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LOCALIZATION OF LAPLACIAN EIGENFUNCTIONS IN CIRCULAR, SPHERICAL, AND ELLIPTICAL DOMAINS*

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Abstract. We consider Laplacian eigenfunctions in circular, spherical, and elliptical domains in order to discuss three kinds of high-frequency localization: whispering gallery modes, bouncing ball modes, and focusing modes. Although the existence of these modes has been known for a class of convex domains, the separation of variables for circular, spherical, and elliptical domains helps us to better understand the "mechanism" of localization, i.e., how an eigenfunction is getting distributed in a small region of the domain and decays rapidly outside this region. Using the properties of Bessel and Mathieu functions, we derive inequalities which imply and clearly illustrate localization. Moreover, we provide an example of a nonconvex domain (an elliptical annulus) for which the high-frequency localized modes are still present. At the same time, we show that there is no localization in most rectangle-like domains. This observation leads us to formulate an open problem of localization in polygonal domains and, more generally, in piecewise smooth convex domains.

Key words. Laplacian eigenfunctions, localization, Bessel and Mathieu functions, diffusion, Laplace operator

AMS subject classifications. 35J05, 35Pxx, 51Pxx, 33C10, 33E10

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1. Introduction.

A hundred years ago, Lord Rayleigh documented an interesting acoustical phenomenon that occurred in the whispering gallery under the dome of Saint Paul's Cathedral in London [1] (see also [2, 3]): a whisper of one person propagated along the curved wall to another person standing near the wall. This acoustical effect and many related wave phenomena can be mathematically described by Laplacian eigenmodes satisfying $-\Delta u = \lambda u$ in a bounded domain, with an appropriate boundary condition:

$$\begin{aligned} u &= 0 \quad \text{(Dirichlet)}, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{(Neumann)}, \\ \frac{\partial u}{\partial n} + hu &= 0 \quad \text{(Robin)}, \end{aligned} \tag{1.1}$$

where $h \geq 0$ is a positive constant and $\partial/\partial n$ is the normal derivative directed outwards from the domain. It turns out that the eigenmodes that are "responsible" for the whispering effect are mostly distributed near the boundary of the domain and almost zero inside. Keller and Rubinow discussed these so-called *whispering gallery modes* and also *bouncing ball* modes. The existence of such *localized* eigenmodes in the limit of large eigenvalues was shown for any two-dimensional domain with arbitrary smooth convex curve as its boundary (so-called high-frequency or high-energy

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localization) [4]. A further semiclassical approximation of Laplacian eigenfunctions in convex domains was developed by Lazutkin [5, 6, 7, 8] (see also [9, 10, 11, 12]). Chen and coworkers analyzed Mathieu and modified Mathieu functions and reported another type of localization named *focusing modes* [13]. These and other localized eigenmodes have been intensively studied for various domains, named quantum billiards [14, 15, 16, 17, 18]. It is also worth mentioning that low-frequency localization of Laplacian eigenfunctions in simple and irregular domains has attracted considerable attention during the last two decades [19, 20, 21, 22, 23, 24].

This paper aims to revisit and illustrate the aforementioned three types of high-frequency localization. For this purpose, we consider circular, spherical, and elliptical domains for which the separation of variables reduces the analysis to the behavior of special functions. Using the properties of Bessel and Mathieu functions, we derive inequalities that clearly show the existence of infinitely many localized eigenmodes in circular, spherical, and elliptical domains with Dirichlet, Neumann, or Robin boundary conditions. More precisely, we call an eigenfunction u of the Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^d$ *L_p-localized* ($p \geq 1$) if it is essentially supported by a small subdomain $\Omega_\alpha \subset \Omega$, i.e.,

$$\frac{\|u\|_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u\|_{L_p(\Omega)}} \ll 1, \qquad \frac{\mu_d(\Omega_\alpha)}{\mu_d(\Omega)} \ll 1, \tag{1.2}$$

where $\|.\|_{L_p}$ is the L_p-norm and μ_d is the Lebesgue measure. We stress that this "definition" is qualitative, as there is no objective criterion for deciding how small these ratios have to be. This is the major problem in defining the notion of localization. For circular, spherical, and elliptical domains, we will show in sections 2 and 3 that both ratios can be made arbitrarily small. In other words, for any prescribed threshold ε there exist a subdomain Ω_α and infinitely many eigenfunctions for which both ratios are smaller than ε. Most importantly, we will provide a simple example of a nonconvex domain for which the high-frequency localization is still present. At the same time, we will show in section 4.1 the absence of localization in most rectangle-like domains. This observation will lead us to formulate an open problem of localization in polygonal domains and, more generally, in piecewise smooth convex domains.

2. Localization in circular and spherical domains.

2.1. Eigenfunctions for circular domains. The rotation symmetry of a disk $\Omega = \{x \in \mathbb{R}^2 : |x| < R\}$ of radius R leads to an explicit representation of the eigenfunctions in polar coordinates:

$$u_{nki}(r,\varphi) = J_n(\alpha_{nk} r/R) \times \begin{cases} \cos(n\varphi), & i = 1, \\ \sin(n\varphi), & i = 2 \ (n \neq 0), \end{cases} \tag{2.1}$$

where $J_n(z)$ are the Bessel functions of the first kind [25, 26, 27] and α_{nk} are the positive zeros of $J_n(z)$ (Dirichlet), $J_n'(z)$ (Neumann), and $J_n'(z)+hJ_n(z)$ (Robin). The eigenfunctions are enumerated by the triple index nki, with $n = 0, 1, 2, \ldots$ counting the order of Bessel functions, $k = 1, 2, 3, \ldots$ counting the positive zeros, and $i = 1, 2$. Since $u_{0k2}(r,\varphi)$ are trivially zero, they are not counted as eigenfunctions. The eigenvalues $\lambda_{nk} = \alpha_{nk}^2/R^2$, which are independent of the last index i, are simple for $n = 0$ and twice degenerate for $n > 0$. In the latter case, the eigenfunction is any nontrivial linear combination of u_{nk1} and u_{nk2}. As we will derive estimates that are independent of the angular coordinate φ, the last index i will be omitted.

2.2. Whispering gallery modes. The disk is the simplest shape for illustrating the whispering gallery and focusing modes. The explicit form (2.1) of eigenfunctions allows one to derive accurate bounds, as shown below. When the index k is fixed while n increases, the Bessel functions $J_n(\alpha_{nk} r/R)$ become strongly attenuated near the origin (as $J_n(z) \sim (z/2)^n/n!$ at small z) and essentially localized near the boundary, yielding whispering gallery modes. In turn, when n is fixed while k increases, the Bessel functions rapidly oscillate, the amplitude of oscillations decreasing toward the boundary. In that case, the eigenfunctions are mainly localized at the origin, yielding focusing modes. These qualitative arguments are rigorously formulated in the following.

THEOREM 2.1. *Let $D = \{x \in \mathbb{R}^2 : |x| < R\}$ be a disk of radius $R > 0$, and $D_{nk} = \{x \in \mathbb{R}^2 : |x| < R d_n/\alpha_{nk}\}$, where $d_n = n - n^{2/3}$ and α_{nk} are the positive zeros of $J_n(z)$ (Dirichlet), $J_n'(z)$ (Neumann), or $J_n'(z) + hJ_n(z)$ for some $h > 0$ (Robin), with $n = 0, 1, 2, \ldots$ denoting the order of Bessel function $J_n(z)$ and $k = 1, 2, 3, \ldots$ counting zeros. Then for any $p \geq 1$ (including $p = \infty$) there exists a universal constant $C_p > 0$ such that for any $k = 1, 2, 3, \ldots$ and any large enough n the Laplacian eigenfunction u_{nk} for Dirichlet, Neumann, or Robin boundary condition satisfies*

$$\frac{\|u_{nk}\|_{L_p(D_{nk})}}{\|u_{nk}\|_{L_p(D)}} < C_p n^{\frac{1}{3}+\frac{2}{3p}} 2^{\frac{-n^{1/3}}{3}}. \tag{2.2}$$

This estimate implies that

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(D_{nk})}}{\|u_{nk}\|_{L_p(D)}} = 0, \qquad \text{while} \qquad \lim_{n\to\infty} \frac{\mu_2(D_{nk})}{\mu_2(D)} = 1. \tag{2.3}$$

The theorem shows the existence of infinitely many Laplacian eigenmodes which are L_p-localized near the boundary ∂D (see Appendix A for a proof). In fact, for any prescribed thresholds for both ratios in (1.2), there exists n_0 such that for all $n > n_0$ the eigenfunctions u_{nk} are L_p-localized. These eigenfunctions are called "whispering gallery eigenmodes" and illustrated in Figure 2.1.

A simple consequence of the above theorem is the following result.

COROLLARY 2.2. *For any $p \geq 1$ and any open subset V compactly included in D (i.e., $\bar{V} \cap \partial D = \emptyset$), one has*

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(V)}}{\|u_{nk}\|_{L_p(D)}} = 0. \tag{2.4}$$

As a consequence,

$$C_p(V) \equiv \inf_{nk} \left\{ \frac{\|u_{nk}\|_{L_p(V)}}{\|u_{nk}\|_{L_p(\Omega)}} \right\} = 0. \tag{2.5}$$

In fact, for any open subset V compactly included in D, there exists n_0 such that, for all $n > n_0$, $V \subset D_{nk}$ so that $\|u_{nk}\|_{L_p(V)} \leq \|u_{nk}\|_{L_p(D_{nk})}$, yielding (2.4).

In the same way, the localization also happens for any circular sector.

In three dimensions, the existence of whispering gallery modes in a ball follows from the following.

THEOREM 2.3. *Let $B = \{x \in \mathbb{R}^3 : |x| < R\}$ be a ball of radius R, and $B_{nk} = \{x \in B : 0 < |x| < R s_n/\alpha_{nk}\}$, where $s_n = (n + 1/2) - (n + 1/2)^{2/3}$ and α_{nk} are the positive zeros of $j_n(z)$ (Dirichlet), $j_n'(z)$ (Neumann), or $j_n'(z) + hj_n(z)$ for some*

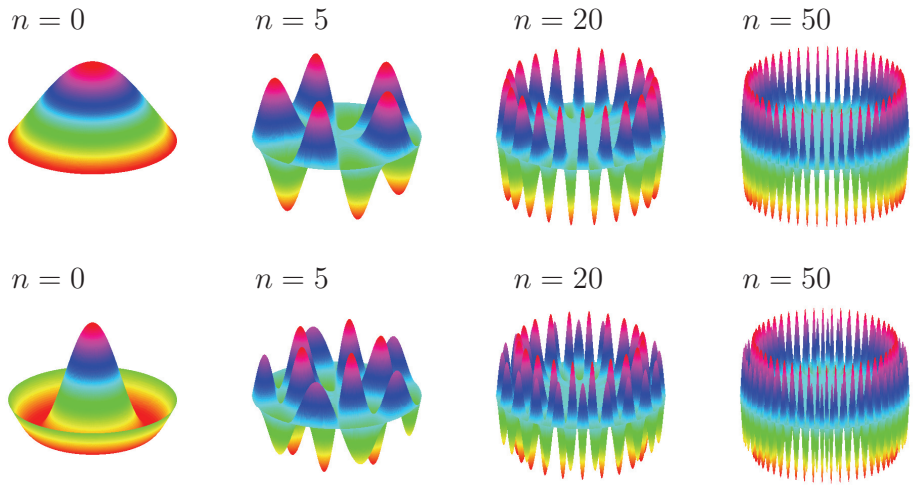

FIG. 2.1. *Formation of whispering gallery modes u_{nk} in the unit disk with Dirichlet boundary condition: for a fixed k ($k = 0$ for top figures and $k = 1$ for bottom figures), an increase of the index n leads to stronger localization of the eigenfunction near the boundary.*

$h > 0$ (Robin), with $n = 0, 1, 2, \ldots$ denoting the order of the spherical Bessel function $j_n(z)$ and $k = 1, 2, 3, \ldots$ counting zeros. Then, for any $p \geq 1$ (including $p = \infty$) there exists a universal constant $\tilde{C}_p > 0$ such that for any $k = 1, 2, 3, \ldots$ and any large enough n the Laplacian eigenfunction u_{nk} with Dirichlet, Neumann, or Robin boundary condition satisfies

$$\frac{\|u_{nk}\|_{L_p(B_{nk})}}{\|u_{nk}\|_{L_p(B)}} < \tilde{C}_p \left(n + \frac{1}{2}\right)^{\frac{1}{3}+\frac{2}{3p}} \exp\left(-\frac{1}{3}\left(n + \frac{1}{2}\right)^{\frac{1}{3}}\right) \qquad (n \gg 1). \tag{2.6}$$

As a consequence,

$$\lim_{n\to\infty} \frac{\|u_{nk}\|_{L_p(B_{nk})}}{\|u_{nk}\|_{L_p(B)}} = 0, \qquad \text{while} \qquad \lim_{n\to\infty} \frac{\mu_3(B_{nk})}{\mu_3(B)} = 1. \tag{2.7}$$

As for the disk, the above results show that infinitely many high-frequency eigenfunctions are L_p-localized near the boundary of the ball (see Appendix B for a proof).

2.3. Focusing modes. The localization of focusing modes at the origin is described by the next claim.

THEOREM 2.4. *For each $R \in (0, 1)$, let $D(R) = \{x \in \mathbb{R}^2 : R < |x| < 1\}$ and D be the unit disk. Then, for any $n = 0, 1, 2, \ldots$, the Laplacian eigenfunction u_{nk} with Dirichlet, Neumann, or Robin boundary condition satisfies*

$$\lim_{k\to\infty} \frac{\|u_{nk}\|_{L_p(D(R))}}{\|u_{nk}\|_{L_p(D)}} = \begin{cases} (1 - R^{2-p/2})^{1/p} & (1 \leq p < 4), \\ 0 & (p > 4). \end{cases} \tag{2.8}$$

The theorem states that for each nonnegative integer n, when the index k increases, the eigenfunctions u_{nk} become more and more L_p-localized near the origin when $p > 4$ (see Appendix A for a proof). These eigenfunctions are called "focusing eigenmodes" and illustrated in Figure 2.2. The theorem shows that the definition of localization is sensitive to the norm: the focusing modes are L_p-localized for $p > 4$

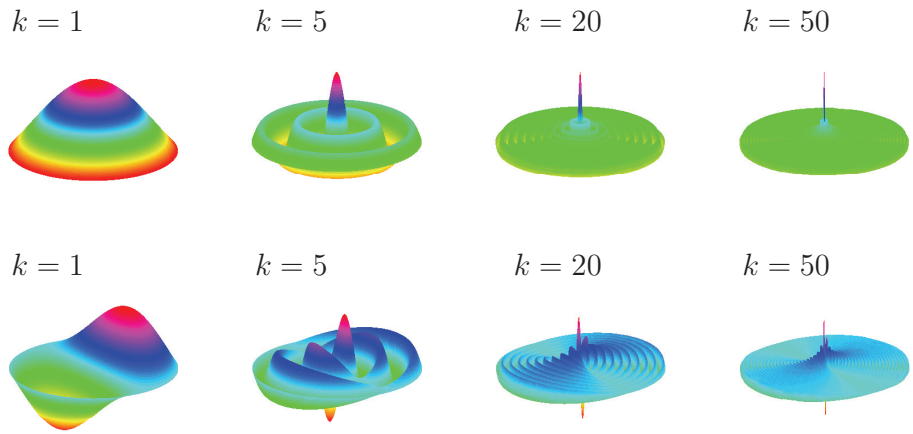

FIG. 2.2. *Formation of focusing modes u_{nk} in the unit disk with Dirichlet boundary condition: for a fixed n ($n = 0$ for top figures and $n = 1$ for bottom figures), an increase of the index k leads to stronger localization of the eigenfunction at the origin.*

(including $p = \infty$), but they are not L_p-localized for $p < 4$. In fact, as the amplitude of oscillations of the focusing modes exhibits a power law decay from the origin toward the boundary (see Figure 2.2), the p values control the behavior of the L_p-norm and determine whether the ratio of these norms vanishes or not in the limit $k \to \infty$.

A similar theorem can be reformulated for a ball in three dimensions.

THEOREM 2.5. *For each $R \in (0,1)$, let $B(R) = \{x \in \mathbb{R}^3 : R < |x| < 1\}$ and B be the unit ball. Then, for any $n = 0, 1, 2, \ldots$, the Laplacian eigenfunction u_{nk} with Dirichlet, Neumann, or Robin boundary condition satisfies*

$$\lim_{k\to\infty} \frac{\|u_{nk}\|_{L_p(B(R))}}{\|u_{nk}\|_{L_p(B)}} = \begin{cases} (1 - R^{3-p})^{1/p} & (1 \le p < 3), \\ 0 & (p > 3). \end{cases} \tag{2.9}$$

See Appendix B for a proof.

3. Localization in elliptical domains.

3.1. Eigenfunctions for elliptical domains. It is convenient to introduce the elliptic coordinates as

$$\begin{cases} x_1 = a \cosh r \cos\theta, \\ x_2 = a \sinh r \sin\theta, \end{cases} \tag{3.1}$$

where $a > 0$ is the prescribed distance between the origin and the foci, and $r \ge 0$ and $0 \le \theta < 2\pi$ are the radial and angular coordinates. A filled ellipse is a domain with $r < R$ so that its points (x_1, x_2) satisfy $x_1^2/A^2 + x_2^2/B^2 < 1$, where R is the elliptic radius and $A = a \cosh R$ and $B = a \sinh R$ are the major and minor semiaxes, respectively. In the elliptic coordinates, the separation of the angular and radial variables leads to Mathieu and modified Mathieu equations, respectively [13, 28]. Periodic solutions of the Mathieu equation are possible for specific characteristic values c. These are denoted as $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ (with $n = 0, 1, 2, \ldots$) and called the angular Mathieu functions of the first and second kind. Each function $\mathrm{ce}_n(\theta, q)$ and $\mathrm{se}_{n+1}(\theta, q)$ corresponds to its own characteristic value c (the relation being implicit; see [29]).

For the radial part, there are two linearly independent solutions for each characteristic value c: two modified Mathieu functions $\mathrm{Mc}_n^{(1)}(r,q)$ and $\mathrm{Mc}_n^{(2)}(r,q)$ correspond to the same c as $\mathrm{ce}_n(\theta,q)$, and two modified Mathieu functions $\mathrm{Ms}_{n+1}^{(1)}(r,q)$ and $\mathrm{Ms}_{n+1}^{(2)}(r,q)$ correspond to the same c as $\mathrm{se}_{n+1}(\theta,q)$. As a consequence, there are four families of eigenfunctions (distinguished by the index $i=1,2,3,4$) in a filled ellipse:

$$\begin{aligned} u_{nk1} &= \mathrm{ce}_n(\theta,q_{nk1})\mathrm{Mc}_n^{(1)}(r,q_{nk1}),\\ u_{nk2} &= \mathrm{ce}_n(\theta,q_{nk2})\mathrm{Mc}_n^{(2)}(r,q_{nk2}),\\ u_{nk3} &= \mathrm{se}_{n+1}(\theta,q_{nk3})\mathrm{Ms}_{n+1}^{(1)}(r,q_{nk3}),\\ u_{nk4} &= \mathrm{se}_{n+1}(\theta,q_{nk4})\mathrm{Ms}_{n+1}^{(2)}(r,q_{nk4}), \end{aligned} \tag{3.2}$$

where the parameters q_{nki} are determined by the boundary condition. For instance, for a filled ellipse of radius R with Dirichlet boundary condition, there are four individual equations for the parameter q_{nki}, for each $n=0,1,2,\ldots$,

$$\begin{aligned} \mathrm{Mc}_n^{(1)}(R,q_{nk1})=0, \quad \mathrm{Mc}_n^{(2)}(R,q_{nk2})=0,\\ \mathrm{Ms}_{n+1}^{(1)}(R,q_{nk3})=0, \quad \mathrm{Ms}_{n+1}^{(2)}(R,q_{nk4})=0, \end{aligned} \tag{3.3}$$

each of them having infinitely many positive solutions q_{nki} enumerated by $k=1,2,\ldots$ [27, 29]. The associated eigenvalues λ_{nki} are determined as

$$\lambda_{nki} = \frac{4q_{nki}}{a^2}. \tag{3.4}$$

The above analysis can be applied almost directly to an elliptical annulus Ω, i.e., a domain between an inner ellipse Γ_1 and an outer ellipse Γ_2, with the same foci. In elliptic coordinates, Ω can be defined by two inequalities: $R_1<r<R_2$ and $0\le\theta<2\pi$, where the prescribed radii R_1 and R_2 determine Γ_1 and Γ_2, respectively.

We consider two families of eigenfunctions in Ω:

$$\begin{aligned} u_{nk1} &= \mathrm{ce}_n(\theta,q_{nk1})\left[a_{nk1}\mathrm{Mc}_n^{(1)}(r,q_{nk1})+b_{nk1}\mathrm{Mc}_n^{(2)}(r,q_{nk1})\right],\\ u_{nk2} &= \mathrm{se}_{n+1}(\theta,q_{nk2})\left[a_{nk2}\mathrm{Ms}_{n+1}^{(1)}(r,q_{nk2})+b_{nk2}\mathrm{Ms}_{n+1}^{(2)}(r,q_{nk2})\right]. \end{aligned} \tag{3.5}$$

The parameters a_{nki}, b_{nki}, and q_{nki} $(i=1,2)$ are set by boundary conditions and the normalization of eigenfunctions. For Dirichlet boundary condition, one solves the following equations:

$$\begin{aligned} \mathrm{Mc}_n^{(1)}(R_1,q_{nk1})\mathrm{Mc}_n^{(2)}(R_2,q_{nk1})-\mathrm{Mc}_n^{(1)}(R_2,q_{nk1})\mathrm{Mc}_n^{(2)}(R_1,q_{nk1})=0,\\ \mathrm{Ms}_{n+1}^{(1)}(R_1,q_{nk2})\mathrm{Ms}_{n+1}^{(2)}(R_2,q_{nk2})-\mathrm{Ms}_{n+1}^{(1)}(R_2,q_{nk2})\mathrm{Ms}_{n+1}^{(2)}(R_1,q_{nk2})=0. \end{aligned} \tag{3.6}$$

For $n=0,1,2,\ldots$, each of these equations has infinitely many solutions q_{nki} enumerated by $k=1,2,3,\ldots$ [29]. The eigenvalues are determined by (3.4).

3.2. Bouncing ball modes. For each $\alpha\in\left(0,\frac{\pi}{2}\right)$ we consider the elliptical sector Ω_α inside an elliptical domain Ω:

$$\Omega_\alpha=\left\{R_1<r<R_2,\ \theta\in(\alpha,\pi-\alpha)\cup(\pi+\alpha,2\pi-\alpha)\right\}.$$

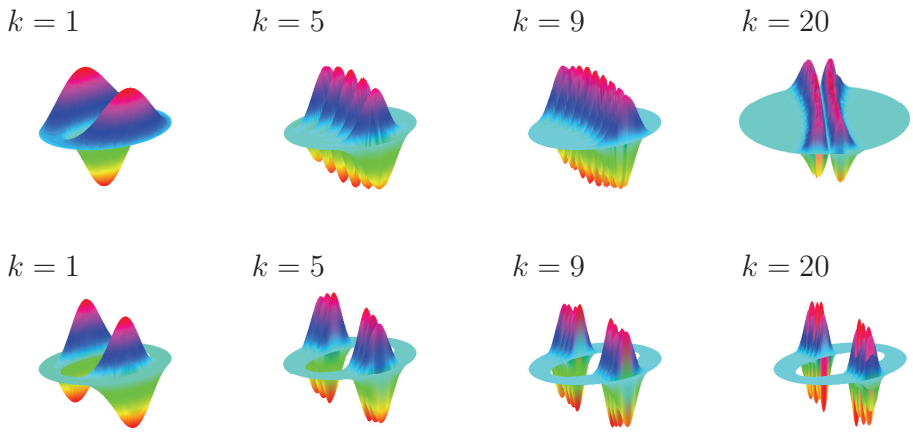

FIG. 3.1. *Formation of bouncing ball modes u_{nki} in a filled ellipse of radius $R = 1$ (top) and an elliptical annulus of radii 0.5 and 1 (bottom), with the focal distance $a = 1$ and Dirichlet boundary condition. For fixed $n = 1$ and $i = 1$, an increase of the index k leads to stronger localization of the eigenfunction near the vertical semiaxis ($K_{max} = 200$; see Appendix C).*

THEOREM 3.1. *Let Ω be a filled ellipse or an elliptical annulus (with a focal distance $a > 0$). For any $\alpha \in \left(0, \frac{\pi}{2}\right)$, $p \geq 1$, and $i = 1, 2, 3, 4$ (for filled ellipse) or $i = 1, 2$ (for elliptical annulus), there exists $\Lambda_\alpha > 0$ such that for any $\lambda_{nki} > \Lambda_\alpha$*

$$\frac{\|u_{nki}\|_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nki}\|_{L_p(\Omega)}} < D_n \left(\frac{16\alpha}{\pi - \alpha/2}\right)^{\frac{1}{p}} \exp\left(-a\sqrt{\lambda_{nki}}\left[\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \sin\alpha\right]\right), \tag{3.7}$$

where

$$D_n = 3\sqrt{\frac{1 + \sin\left(\frac{3\pi}{8} + \frac{\alpha}{4}\right)}{\left[\tan\left(\frac{\pi}{16} - \frac{\alpha}{8}\right)\right]^n}}. \tag{3.8}$$

See Appendix D for a proof; a similar exponential bound for the L_2-norm was recently derived in [30]. Given that $\lambda_{nki} \to \infty$ as k increases (for any fixed n and i), while the area of Ω_α can be made arbitrarily small by sending $\alpha \to \pi/2$, the theorem implies that there are infinitely many eigenfunctions u_{nki} which are L_p-localized in the elliptical sector Ω_α:

$$\lim_{k\to\infty} \frac{\|u_{nki}\|_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nki}\|_{L_p(\Omega)}} = 0. \tag{3.9}$$

These eigenfunctions are called bouncing ball modes and illustrated in Figure 3.1. We note that similar results were already known for a filled ellipse and, more generally, for convex planar domains with smooth boundary [4, 13]. Although our estimates are specific to elliptical shapes, they are explicit, simpler and also applicable to L_p-norms and to elliptical annuli, i.e., nonconvex domains.

The quality of the above estimates was checked numerically. Figure 3.2 shows the ratio $\frac{\|u_{nk1}\|_{L_2(\Omega\setminus\Omega_\alpha)}}{\|u_{nk1}\|_{L_2(\Omega_\alpha)}}$ and its upper bound for two families of eigenfunctions in a filled ellipse and an elliptical annulus. One can clearly see the rapid exponential decay of this ratio when k increases, which implies localization in a thin sector around the vertical (minor) axis. Note that the upper bound is not sharp and can be further improved.

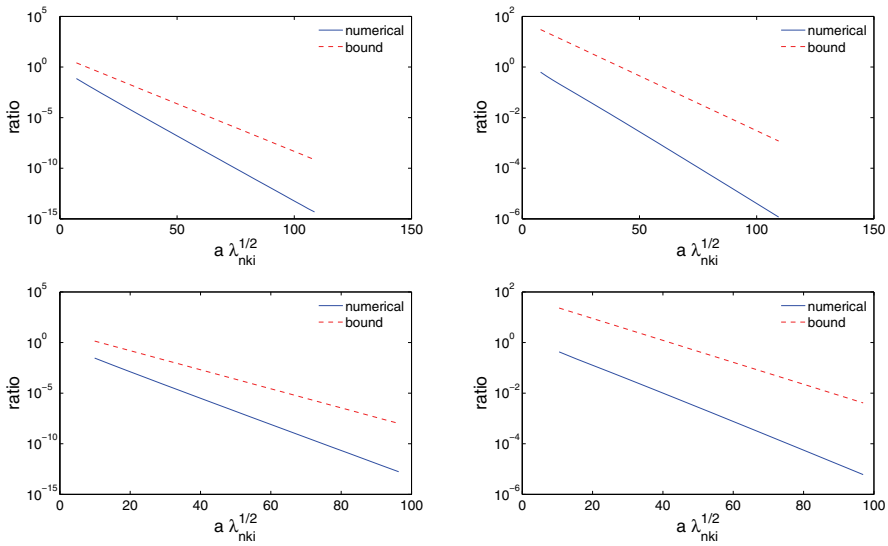

FIG. 3.2. *The ratio* $\frac{\|u_{nk1}\|_{L_2(\Omega\setminus\Omega_\alpha)}}{\|u_{nk1}\|_{L_2(\Omega_\alpha)}}$ *(solid line) and its upper bound* (3.7) *(dashed line) in a filled ellipse of radius* $R = 1$ *and focal distance* $a = 1$ *(top) and in an elliptical annulus of radii* $R = 0.5$ *and* $R = 1$ *and focal distance* $a = 1$ *(bottom), with* $n = 0$ *and* $\alpha = \pi/4$ *(left) and* $n = 1$ *and* $\alpha = \pi/3$ *(right). For numerical computation of Mathieu functions, we used* $K_{max} = 200$ *(see Appendix* C*).*

4. Discussion. The explicit estimates from previous sections provide us with simple examples of domains for which there are infinitely many L_p-localized eigenfunctions, according to the definition (1.2). Most importantly, high-frequency localization may occur in both convex and nonconvex domains. This observation relaxes, at least for elliptical domains, the condition of convexity that was significant for the construction of whispering gallery and bouncing ball modes by Keller and Rubinow [4] and for semiclassical approximations by Lazutkin [7, 8]. At the same time, these approximations suggest the existence of L_p-localized eigenfunctions for a large class of domains. How large is this class? What are the relevant conditions on the domain? To our knowledge, these questions are open. In order to highlight the relevance of these questions, it is instructive to give an example of domains for which there is no localization.

4.1. Rectangle-like domains. Rectangle-like domains, $\Omega = (0, \ell_1)\times\cdots\times(0, \ell_d) \subset \mathbb{R}^d$ (with the sizes $\ell_i > 0$), may seem to present the simplest shape for studying the Laplacian eigenfunctions as they are factored and expressed through sines (Dirichlet), cosines (Neumann), or their combination (Robin):

$$u_{n_1,\dots,n_d}(x_1,\dots,x_d) = u^{(1)}_{n_1}(x_1)\dots u^{(d)}_{n_d}(x_d), \qquad \lambda_{n_1,\dots,n_d} = \lambda^{(1)}_{n_1} + \cdots + \lambda^{(d)}_{n_d}, \tag{4.1}$$

with the multiple index $n_1,\dots,n_d$, and $u^{(i)}_{n_i}(x_i)$ and $\lambda^{(i)}_{n_i}$ $(i = 1,\dots,d)$ corresponding to the one-dimensional problem on the interval $(0, \ell_i)$:

$$u^{(i)}_n(x) = \sin(\pi n x/\ell_i), \quad \lambda^{(i)}_n = \pi^2 n^2/\ell_i^2, \quad n = 1, 2, 3, \dots \qquad \text{(Dirichlet)},$$
$$u^{(i)}_n(x) = \cos(\pi n x/\ell_i), \quad \lambda^{(i)}_n = \pi^2 n^2/\ell_i^2, \quad n = 0, 1, 2, \dots \qquad \text{(Neumann)}.$$

(Robin boundary condition will not be considered here.) The situation is indeed elementary for rectangle-like domains for which all eigenvalues are simple.

THEOREM 4.1. *Let $\Omega = (0, \ell_1) \times \cdots \times (0, \ell_d) \subset \mathbb{R}^d$ be a rectangle-like domain with sizes $\ell_1 > 0, \ldots, \ell_d > 0$ such that*

$$\ell_i^2/\ell_j^2 \notin \mathbb{Q} \qquad \forall\, i \neq j \tag{4.2}$$

($\mathbb{Q}$ denoting the set of rational numbers). Then for any $p \geq 1$ and any open subset $V \subset \Omega$,

$$C_p(V) = \inf_{n_1,\ldots,n_d} \left\{ \frac{\|u_{n_1,\ldots,n_d}\|_{L_p(V)}}{\|u_{n_1,\ldots,n_d}\|_{L_p(\Omega)}} \right\} > 0. \tag{4.3}$$

The proof is elementary (see Appendix E) and relies on the fact that all the eigenvalues are simple due to condition (4.2). The fact that $C_p(V) > 0$ for any open subset V means that there is no eigenfunction that could fully "avoid" any location inside the domain; i.e., there is no L_p-localized eigenfunction. Since the set of rational numbers has zero Lebesgue measure, the condition (4.2) is fulfilled almost surely if one would choose a rectangle-like domain randomly. In other words, for most rectangle-like domains there is no L_p-localized eigenfunction.

When at least one ratio ℓ_i^2/ℓ_j^2 is rational, certain eigenvalues are degenerate, and the associated eigenfunctions become linear combinations of products of sines or cosines. For instance, for the square with $\ell_1 = \ell_2 = \pi$ and Dirichlet boundary condition, the eigenvalue $\lambda_{1,2} = 1^2 + 2^2$ is twice degenerate, and $u_{1,2}(x_1, x_2) = c_1 \sin(x_1)\sin(2x_2) + c_2 \sin(2x_1)\sin(x_2)$, with arbitrary constants c_1 and c_2 ($c_1^2 + c_2^2 \neq 0$). Although the computation is still elementary for each eigenfunction, it is unknown whether the infimum $C_p(V)$ from (4.3) is strictly positive or not for arbitrary rectangle-like domain Ω and any open subset V. The most general known result for a rectangle $\Omega = (0, \ell_1) \times (0, \ell_2)$ states that $C_2(V) > 0$ for any $V \subset \Omega$ of the form $V = (0, \ell_1) \times \omega$, where ω is any open subset of $(0, \ell_2)$ [31]. Even for the unit square, the statement $C_p(V) > 0$ for any open subset V seems to be an open problem. More generally, one may wonder whether $C_p(V)$ is strictly positive or not for any open subset V in polygonal convex domains or in piecewise smooth convex domains. To our knowledge, these questions are open.

5. Conclusion. We have revived the classical problem of high-frequency localization of Laplacian eigenfunctions. For circular, spherical, and elliptical domains, we derived the inequalities for L_p-norms of the Laplacian eigenfunctions that clearly illustrate the emergence of whispering gallery, bouncing ball, and focusing eigenmodes. We gave an alternative proof for the existence of bouncing ball modes in elliptical domains, relying on the properties of Mathieu functions and well applicable to elliptical annuli. As a consequence, bouncing ball modes also exist in nonconvex domains. At the same time, we showed that there is no localization in most rectangle-like domains, which led us to formulate the problem of how to characterize the class of domains admitting high-frequency localization. In particular, the roles of convexity and smoothness have to be further investigated. The problems of localization in polygonal convex domains or, more generally, in piecewise smooth convex domains are open.

Appendix A. Proofs for a disk. The proof of Theorem 2.1 is based on several estimates for Bessel functions and their roots, which we recall in the following lemmas. In this appendix, $j_{\nu,k}$ and $j'_{\nu,k}$ denote all positive zeros (enumerated by

$k = 1, 2, 3, \ldots$ in an increasing order) of the Bessel function $J_\nu(x)$ and its derivative $J'_\nu(x)$, respectively.

LEMMA A.1. *For any $n = 1, 2, 3, \ldots$ and any $\varepsilon \in (0, 2/3)$, the Bessel function $J_n(x)$ satisfies* [32]

$$0 < J_n(nz) < 2^{-\frac{n^\varepsilon}{3}} \qquad \forall\ z \in (0, 1 - n^{\varepsilon - \frac{2}{3}}). \tag{A.1}$$

LEMMA A.2. *The first zeros $j_{n,1}$ and $j'_{n,1}$ with $n = 1, 2, \ldots$ satisfy* [25, 33]

$$n < j'_{n,1} < j_{n,1} < \sqrt{n+1}\left(\sqrt{n+2} + 1\right). \tag{A.2}$$

LEMMA A.3. *For large enough n, the following asymptotic relations hold* [25]:

$$J_n(n) = C'_1 n^{-1/3} + O(n^{-5/3}) \qquad \left(C'_1 = \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi} \approx 0.4473\right), \tag{A.3}$$

$$j'_{n,1} = n + n^{1/3}\ C'_2 + O(n^{-1/3}) \qquad (C'_2 = 0.808618\ldots). \tag{A.4}$$

As a consequence, taking smaller constants (e.g., $C_1 = 0.447$ and $C_2 = 0.8086$), one gets lower bounds for large enough n:

$$J_n(n) > C_1 n^{-1/3} \qquad (n \gg 1), \tag{A.5}$$

$$j'_{n,1} > n + C_2 n^{1/3} \qquad (n \gg 1). \tag{A.6}$$

LEMMA A.4. *For fixed k and large ν, the Olver's expansion holds* [34, 35, 36]:

$$\begin{aligned} j_{\nu,k} = \nu + \delta_k \nu^{1/3} + \frac{3}{10}\delta_k^2 \nu^{-1/3} + \frac{5 - \delta_k^3}{350}\nu^{-1} - \frac{479\delta_k^4 + 20\delta_k}{63000}\nu^{-5/3} \\ + \frac{20231\delta_k^5 - 27550\delta_k^2}{8085000}\nu^{-7/3} + O(\nu^{-3}), \end{aligned} \tag{A.7}$$

where $\delta_k = -a_k 2^{-1/3} > 0$ and a_k are the negative zeros of the Airy function (e.g., $\delta_1 = 1.855757\ldots$). Taking $c_k = \delta_k + \epsilon$ (e.g., $\epsilon = 1$), one gets the upper bounds for $j_{\nu,k}$ for ν large enough:

$$j_{\nu,k} < \nu + c_k \nu^{1/3} \qquad (\nu \gg 1). \tag{A.8}$$

LEMMA A.5. *For fixed ν and large k, the McMahon's expansion holds* [25, p. 506]:

$$j_{\nu,k} = k\pi + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right) - \frac{4\nu^2 - 1}{8(k\pi + \pi(\nu - 1/2)/2)} + O\left(\frac{1}{k^3}\right). \tag{A.9}$$

LEMMA A.6. *The absolute extrema of any Bessel function $J_\nu(z)$ progressively decrease* [25, p. 488], *i.e.,*

$$|J_\nu(j'_{\nu,1})| > |J_\nu(j'_{\nu,2})| > |J_\nu(j'_{\nu,3})| > \cdots. \tag{A.10}$$

LEMMA A.7. *The kth positive zero α_{nk} of the function $J'_n(z) + hJ_n(z)$ for any $h > 0$ lies between the kth positive zeros $j_{n,k}$ and $j'_{n,k}$ of the Bessel function $J_n(z)$ and its derivative $J'_n(z)$:*

$$j'_{n,k} < \alpha_{nk} < j_{n,k}. \tag{A.11}$$

Proof. This lemma is a direct consequence of the minimax principle that ensures the monotonic increase of eigenvalues with the parameter h [37]. ∎

Using these lemmas, we prove Theorem 2.1.

Proof of Theorem 2.1. The proof formalizes the idea that the eigenfunction u_{nk} is small in the large subdomain $D_{nk} = \{x \in D : |x| < Rd_n/\alpha_{nk}\}$ (with $d_n = n - n^{2/3}$) and large in the small subdomain $A_{nk} = \{x \in D : Rn/\alpha_{nk} < |x| < Rj'_{n,1}/\alpha_{nk}\}$. Since $A_{nk} \subset D$, we have for $1 \le p < \infty$

$$\frac{\|u_{nk}\|^p_{L_p(D_{nk})}}{\|u_{nk}\|^p_{L_p(D)}} < \frac{\|u_{nk}\|^p_{L_p(D_{nk})}}{\|u_{nk}\|^p_{L_p(A_{nk})}} = \frac{\int_0^{Rd_n/\alpha_{nk}} dr\ r\ |J_n(r\alpha_{nk}/R)|^p}{\int_{Rn/\alpha_{nk}}^{Rj'_{n,1}/\alpha_{nk}} dr\ r\ |J_n(r\alpha_{nk}/R)|^p}$$
$$= \frac{\int_0^{d_n} dzz\ |J_n(z)|^p}{\int_n^{j'_{n,1}} dzz\ |J_n(z)|^p}.$$

The numerator can be bounded by the inequality (A.1) with $\epsilon = 1/3$:

$$\int_0^{d_n} dzz\ |J_n(z)|^p < \left(2^{-n^{1/3}/3}\right)^p\ \frac{d_n^2}{2} < 2^{-pn^{1/3}/3}\ \frac{n^2}{2} \qquad (n = 1, 2, 3, \ldots).$$

In order to bound the denominator, we use the inequalities (A.5), (A.6) and the fact that $J_n(z)$ increases on the interval $[n, j'_{n,1}]$ (up to the first maximum at $j'_{n,1}$):

$$\int_n^{j'_{n,1}} dzz\ |J_n(z)|^p > |J_n(n)|^p \frac{(j'_{n,1})^2 - n^2}{2} > [C_1 n^{-1/3}]^p\ \frac{(n + C_2 n^{1/3})^2 - n^2}{2}$$
$$> C_1^p C_2 n^{(4-p)/3}$$

for n large enough, from which

$$\frac{\|u_{nk}\|_{L_p(D_{nk})}}{\|u_{nk}\|_{L_p(A_{nk})}} < \frac{n^{\frac{1}{3}+\frac{2}{3p}} 2^{-n^{1/3}/3}}{C_1 (2C_2)^{1/p}} \qquad (n \gg 1),$$

which implies (2.2).

For $p = \infty$, one has

$$\frac{\|u_{nk}\|_{L_\infty(D_{nk})}}{\|u_{nk}\|_{L_\infty(D)}} < \frac{\|u_{nk}\|_{L_\infty(D_{nk})}}{\|u_{nk}\|_{L_\infty(A_{nk})}} = \frac{\max_{0<z<d_n} |J_n(z)|}{\max_{n<z<j'_{n,1}} |J_n(z)|}.$$

Using the same bounds as above, one gets

$$\max_{0<z<d_n} |J_n(z)| < 2^{-n^{1/3}/3}, \qquad \max_{n<z<j'_{n,1}} |J_n(z)| > J_n(n) > C_1 n^{-1/3},$$

which implies (2.2).

Finally, from Lemmas A.4 and A.7, we have

$$1 > \frac{\mu_2(D_{nk})}{\mu_2(D)} = \left(\frac{d_n}{\alpha_{nk}}\right)^2 > \frac{d_n^2}{j_{n,k}^2} > \frac{(n - n^{2/3})^2}{(n + c_k n^{1/3})^2} \qquad (n \gg 1),$$

so that the ratio of the areas tends to 1 as n goes to infinity. ∎

Let us now prove Theorem 2.4.

Proof of Theorem 2.4. For $p = \infty$, the explicit representation (2.1) of eigenfunctions leads to

$$\frac{\|u_{nk}\|_{L_\infty(D(R))}}{\|u_{nk}\|_{L_\infty(D)}} = \frac{\max_{r\in[R,1]} |J_n(\alpha_{nk} r)|}{\max_{r\in[0,1]} |J_n(\alpha_{nk} r)|} = \frac{\max_{r\in[R,1]} |J_n(\alpha_{nk} r)|}{\left|J_n(j'_{n,1})\right|}, \tag{A.12}$$

where we used the fact that the first maximum (at $j'_{n,1}$) is the largest (Lemma A.6). Since $\lim_{k\to\infty} \alpha_{nk} = \infty$, the Bessel function $J_n(\alpha_{nk} r)$ with $k \gg 1$ can be approximated in the interval $[R, 1]$ as [27]

$$J_n(\alpha_{nk} r) \approx \sqrt{\frac{2}{\pi \alpha_{nk} r}} \cos\left(\alpha_{nk} r - \frac{n\pi}{2} - \frac{\pi}{4}\right). \tag{A.13}$$

This means that there exists a positive integer K_0 and a constant $A_0 > 0$ (e.g., $A_0 = 3/\pi$) such that

$$|J_n(\alpha_{nk} r)| < \sqrt{\frac{A_0}{\alpha_{nk} r}} \le \sqrt{\frac{A_0}{\alpha_{nk} R}} \qquad \forall r \in [R, 1],\ k > K_0. \tag{A.14}$$

Given that the denominator in (A.12) is fixed, while the numerator decays as $\alpha_{nk}^{-1/2}$, one gets (2.8) for $p = \infty$.

For $p > 4$, the ratio of L_p-norms is

$$\frac{\|u_{nk}\|^p_{L_p(D(R))}}{\|u_{nk}\|^p_{L_p(D)}} = \frac{\int_{\alpha_{nk} R}^{\alpha_{nk}} dr\ r\ |J_n(r)|^p}{\int_0^{\alpha_{nk}} dr\ r\ |J_n(r)|^p}. \tag{A.15}$$

The inequality (A.14) allows one to bound the numerator as

$$\int_{\alpha_{nk} R}^{\alpha_{nk}} dr\ r\ |J_n(r)|^p \le A_0^{p/2} \int_{\alpha_{nk} R}^{\alpha_{nk}} dr\ r^{1-p/2} = \frac{A_0^{p/2}[R^{2-p/2} - 1]}{p/2 - 2}\ \alpha_{nk}^{2-p/2},$$

while the denominator can be simply bounded from below by a constant

$$\int_0^{\alpha_{nk}} dr\ r\ |J_n(r)|^p \ge \int_0^1 dr\ r\ |J_n(r)|^p.$$

As a consequence, the ratio of L_p-norms in (2.8) goes to 0 as k increases.

For $1 \le p < 4$, one can write

$$\frac{\|u_{nk}\|^p_{L_p(D(R))}}{\|u_{nk}\|^p_{L_p(D)}} = 1 - \frac{f_{p,n}(\alpha_{nk} R)}{f_{p,n}(\alpha_{nk})},$$

where

$$f_{p,n}(z) \equiv \int_0^z dr\ r\ |J_n(r)|^p. \tag{A.16}$$

As discussed in Remark A.1, the function $f_{p,n}(z)$ behaves asymptotically as $z^{2-p/2}$ for large z; i.e., there exists $0 < c_{p,n} < \infty$ such that for any $\varepsilon > 0$ there exists $z_0 > 0$ such that for any $z > z_0$ (cf. (A.17))

$$(c_{p,n} - \varepsilon) z^{2-p/2} \le f_{p,n}(z) \le (c_{p,n} + \varepsilon) z^{2-p/2},$$

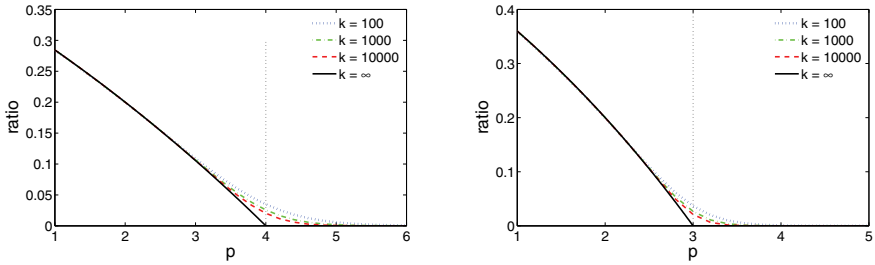

FIG. A.1. *The ratio* $\|u_{nk}\|^p_{L_p(D(R))}/\|u_{nk}\|^p_{L_p(D)}$ *as a function of* p *for* $n = 1$ *and* $R = 0.8$, *in two dimensions (left) and three dimensions (right). Three color curves correspond to three eigenfunctions with* $k = 100$, $k = 1000$, *and* $k = 10000$, *while the solid black curve shows the theoretical limit as* $k \to \infty$ *given by* (2.8) *in two dimensions and* (2.9) *in three dimensions. Note a slower convergence (deviations) for* $p \sim 4$ *in two dimensions and* $p \sim 3$ *in three dimensions.*

from which one immediately deduces

$$\frac{c_{p,n} - \varepsilon}{c_{p,n} + \varepsilon} R^{2-p/2} \leq \frac{f_{p,n}(\alpha_{nk} R)}{f_{p,n}(\alpha_{nk})} \leq \frac{c_{p,n} + \varepsilon}{c_{p,n} - \varepsilon} R^{2-p/2}.$$

As a consequence, for any $R < 1$, one can always choose ε such that the right-hand side is strictly smaller than 1, so that the ratio of L_p-norms is then strictly positive. Moreover, the limiting value is $1 - R^{2-p/2}$, which completes the proof of (2.8) for $1 \leq p < 4$. ∎

Remark A.1. For $1 \leq p < 4$, the function $f_{p,n}(z)$ defined by (A.16) asymptotically behaves as $z^{2-p/2}$ for large z; i.e., the limit

$$c_{p,n} = \lim_{z \to \infty} \frac{f_{p,n}(z)}{z^{2-p/2}} \tag{A.17}$$

exists and is finite and strictly positive: $0 < c_{p,n} < \infty$.

Although this result is naturally expected from the asymptotic behavior (A.13) of Bessel functions, its rigorous proof is beyond the scope of this paper. An upper bound for the limit (i.e., $c_{p,n} < \infty$) can be easily deduced from inequality (A.14). A lower strictly positive bound (i.e., $c_{p,n} > 0$) would require more careful estimations. The most difficult part consists of proving the existence of the limit, as the numerical computation of the function $f_{p,n}(z)/z^{2-p/2}$ at large z shows its oscillatory behavior with a slowly decaying amplitude. Since the statement of Theorem 2.4 for $1 \leq p < 4$ relies on this conjectural result, (2.8) was also checked numerically and is presented in Figure A.1.

Appendix B. Proofs for a ball. In this section, we generalize the previous estimates to a ball $B = \{x \in \mathbb{R}^3 : |x| < R\}$ of radius $R > 0$. We recall that the Laplacian eigenfunctions in spherical coordinates are

$$u_{nkl}(r, \theta, \varphi) = j_n(\alpha_{nk} r/R) P_n^l(\cos\theta) e^{il\varphi}, \tag{B.1}$$

where $j_n(z)$ are the spherical Bessel functions of the first kind (not to be confused with zeros $j_{n,k}$),

$$j_n(z) \equiv \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z), \tag{B.2}$$

$P_n^l(z)$ are associated Legendre polynomials, and α_{nk} are the positive zeros of $j_n(z)$ (Dirichlet), $j_n'(z)$ (Neumann), or $j_n'(z) + hj_n(z)$ for some $h > 0$ (Robin), which are enumerated by $k = 1, 2, 3, \ldots$ for each $n = 0, 1, 2, \ldots$. We will derive estimates that do not depend on the angular coordinates θ and φ, so that the last index l will be omitted.

We start by recalling and extending several classical estimates.

LEMMA B.1. *For any $\nu \in \mathbb{R}_+$ and any $x \in (0,1)$, the Kapteyn inequality holds* [38]:

$$0 < J_\nu(\nu x) < \frac{x^\nu \exp(\nu\sqrt{1-x^2})}{\left(1+\sqrt{1-x^2}\right)^\nu}. \tag{B.3}$$

Now we can prove the following claim.

LEMMA B.2. *For any $\nu > 1$ and $0 < \epsilon < 2/3$ one has*

$$0 < j_{\nu-\frac{1}{2}}(x) < \sqrt{\frac{\pi}{2\nu}}\ \exp\left(\frac{2}{3} - \frac{1}{3}\nu^\epsilon\right) \qquad \forall\ x \in (0, \nu - \nu^{\epsilon+1/3}). \tag{B.4}$$

Proof. Using the Kapteyn inequality (B.3) and taking $x = \nu z$ with $z \in (0, 1 - \nu^{\epsilon-2/3})$, one has

$$j_{\nu-1/2}(\nu z) = \sqrt{\frac{\pi}{2\nu}}\frac{J_\nu(\nu z)}{\sqrt{z}} < \sqrt{\frac{\pi}{2\nu}} f(z), \qquad \text{with } f(z) \equiv \frac{z^{\nu-1/2} e^{\nu\left(1-z^2\right)^{1/2}}}{\left(1+\sqrt{1-z^2}\right)^\nu}.$$

Substituting $u = \sqrt{1-z^2} \in (0,1)$, one gets

$$f(z) = \left(\frac{\left(1-u^2\right)^{1-\frac{1}{2\nu}} e^{2u}}{(1+u)^2}\right)^{\frac{\nu}{2}} = \frac{e^{\frac{u}{2}}}{(1+u)^{\frac{1}{2}}}\left[\left(\frac{1-u}{1+u}\right)e^{2u}\right]^{\frac{\nu}{2}-\frac{1}{4}}.$$

Using the inequality

$$\frac{1-u}{1+u}\ e^{2u} < 1 - \frac{2}{3}u^3 \qquad \forall\ u \in (0,1),$$

one gets

$$f(z) < \frac{e^{\frac{1}{2}}}{(1+0)^{\frac{1}{2}}}\left[1 - \frac{2}{3}u^3\right]^{\frac{\nu}{2}-\frac{1}{4}} < e^{\frac{1}{2}}\left[1 - \frac{2}{3}u^3\right]^{\frac{\nu}{2}-\frac{1}{4}}.$$

Since $z < 1 - \nu^{\epsilon-2/3}$, one has

$$u = \sqrt{1-z^2} > \sqrt{1-z} > \nu^{\frac{\epsilon}{2}-\frac{1}{3}} > \nu^{\frac{\epsilon}{3}-\frac{1}{3}},$$

from which

$$f(z) < e^{\frac{1}{2}}\left[1 - \frac{2}{3}\nu^{\epsilon-1}\right]^{\frac{\nu}{2}-\frac{1}{4}} < e^{\frac{1}{2}}\left[\left(1 - \frac{2}{3}\nu^{\epsilon-1}\right)^{\frac{3}{2}\nu^{1-\epsilon}}\right]^{\frac{\frac{\nu}{2}-\frac{1}{4}}{\frac{3}{2}\nu^{1-\epsilon}}}.$$

Since

$$(1-x)^{\frac{1}{x}} < e^{-1} \quad \forall x \in (0,1), \quad \text{and} \quad 0 < \frac{2}{3}\nu^{\epsilon-1} < \frac{2}{3} < 1,$$

one finally gets

$$f(z) < \exp\left(\frac{1}{2} - \frac{\frac{\nu}{2} - \frac{1}{4}}{\frac{3}{2}\nu^{1-\epsilon}}\right) < \exp\left(\frac{1}{2} + \frac{1}{6}\nu^{\epsilon-1} - \frac{1}{3}\nu^{\epsilon}\right) < \exp\left(\frac{2}{3} - \frac{1}{3}\nu^{\epsilon}\right),$$

which completes the proof. ∎

As a consequence, taking $\nu = n + 1/2$ and $\epsilon = 1/3$, one has the following.

LEMMA B.3. *For $n = 1, 2, \ldots$ and any $z \in (0, n + 1/2 - (n + 1/2)^{2/3})$,*

$$j_n(z) < \sqrt{\frac{\pi}{2n+1}}\ \exp\left(\frac{2}{3} - \frac{1}{3}\left(n + \frac{1}{2}\right)^{\frac{1}{3}}\right). \tag{B.5}$$

The lemmas for Bessel functions and their zeros from Appendix A allow one to get similar estimates for spherical Bessel functions $j_n(z)$, their positive zeros $\gamma_{n,k}$, and the positive zeros $\gamma'_{n,k}$ of $j'_n(z)$. They are summarized in the following result.

LEMMA B.4. *For n large enough,*

$$j_n(n + 1/2) > \tilde{C}_1(n + 1/2)^{-5/6} \qquad (\tilde{C}_1 = \sqrt{\pi/2}\ C_1), \tag{B.6}$$

$$\gamma_{n,k} < (n + 1/2) + \tilde{c}_k(n + 1/2)^{1/3}, \tag{B.7}$$

$$\gamma'_{n,1} > n + 1/2 + \tilde{C}_2(n + 1/2)^{1/3} \qquad (\tilde{C}_2 = 0.80), \tag{B.8}$$

$$\gamma'_{n,k} < \alpha_{nk} < \gamma_{n,k}. \tag{B.9}$$

Proof. From Lemma A.3, we have

$$j_{\nu-1/2}(\nu) = \sqrt{\frac{\pi}{2\nu}} J_\nu(\nu) > \sqrt{\frac{\pi}{2\nu}} C_1 \nu^{-1/3} = \tilde{C}_1 \nu^{-5/6},$$

from which (B.6) follows by taking $\nu = n + 1/2$.

The zeros $\gamma_{n,k}$ of the spherical Bessel function $j_n(z)$ are also the zeros of the Bessel function $J_{n+1/2}(z)$, so that (B.7) follows directly from (A.8) for $\nu = n + 1/2$ large enough.

The inequality (B.8) follows from the asymptotic expansion of $\gamma'_{n,1}$ for large n [27, p. 441]:

$$\begin{aligned}\gamma'_{n,1} = n + 1/2 &+ 0.8086165(n + 1/2)^{1/3} - 0.236680(n + 1/2)^{-1/3} \\ &- 0.20736(n + 1/2)^{-1} + 0.0233(n + 1/2)^{-5/6} + \cdots \qquad (n \gg 1).\end{aligned}$$

Taking $\tilde{C}_2 = 0.80$, one gets the inequality (B.8).

Finally, the inequalities (B.9) follow from the general minimax principle as for the disk. ∎

We also prove that the first maximum of the spherical Bessel function at $\gamma'_{n,1}$ is the largest (although this is a classical fact, we did not find an explicit reference).

LEMMA B.5. *For an integer $n \geq 0$, one has*

$$\max_{x \in (0,\infty)} j_n(x) = j_n(\gamma'_{n,1}). \tag{B.10}$$

Proof. The spherical Bessel function $j_n(x)$ satisfies

$$x^2 j''_n + 2x j'_n + [x^2 - (n + 1)n] j_n = 0.$$

Denoting $\kappa = n(n+1)$, one can rewrite this equation as

$$j_n''(x) = -\frac{2xj_n'(x) + \left(x^2-\kappa\right) j_n(x)}{x^2},$$

from which

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^2}{x^2-\kappa}\left(j_n'(x)\right)^2\right] &= \frac{d}{dx}\left[\left(j_n'(x)\right)^2 + \frac{\kappa}{x^2-\kappa}\left(j_n'(x)\right)^2\right] \\
&= 2j_n'(x)j_n''(x) + \kappa\left[\frac{2(x^2-\kappa)j_n'(x)j_n''(x) - 2x\left(j_n'(x)\right)^2}{\left(x^2-\kappa\right)^2}\right] \\
&= \frac{2x^2}{x^2-\kappa}j_n'(x)j_n''(x) - \frac{2x\kappa}{\left(x^2-\kappa\right)^2}\left(j_n'(x)\right)^2 \\
&= -\frac{2j_n'(x)}{x^2-\kappa}\left[2xj_n'(x) + \left(x^2-\kappa\right) j_n(x)\right] - \frac{2x\kappa}{\left(x^2-\kappa\right)^2}\left(j_n'(x)\right)^2 \\
&= -2j_n(x)j_n'(x) - \frac{2x}{\left(x^2-\kappa\right)^2}\left(j_n'(x)\right)^2\left[2(x^2-\kappa)+\kappa\right] \\
&= -\frac{2x}{\left(x^2-\kappa\right)^2}\left(j_n'(x)\right)^2\left[2x^2-\kappa\right] - \frac{d}{dx}\left[j_n^2(x)\right].
\end{aligned}$$

Now, if we set

$$\Lambda_n(x) = j_n^2(x) + \left[\frac{x^2}{x^2-\kappa}\left(j_n'(x)\right)^2\right],$$

then

$$\frac{d}{dx}\Lambda_n(x) = -\frac{2x}{\left(x^2-\kappa\right)^2}\left(j_n'(x)\right)^2\left[2x^2-\kappa\right] < 0$$

for all $x > \sqrt{\frac{n(n+1)}{2}}$; i.e., $\Lambda_n(x)$ monotonically decreases. Given that $\Lambda_n(\gamma_{n,k}') = j_n^2(\gamma_{n,k})$ and

$$\sqrt{\frac{n(n+1)}{2}} < \gamma_{n,1}' < \gamma_{n,2}' < \cdots,$$

we get the conclusion. ∎

Now, we can prove Theorem 2.3.

Proof of Theorem 2.3. As earlier, the proof formalizes the idea that the eigenfunction u_{nk} is small in the large subdomain $B_{nk} = \{x \in B : |x| < Rs_n/\alpha_{nk}\}$ (with $s_n = (n+1/2) - (n+1/2)^{2/3}$) and large in the small subdomain $A_{nk} = \{x \in B : R(n+1/2)/\alpha_{nk} < |x| < R\gamma_{n,1}'/\alpha_{nk}\}$. Since $A_{nk} \subset B$, we have for $1 \le p < \infty$

$$\begin{aligned}
\frac{\|u_{nk}\|_{L_p(B_{nk})}^p}{\|u_{nk}\|_{L_p(B)}^p} &< \frac{\|u_{nk}\|_{L_p(B_{nk})}^p}{\|u_{nk}\|_{L_p(A_{nk})}^p} \\
&= \frac{\int_0^{Rs_n/\alpha_{nk}} dr\ r^2\ |j_n(r\alpha_{nk}/R)|^p}{\int_{R(n+1/2)/\alpha_{nk}}^{R\gamma_{n,1}'/\alpha_{nk}} dr\ r^2\ |j_n(r\alpha_{nk}/R)|^p} = \frac{\int_0^{s_n} dz z^2\ |j_n(z)|^p}{\int_{n+1/2}^{\gamma_{n,1}'} dz z^2\ |j_n(z)|^p}.
\end{aligned}$$

The numerator can be bounded by the inequality (B.5):

$$\int_0^{s_n} dz z^2\ |j_n(z)|^p < \left(\frac{\pi}{2n+1}\right)^{p/2} \exp\left(\frac{2p}{3} - \frac{p}{3}\left(n+\frac{1}{2}\right)^{1/3}\right)\frac{s_n^3}{3}$$
$$< \frac{(\pi/2)^{p/2}}{3}\exp\left(\frac{2p}{3} - \frac{p}{3}\left(n+\frac{1}{2}\right)^{1/3}\right)\left(n+\frac{1}{2}\right)^{3-p/2}$$

($n = 1, 2, 3, \ldots$). In order to bound the denominator, we use the inequalities (B.6), (B.8) and the fact that $j_n(z)$ increases on the interval $[n+1/2, \gamma'_{n,1}]$ (up to the first maximum at $\gamma'_{n,1}$):

$$\int_{n+1/2}^{\gamma'_{n,1}} dz z^2\ |j_n(z)|^p > [j_n(n+1/2)]^p \frac{(\gamma'_{n,1})^3 - (n+1/2)^3}{3}$$
$$> [\tilde{C}_1(n+1/2)^{-5/6}]^p\ \frac{(n+1/2+\tilde{C}_2(n+1/2)^{1/3})^3 - (n+1/2)^3}{3}$$
$$> \tilde{C}_1^p \tilde{C}_2 (n+1/2)^{7/3-5p/6}$$

for n large enough, from which

$$\frac{\|u_{nk}\|_{L_p(B_{nk})}}{\|u_{nk}\|_{L_p(A_{nk})}} < \frac{\sqrt{\pi/2}}{\tilde{C}_1(3\tilde{C}_2)^{1/p}}\exp\left(\frac{2}{3} - \frac{1}{3}\left(n+\frac{1}{2}\right)^{1/3}\right)\left(n+\frac{1}{2}\right)^{1/3+2/(3p)} \qquad (n \gg 1),$$

which implies (2.6). The case $p = \infty$ is treated similarly. Finally, from Lemma B.4, we have for n large enough that

$$1 > \frac{\mu_3(B_{nk})}{\mu_3(B)} = \left(\frac{s_n}{\alpha_{nk}}\right)^3 > \frac{s_n^3}{\gamma_{n,k}^3} > \frac{(n+1/2-(n+1/2)^{2/3})^3}{(n+1/2+\tilde{c}_k(n+1/2)^{1/3})^3},$$

so that the ratio of volumes tends to 1 as n goes to infinity. ∎

The proof of Theorem 2.5 for a ball is similar to that of Theorem 2.4.

Proof of Theorem 2.5. For $p = \infty$, the explicit representation (B.1) of eigenfunctions leads to

$$\text{(B.11)} \qquad \frac{\|u_{nk}\|_{L_\infty(B(R))}}{\|u_{nk}\|_{L_\infty(B)}} = \frac{\max_{r\in[R,1]}|j_n(\alpha_{nk}r)|}{\max_{r\in[0,1]}|j_n(\alpha_{nk}r)|} = \frac{\max_{r\in[R,1]}|j_n(\alpha_{nk}r)|}{|j_n(\gamma'_{n,1})|},$$

where we used the fact that the first maximum (at $\gamma'_{n,1}$) is the largest (Lemma B.5). Since $\lim_{k\to\infty}\alpha_{nk} = \infty$, the spherical Bessel function $j_n(\alpha_{nk}r)$ with $k \gg 1$ can be approximated in the interval $[R, 1]$ as (see [27])

$$\text{(B.12)} \qquad j_n(\alpha_{nk}r) = \sqrt{\frac{\pi}{2\alpha_{nk}r}}\ J_{n+1/2}(\alpha_{nk}r) \approx \frac{1}{\alpha_{nk}r}\cos\left(\alpha_{nk}r - \frac{(n+1)\pi}{2}\right).$$

This means that there exists a positive integer K_0 and a constant $A_0 > 0$ (e.g., $A_0 = 2$) such that

$$\text{(B.13)} \qquad |j_n(\alpha_{nk}r)| < \frac{A_0}{\alpha_{nk}r} \le \frac{A_0}{\alpha_{nk}R} \qquad \forall\ r \in [R,1],\ k > K_0.$$

Given that the denominator in (B.11) is fixed, while the numerator decays as α_{nk}^{-1}, one gets (2.9).

For $p > 3$, the ratio of L_p-norms is

$$\frac{\|u_{nk}\|^p_{L_p(D(R))}}{\|u_{nk}\|^p_{L_p(D)}} = \frac{\int_{\alpha_{nk}R}^{\alpha_{nk}} dr\ r^2\ |j_n(r)|^p}{\int_0^{\alpha_{nk}} dr\ r^2\ |j_n(r)|^p}. \tag{B.14}$$

The inequality (B.13) allows one to bound the numerator as

$$\int_{\alpha_{nk}R}^{\alpha_{nk}} dr\ r^2\ |j_n(r)|^p \le A_0^p \int_{\alpha_{nk}R}^{\alpha_{nk}} dz\ z^{2-p} = \frac{A_0^p[R^{3-p}-1]}{p-3}\ \alpha_{nk}^{3-p},$$

while the denominator can be simply bounded from below by a constant

$$\int_0^{\alpha_{nk}} dr\ r^2\ |j_n(r)|^p \ge \int_0^1 dr\ r^2\ |j_n(r)|^p.$$

As a consequence, the ratio of L_p-norms goes to 0 as k increases.

For $1 \le p < 3$, one can write

$$\frac{\|u_{nk}\|^p_{L_p(B(R))}}{\|u_{nk}\|^p_{L_p(B)}} = 1 - \frac{\tilde{f}_{p,n}(\alpha_{nk}R)}{\tilde{f}_{p,n}(\alpha_{nk})},$$

where

$$\tilde{f}_{p,n}(z) \equiv \int_0^z dr\ r^2\ |j_n(r)|^p. \tag{B.15}$$

As discussed in Remark B.1, the function $\tilde{f}_{p,n}(z)$ behaves asymptotically as z^{3-p} for large z; i.e., there exists $0 < \tilde{c}_{p,n} < \infty$ such that for any $\varepsilon > 0$ there exists $z_0 > 0$ such that for any $z > z_0$ (cf. (B.16))

$$(\tilde{c}_{p,n} - \varepsilon)z^{3-p} \le \tilde{f}_{p,n}(z) \le (\tilde{c}_{p,n} + \varepsilon)z^{3-p},$$

from which one immediately deduces

$$\frac{\tilde{c}_{p,n} - \varepsilon}{\tilde{c}_{p,n} + \varepsilon} R^{3-p} \le \frac{\tilde{f}_{p,n}(\alpha_{nk}R)}{\tilde{f}_{p,n}(\alpha_{nk})} \le \frac{\tilde{c}_{p,n} + \varepsilon}{\tilde{c}_{p,n} - \varepsilon} R^{3-p}.$$

As a consequence, for any $R < 1$, one can always choose ε such that the right-hand side is strictly smaller than 1 so that the ratio of L_p-norms is then strictly positive. Moreover, the limiting value is $1 - R^{3-p}$, which completes the proof of (2.9) for $1 \le p < 3$. ∎

Remark B.1. The function $\tilde{f}_{p,n}(z)$ defined by (B.15) asymptotically behaves as z^{3-p} for large z; i.e., the limit

$$\tilde{c}_{p,n} = \lim_{z\to\infty} \frac{\tilde{f}_{p,n}(z)}{z^{3-p}} \tag{B.16}$$

exists and is finite and strictly positive: $0 < \tilde{c}_{p,n} < \infty$. As for Remark A.1, a rigorous proof of this result is beyond the scope of this paper. The asymptotic behavior from (2.9) for focusing modes is illustrated in Figure A.1.

Appendix C. Analysis of Mathieu functions. Many algorithms have been proposed for a numerical computation of Mathieu functions [39, 40, 41, 42]. The main

difficulty is the computation of Mathieu characteristic numbers (MCNs). Alhargan introduced a complete method for calculating the MCNs for Mathieu functions of integer orders by using recurrence relations for MCNs [39]. His algorithm is a good compromise between complexity, accuracy, speed, and ease of use. Nevertheless, for illustrative purposes of the paper, we used a simpler approach by Zhang and Jin [43]. In this approach, the problem of calculating expansion coefficients of Mathieu functions is reduced to an eigenproblem for sparse tridiagonal matrices. We have rebuilt the computation of Mathieu functions and modified Mathieu functions and checked the accuracy of the numerical algorithm by comparing their values to those published in the literature [13, 43, 44]. We also checked that the truncation of the underlying tridiagonal matrices to the size $K_{max} = 200$ was enough for getting very accurate results, at least for the examples presented in the paper.

Appendix D. Asymptotic behavior of Mathieu functions for large q. The large q asymptotic expansions of $\mathrm{ce}_n(z,q)$ and $\mathrm{se}_{n+1}(z,q)$ for $z \in [0, \frac{\pi}{2})$ and $n = 0, 1, 2, \ldots$ are [29, 45]

$$\mathrm{ce}_n(z,q) = C_n(q)\left(e^{2\sqrt{q}\sin z} h_n^+(z) \sum_{k=0}^{\infty} \frac{f_k^+(z)}{q^{k/2}} + e^{-2\sqrt{q}\sin z} h_n^-(z) \sum_{k=0}^{\infty} \frac{f_k^-(z)}{q^{k/2}}\right), \tag{D.1}$$

$$\mathrm{se}_{n+1}(z,q) = S_{n+1}(q)\left(e^{2\sqrt{q}\sin z} h_n^+(z) \sum_{k=0}^{\infty} \frac{f_k^+(z)}{q^{k/2}} - e^{-2\sqrt{q}\sin z} h_n^-(z) \sum_{k=0}^{\infty} \frac{f_k^-(z)}{q^{k/2}}\right), \tag{D.2}$$

where

$$h_n^+(z) = 2^{n+\frac{1}{2}} \frac{\left[\cos\left(\frac{1}{2}z + \frac{\pi}{4}\right)\right]^{2n+1}}{(\cos z)^{n+1}} = \sqrt{\frac{(1-\sin z)^n}{(1+\sin z)^{n+1}}}, \tag{D.3}$$

$$h_n^-(z) = 2^{n+\frac{1}{2}} \frac{\left[\sin\left(\frac{1}{2}z + \frac{\pi}{4}\right)\right]^{2n+1}}{(\cos z)^{n+1}} = \sqrt{\frac{(1+\sin z)^n}{(1-\sin z)^{n+1}}}, \tag{D.4}$$

and the coefficients $C_n(q)$ and $S_{n+1}(q)$ are given explicitly in [29]. The coefficients $f_k^{\pm}(z)$ can be computed through the recursive formulas given in [45], e.g.,

$$f_0^{\pm}(z) = 1, \qquad f_1^{\pm}(z) = \frac{2n+1 \mp \left(n^2+n+1\right)\sin z}{8\cos(z)^2}.$$

When q is large enough, one can truncate the asymptotic expansions (D.1), (D.2) by keeping only two terms ($k = 0, 1$) and get accurate approximations for ce_n and se_{n+1}, as illustrated in Figure D.1.

It is convenient to define the functions

$$G_n^{\pm}(z,q) = h_n^+(z) \pm e^{-4\sqrt{q}\sin(z)} h_n^-(z) + h_n^+(z) \sum_{k=1}^{\infty} \frac{f_k^+(z)}{q^{k/2}} \pm e^{-4\sqrt{q}\sin(z)} h_n^-(z) \sum_{k=1}^{\infty} \frac{f_k^-(z)}{q^{k/2}},$$

in order to write

$$\mathrm{ce}_n(z,q) = C_n(q) e^{2\sqrt{q}\sin(z)} G_n^+(z,q), \qquad \mathrm{se}_{n+1}(z,q) = S_{n+1}(q) e^{2\sqrt{q}\sin(z)} G_n^-(z,q).$$

In what follows, we will estimate the functions $G_n^{\pm}(z,q)$ by their leading terms, given that the remaining part is getting small for large q.

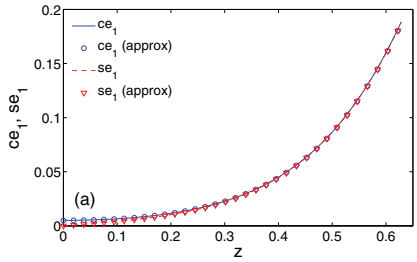

FIG. D.1. *The functions* $\mathrm{ce}_1(z,q)$ *and* $\mathrm{se}_1(z,q)$ *(solid and dashed lines), computed by our algorithm with* $K_{max} = 200$*, and their approximations (circles and triangles) by the asymptotic expansions* (D.1), (D.2) *truncated to two terms* ($k = 0, 1$)*, with* $q = 20$.

LEMMA D.1. *For* $\gamma \in \left(0, \frac{\pi}{2}\right)$ *there exists* $N_\gamma > 0$ *such that for* $q > N_\gamma$ *and* $z \in (0, \gamma)$

$$\left| \sum_{k=1}^{\infty} \frac{f_k^{\pm}(z)}{q^{k/2}} \right| < \frac{1}{2}. \tag{D.5}$$

Now, we establish the upper and lower bounds for the functions $G_n^{\pm}$.

LEMMA D.2. *Let* $\alpha \in \left(0, \frac{\pi}{2}\right)$, $\gamma \in \left(\alpha, \frac{\pi}{2}\right)$. *Then, there exists* $N_\gamma > 0$ *such that for any* $\beta \in (\alpha, \gamma)$ *and* $q > N_\gamma$

$$\left|G_n^{\pm}(z_1, q)\right| < \frac{3}{2}\left(1 + h_n^-(\alpha) e^{-4\sqrt{q}\sin(z_1)}\right) \qquad \forall\ z_1 \in (0, \alpha), \tag{D.6}$$

$$\left|G_n^{\pm}(z_2, q)\right| > \frac{1}{2} h_n^+(\gamma) \qquad \forall\ z_2 \in (\beta, \gamma). \tag{D.7}$$

Proof. From Lemma D.1, there exists $N_\gamma > 0$ such that for $q > N_\gamma$ and $z_1 \in (0, \alpha)$ one has

$$\left|G_n^{\pm}(z_1, q)\right| < \frac{3}{2}\left(h_n^+(z_1) + h_n^-(z_1) e^{-4\sqrt{q}\sin(z_1)}\right) < \frac{3}{2}\left(1 + h_n^-(\alpha) e^{-4\sqrt{q}\sin(z_1)}\right).$$

For $q > N_\gamma$ and $z_2 \in (\beta, \gamma)$ one has

$$\left|G_n^{+}(z_2, q)\right| > \frac{1}{2}\left(h_n^+(z_2) + h_n^-(z_2) e^{-4\sqrt{q}\sin(z_2)}\right) > \frac{1}{2} h_n^+(\gamma) > 0$$

and

$$\left|G_n^{-}(z_2, q) - \left(h_n^+(z_2) - h_n^-(z_2) e^{-4\sqrt{q}\sin(z_2)}\right)\right| < \frac{1}{2}\left(h_n^+(z_2) + h_n^-(z_2) e^{-4\sqrt{q}\sin(z_2)}\right).$$

The last inequality implies

$$\left|G_n^{-}(z_2, q)\right| > \min\left\{\left(h_n^+(z_2) - h_n^-(z_2) e^{-4\sqrt{q}\sin(z_2)}\right), \frac{1}{2}\left(h_n^+(z_2) - 3h_n^-(z_2) e^{-4\sqrt{q}\sin(z_2)}\right)\right\}.$$

Since $h_n^-(z_2) > 0$ and $h_n^+(z_2)$ is a decreasing function, one gets

$$\left|G_n^{-}(z_2, q)\right| > \frac{1}{2} h_n^+(\gamma),$$

which completes the proof. ∎

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. We first consider the case $i = 1$. Using the symmetric properties of Mathieu functions [43], one has

$$\frac{\|u_{nk1}\|^p_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nk1}\|^p_{L_p(\Omega_\alpha)}} = \frac{\int_0^\alpha |\mathrm{ce}_n(z_1, q_{nk1})|^p dz_1}{\int_\alpha^{\pi/2} |\mathrm{ce}_n(z_2, q_{nk1})|^p dz_2}.$$

Choosing $\beta = \frac{\pi}{4} + \frac{\alpha}{2}$ and $\gamma = \frac{3\pi}{8} + \frac{\alpha}{4}$, one gets

$$\frac{\int_0^\alpha |\mathrm{ce}_n(z_1, q_{nk1})|^p dz_1}{\int_\alpha^{\pi/2} |\mathrm{ce}_n(z_2, q_{nk1})|^p dz_2} < \frac{\int_0^\alpha |\mathrm{ce}_n(z_1, q_{nk1})|^p dz_1}{\int_\beta^{\gamma} |\mathrm{ce}_n(z_2, q_{nk1})|^p dz_2}.$$

From Lemma D.2, there exists $N_\gamma > 0$ such that for $q > N_\gamma$,

$$\begin{aligned}\int_0^\alpha \quad |\mathrm{ce}_n(z_1, q)|^p dz_1 &= (C_n(q))^p \int_0^\alpha e^{2p\sqrt{q}\sin z_1} |G_n^+(z_1, q)|^p dz_1 \\ &< (C_n(q))^p \left(\frac{3}{2}\right)^p \int_0^\alpha \left(\sum_{k=0}^{p} \binom{p}{k} [e^{2\sqrt{q}\sin z_1}]^{p-2k} (h_n^-(\alpha))^k\right) dz_1 \\ &\le \alpha (C_n(q))^p \left(\frac{3}{2}\right)^p \left(\sum_{k=0}^{[p/2]} \binom{p}{k} [e^{2\sqrt{q}\sin\alpha}]^{p-2k} (h_n^-(\alpha))^k + \sum_{k=[p/2]+1}^{p} \binom{p}{k} (h_n^-(\alpha))^k\right),\end{aligned}$$

where the terms $e^{m\sqrt{q}\sin z_1}$ were bounded by $e^{m\sqrt{q}\sin\alpha}$ for $m > 0$, and by 1 for $m \le 0$ (here $[x]$ denotes the integer part of x). In addition,

$$\begin{aligned}\int_\beta^\gamma |\mathrm{ce}_n(z_2, q)|^p dz_2 &> (C_n(q))^p \left(\frac{1}{2}\right)^p (h_n^+(\gamma))^p \int_\beta^\gamma e^{2p\sqrt{q}\sin z_2} dz_2 \\ &> (C_n(q))^p \left(\frac{1}{2}\right)^p (h_n^+(\gamma))^p (\gamma - \beta) e^{2p\sqrt{q}\sin\beta},\end{aligned}$$

from which

$$\begin{aligned}&\frac{\|u_{nk1}\|^p_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nk1}\|^p_{L_p(\Omega_\alpha)}} < \frac{3^p\alpha}{(\gamma-\beta)(h_n^+(\gamma))^p} e^{-2p\sqrt{q_{nk1}}(\sin\beta - \sin\alpha)} \\ &\cdot \left(1 + \left[\sum_{k=1}^{[p/2]} \binom{p}{k} (e^{2\sqrt{q}\sin\alpha})^{-2k} (h_n^-(\alpha))^k + e^{-2p\sqrt{q}\sin\alpha} \sum_{k=[p/2]+1}^{p} \binom{p}{k} (h_n^-(\alpha))^k\right]\right).\end{aligned}$$

Taking q large enough, one can make the terms in large brackets smaller than any prescribed threshold ϵ. For $\epsilon = 1$, one can simplify the estimate as

$$\frac{\|u_{nk1}\|^p_{L_p(\Omega\setminus\Omega_\alpha)}}{\|u_{nk1}\|^p_{L_p(\Omega_\alpha)}} < 2\frac{3^p\alpha}{(\gamma-\beta)(h_n^+(\gamma))^p} \exp\left[-2p\sqrt{q_{nk1}}(\sin\beta - \sin\alpha)\right].$$

Substituting $\beta = \pi/4 + \alpha/2$ and $\gamma = 3\pi/8 + \alpha/4$, one gets (3.7) after trigonometric simplifications.

For $i = 2$, one can use similar estimates for se_{n+1}. ∎

Appendix E. No localization in rectangle-like domains. Theorem 4.1 relies on the following simple estimate.

LEMMA E.1. *For $0 \le a < b$ and any positive integer m, one has*

$$\begin{aligned} \int_a^b |\sin(mx)|dx &\ge \int_a^b \sin^2(mx)dx \ge \epsilon(a,b) > 0, \\ \int_a^b |\cos(mx)|dx &\ge \int_a^b \cos^2(mx)dx \ge \epsilon(a,b) > 0. \end{aligned} \tag{E.1}$$

where

$$\epsilon(a,b) = \min\left\{\frac{b-a}{4}, \frac{b-a}{2} - \frac{1}{2}\left|\frac{\sin(n(b-a))}{n}\right| : n = 1,2,\dots,\left[\frac{2}{b-a}\right]\right\} > 0. \tag{E.2}$$

It is important to stress that the lower bound $\epsilon(a,b)$ does not depend on m. The proof of this lemma is elementary.

The proof of Theorem 4.1 is a simple consequence.

Proof of Theorem 4.1. The condition (4.2) ensures that all the eigenvalues are simple so that each eigenfunction is

$$u_{n_1,\dots,n_d}(x_1,\dots,x_d) = \begin{cases} \sin(\pi n_1 x_1/\ell_1)\dots\sin(\pi n_d x_d/\ell_d) & \text{(Dirichlet)}, \\ \cos(\pi n_1 x_1/\ell_1)\dots\cos(\pi n_d x_d/\ell_d) & \text{(Neumann)}. \end{cases}$$

For any open subset V there exists a ball included in V, and thus there exists a rectangle-like domain $\Omega_V = [a_1,b_1]\times\dots\times[a_d,b_d] \subset V$, with $0 \le a_i < b_i \le \ell_i$ for all $i = 1,\dots,d$. The L_1-norm of u in V can be estimated as

$$\begin{aligned} \|u_{n_1,\dots,n_d}\|_{L_1(V)} \ge \|u_{n_1,\dots,n_d}\|_{L_1(\Omega_V)} &= \prod_{i=1}^d \int_{a_i}^{b_i} dx_i \begin{cases} |\sin(\pi n_i x_i/\ell_i)| \\ |\cos(\pi n_i x_i/\ell_i)| \end{cases} \\ = \frac{\ell_1\dots\ell_d}{\pi^d}\prod_{i=1}^d \int_{\pi a_i/\ell_i}^{\pi b_i/\ell_i} dx_i \begin{cases} |\sin(n_i x_i)| \\ |\cos(n_i x_i)| \end{cases} &\ge \frac{\ell_1\dots\ell_d}{\pi^d}\prod_{i=1}^d \epsilon(\pi a_i/\ell_i, \pi b_i/\ell_i), \end{aligned}$$

where the last inequality results from (E.1). To complete the proof, one uses the Jensen inequality for L_p-norms and $\mu_d(V) \ge \mu_d(\Omega_V) = (b_1-a_1)\dots(b_d-a_d)$:

$$\begin{aligned} \frac{\|u_{n_1,\dots,n_d}\|_{L_p(V)}}{\|u_{n_1,\dots,n_d}\|_{L_p(\Omega)}} &> \frac{\|u_{n_1,\dots,n_d}\|_{L_1(V)}(\mu_d(V))^{\frac{1}{p}-1}}{\|u_{n_1,\dots,n_d}\|_{L_\infty(\Omega)}(\mu_d(\Omega))^{\frac{1}{p}}} \\ &\ge \frac{1}{\pi^d}\prod_{i=1}^d \left(\frac{b_i-a_i}{\ell_i}\right)^{\frac{1}{p}-1} \epsilon\left(\pi\frac{a_i}{\ell_i}, \pi\frac{b_i}{\ell_i}\right) > 0. \end{aligned}$$

Since the right-hand side is strictly positive and independent of $n_1,\dots,n_d$, the infimum of the left-hand side over all eigenfunctions is strictly positive. ∎

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