formula (29). Using only the definition (23), we find

$$\alpha_n = \alpha_l \alpha_{n-l+1} - \alpha_{l-1}\alpha_{n-l} \tag{53}$$

for any $l \le n$. Also using (24), we have

$$\beta_n = \alpha_{n+1} - \varphi\alpha_n.$$

Let us prove (29). According to (27), we have

$$c_n = \frac{\beta_{N_u-n}}{\beta_{N_u}}\left(c_0 + \sum_{l=1}^{n}\alpha_l\Delta c_l\right) + \frac{1}{\beta_{N_u}}\sum_{l=1}^{N_u-n}\Delta c_{n+l}(\beta_{N_u-n}\alpha_{n+l} - \beta_{N_u}\alpha_l).$$

Now we should simplify the difference in brackets in the last sum.

$$\beta_{N_u-n}\alpha_{n+l} - \beta_{N_u}\alpha_l = (\alpha_{N_u-n+1} - \varphi\alpha_{N_u-n})\alpha_{n+l} - (\alpha_{N_u+1} - \varphi\alpha_{N_u})\alpha_l. \tag{54}$$

Consider the difference $\Delta = \alpha_{N_u-n+1}\alpha_{n+l} - \alpha_{N_u+1}\alpha_l$. Using the property (53), we can reduce the index $(n+l)$ in the first term and (N_u+1) – in the second term,

$$\Delta = \alpha_{N_u-n+1}(\alpha_l\alpha_{n+1} - \alpha_{l-1}\alpha_n) - (\alpha_{n+1}\alpha_{N_u-n+1} - \alpha_n\alpha_{N_u-n})\alpha_l =$$

$$= \alpha_n(\alpha_{N_u-n}\alpha_l - \alpha_{N_u-n+1}\alpha_{l-1}) = \alpha_n\alpha_{N_u-n-l+1}$$

(we used the property (53) in the last equality). Thus, we can represent (54) as

$$\beta_{N_u-n}\alpha_{n+l} - \beta_{N_u}\alpha_l = \alpha_n\alpha_{N_u-n-l+1} - \varphi\alpha_n\alpha_{N_u-n-l} = \alpha_n\beta_{N_u-n-l},$$

hence we find the formula (29),

$$c_n = f_n^{(N_u)}\left(c_0 + \sum_{l=1}^{n}\alpha_l\Delta c_l\right) + \alpha_n\sum_{l=n+1}^{N_u} f_l^{(N_u)}\Delta c_l.$$

References

[1] W.Feller *An Introduction to Probability Theory and Its Applications*, Second Edition (1971)

[2] F.Spitzer *Principles of Random Walk* (1964)

[3] B.Sapoval *General Formulation of Laplacian Transfer Across Irregular Surfaces*, Phys.Rev.Lett.73, 3314 (1994)

[4] B.Sapoval, M.Filoche, K.Karamanos, R.Brizzi *Can one hear the shape of an electrode? I. Numerical study of the active zone in Laplacian tranfer*, Eur. Phys. J. B 9, 739-753 (1999)

[5] M.Filoche, B.Sapoval *Can one hear the shape of an electrode? II. Theoretical study of the Laplacian tranfer*, Eur. Phys. J. B 9, 755-763 (1999)

[6] D.S.Grebenkov *Propriétés de l'opérateur d'auto-transport brownien*, Rapport de stage à l'Ecole Polytechnique, France (2000)

[7] D.S.Grebenkov, B.Sapoval, M.Filoche *Spectral Properties of the Brownian Self-Transport Operator* (in press)