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1. Introduction

Stylized facts from the financial markets include heavy tails, extreme correlations and the leverage effect (Bouchaud and Potters 2000, 2001, Cont 2001). Various studies suggest that the power-law nature of financial returns, $\mathbb{P}\{|r| > x\} \propto x^{-\alpha}$ (at large x) with a quasi-universal exponent α close to 3, explains most extreme events, including crashes (Gabaix *et al*. 2003). All these extreme events in the time series affect the estimation of the tail dependence measure (Davis and Mikosch 2009). In particular, when the market is facing extreme negative returns, the level of correlation is significantly higher than during extreme positive returns (Longin and Solnik 2001, Ang and Bekaert 2002). This pattern can be explained by the asymmetry of the volatility which is partly due to the so-called 'leverage effect'. The leverage effect is characterized by a surge in volatility and a subsequent drop in the stock price (Black 1976, Christie 1982, Campbell and Hentchel 1992, Bekaert and Wu 2000).

The leverage effect puzzle is well documented in the recent literature (Perello and Masoliver 2003, Bollerslev *et al*. 2006, Qiu *et al*. 2006, Cevdet *et al*. 2007, Aït-Sahalia *et al*. 2013). Various approaches have been suggested to include the leverage effect.

1.1. Continuous-time models

The Constant Elasticity of Variance model (CEV) developed by Cox (1975) was the first model that explicitly described the relation between volatility and price. The elasticity stock price exponent allows the instantaneous variance of the percentage price change to be a direct inverse function of the stock price. Harvey and Shephard (1996) developed a stochastic volatility model with leverage effect to be estimated by a quasi-maximum likelihood procedure. Hagan *et al*. (2000) proposed a generalization of the CEV dynamics known as

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the SABR model. Carr and Wu (2009) proposed to model the equity-to-asset ratio by a simple CEV process in which large discontinuous shocks are modeled as a jump process to capture self-exciting behavior. Veraart and Veraart (2009) modeled the leverage effect as a stochastic process by introducing an additional source of randomness into the stochastic volatility model. More recently, Filimonov and Sornette (2011) developed a model describing critical events in self-organized systems such as financial markets. It relies on a self-excited multifractal process with an explicit dependence of the dynamics of the process on both external events and internal memory. It allows for the leverage effect as an intrinsic property.

1.2. *Discrete-time models*

In the ARCH literature, the conditional variance is specified to be a function of the size and the sign of the returns. One can mention the Exponential-GARCH (Nelson 1991), the GJR-GARCH (Glosten *et al.* 1993), the Asymmetric Power-ARCH (Ding *et al.* 1993), Nonlinear-GARCH (Engle and Ng 1993) and the Threshold-GARCH (Zakoïan 1994).

To our knowledge, both the continuous-time and discrete-time approaches remain focused mainly on theoretical issues. Bouchaud *et al.* (2001) were the first to study in detail the dynamics of the leverage effect adapted to stocks and stock indices. In particular, they introduced and measured the 'return–volatility correlation function' for stocks and indices (figure 1). They derived two different dynamics for the systematic and specific risk and found a moderate (strong) correlation with a decay period of 50 (10) days for individual stocks (indices). The correlations for indices are therefore stronger than for stocks, despite the fact that a stock index is merely a portfolio of stocks. They argue that the leverage effect for stocks stems from a simple retarded effect, in which price variations are calibrated not on the instantaneous value of the price but on an exponential moving average of the price. Their retarded volatility model adequately represents the leverage effect for individual stocks, which are mainly characterized by idiosyncratic (or specific) risks. However, they recognize that it is no longer the case for stock indices, which are only characterized by systematic risk, whose leverage is driven by another phenomenon, the so-called 'panic effect'. This panic effect is partly explained by an increase of the correlation between single stocks (Reigneron *et al.* 2011). In fact, a fast decline of the market leads to the panic of investors and to an increase of the systematic risk of each single stock, which may become the dominant factor when the panic is extreme. As emphasized by Allez and Bouchaud (2011), "during large swings of the index, the market exposure of stocks becomes the dominant factor".

Relying on the retarded volatility model of Bouchaud *et al.* (2001), we develop a new volatility model, simple to implement, that (i) conserves the retarded effect property and (ii) adequately extends the leverage effect from the specific single stock case to the systematic case of stock indices, taking into account the panic effect. This model is well suited not only for stock indices but also for single stocks, particularly in times of distress during which stocks are mainly affected by the systematic risk. This advantage enables the model to be as reactive as the implied volatility, i.e. there is no delay between the implied volatility changes and our model: they remain highly correlated. This important property makes it different from most previously published theoretical models.

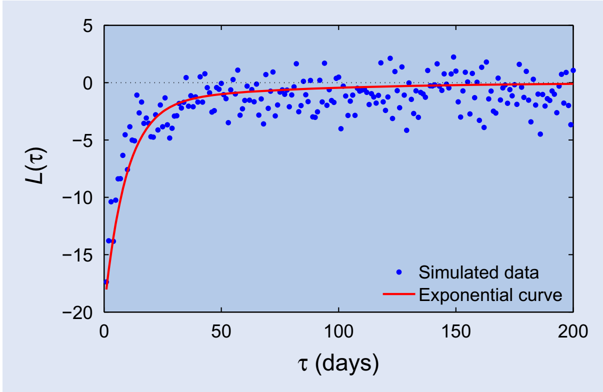

Figure 1. The return–volatility correlation function (leverage effect), $\mathcal{L}(\tau) \equiv \langle[\delta x(t+\tau)]^2\delta x(t)\rangle/\langle[\delta x(t)]^2\rangle^2$, with $\delta x(t) = \Delta I(t)/I(t)$, for simulated data (symbols), and the exponential curve $\mathcal{L}(\tau) \approx -A_I \exp(-\tau/T_I)$, with $A_I = 18$ and $T_I = 9.3$ (days), as introduced and calibrated by Bouchaud *et al.* (2001). The simulated data were obtained from a Monte Carlo generated price series of 10,000 points, with $I(t+1) = I(t)(1+\sigma_I(t)\varepsilon_t)$, in which i.i.d. $\varepsilon_t \sim \mathcal{N}(0,1)$, and $\sigma_I(t)$ is given by equation (7) with a constant $\tilde{\sigma}_I(t) = 0.01$. This figure supports the claim that the reactive volatility model captures the leverage effect, using the correct parameters A_I and T_I calibrated from the empirical stock indices.

The reactivity of the model is first tested against the European volatility index V2X and is also compared with two classical models selected as benchmarks: the GARCH model and the standard volatility estimate based on the square root of the exponential moving average of squared returns. The comparison to the volatility index is chosen because volatility is not directly observable and market participants prefer using implied volatility indices as market volatility proxies. Thus, how well the model captures the dynamics of such a proxy may be an adequate gauge of quality.

The robustness of the model is then tested using extreme events. An empirical study is performed on the 470 most liquid European stocks over the last decade. We investigate extreme systematic and specific risks that could be responsible for massive losses and are therefore important for investors. Our results suggest that market shocks are better assimilated into the reactive volatility model.

The article is organized as follows. Section 2 describes the reactive volatility model. Section 3 analyses the empirical robustness of the model near extreme events. Section 4 summarizes the main findings and provides some concluding remarks.

2. The reactive volatility model

The reactive volatility model takes into account two different dynamic features of the leverage effect: it models the panic effect related to the systematic risk and combines it with the retarded volatility model of Bouchaud *et al.* (2001) that accurately describes the slow dynamics of the leverage effect for the specific risk. We focus first on the stock index case since it allows us to introduce the model of the panic effect that governs the systematic risk. Second, we treat the single stock case that combines both the specific and systematic risks. We show how the retarded volatility model is combined with the panic effect in order to model the leverage effect of the specific and systematic risks concurrently.

2.1. *Model of the panic effect describing the systematic risk: the case of the stock index*

Let $I(t)$ be a stock index at equally spaced, discrete times t. It is well known that arithmetic returns, $\Delta I(t)/I(t)$, are heteroscedastic, partly due to price–volatility correlations. The goal is to define a convenient estimator, a level $L(t)$, of the stock index $I(t)$ such that the re-normalized arithmetic returns, $\Delta I(t)/L(t)$, become more homoscedastic.

Let us introduce two stock index levels as exponential moving averages (EMAs) with two characteristic time scales: a slow level $L_s(t)$ and a fast level $L_f(t)$. These EMAs can be computed using standard linear relations:

$$L_s(t+1) = (1-\lambda_s)L_s(t) + \lambda_s I(t+1), \tag{1}$$

$$L_f(t+1) = (1-\lambda_f)L_f(t) + \lambda_f I(t+1), \tag{2}$$

where λ_s and λ_f are the weighting parameters of the EMAs. The appropriate values of $\lambda_s = 0.0241$ and $\lambda_f = 0.1484$ are extracted from the measurement of the return–volatility correlation function of Bouchaud *et al.* (2001) (see the caption to figure 1 for details). The slow parameter corresponds to the relaxation time of the retarded effect for the specific risk, whereas the fast parameter corresponds to the relaxation time of the panic effect for the systematic risk. These two relaxing times are rather universal as they are stable in time and remain close to each other for different mature stock markets.

For practical purposes, a filter is introduced to make the estimator more robust against outliers or extreme instantaneous variations of the stock index. We set

$$\hat{L}_s(t+1) = I(t+1)\left(1 + F_\phi\left(\frac{L_s(t+1) - I(t+1)}{I(t+1)}\right)\right), \tag{3}$$

where the filter function $F_\phi(z)$ is proportional to z for small z and saturated to a constant for large $|z|$. We expect that the leverage effect is linear up to a certain point. We choose

$$F_\phi(z) = \frac{\tanh(\phi z)}{\phi},$$

where ϕ is a parameter that determines the region of linearity of the filter, i.e. a smaller ϕ corresponds to a wider linearity region.† At $\phi = 0$, there is no filter, $F_0(z) = z$ and $\hat{L}_s(t+1) = L_s(t+1)$. This filter has only a very minor impact on the results and is useful in practice to level off a couple of very extreme events. One could use any other S-shaped function and it would be expected to give similar results. Finally, the main equation for the stock index level $L(t)$, in which the fast level is modulated by the filtered slow level, is defined as

$$L(t+1) = \hat{L}_s(t+1)\left(1 + F_\phi\left(\left(\frac{L_f(t+1)}{I(t+1)}\right)^{\ell} - 1\right)\right), \tag{4}$$

where ℓ is the leverage parameter that describes the amplitude of the leverage effect between stock returns and volatility.

The leverage parameter ℓ is set to 8 to reproduce the double-exponential fit of the return–volatility correlation function in the stock index case (see the caption to figure 1 for details). For instance, the value of $\ell = 8$ means that if the index varies by 1%, the volatility is expected to vary by $-\ell \cdot 1\% = -8\%$. This can be seen using a Taylor expansion: if there is no filter ($\phi = 0$) and $L_f(t+1)$ is close to $I(t+1)$, a Taylor expansion of equation (4) yields a simpler form:

$$L(t+1) \simeq \hat{L}_s(t+1)\left(1 + \ell\frac{L_f(t+1) - I(t+1)}{L_f(t+1)}\right). \tag{5}$$

The Taylor expansion shows that the panic effect can be modeled in a simple way with two exponential moving averages, whereas the retarded volatility model needs only one. The level $L(t)$ is the slow moving average of the retarded volatility model which is modulated by the relative excess of the systematic risk $I(t+1)$ compared with its fast moving average $L_f(t+1)$. This modulation is amplified by the ℓ parameter, which describes the 'capacity' of the market to panic: $\ell = 0$ corresponds to a market that never panics. It seems that the ℓ parameter is also universal as it is stable over time and is approximately the same for different mature stock markets.

The introduction of the leverage effect into equation (4) allows one to use the 'corrected' level $L(t)$ instead of $I(t)$, which leads to more homoscedastic stock index returns $\Delta I(t)/L(t)$ than $\Delta I(t)/I(t)$, as tested below.

An estimator of the re-normalized variance $\tilde{\sigma}_I^2$ is obtained through an EMA based on $L(t)$:

$$\tilde{\sigma}_I^2(t+1) = (1-\lambda_\sigma)\tilde{\sigma}_I^2(t) + \lambda_\sigma\left(\frac{\Delta I(t+1)}{L(t+1)}\right)^2, \tag{6}$$

where λ_σ is a weighting parameter. λ_σ has to be chosen as a compromise between the estimation accuracy of the standard deviation of the re-normalized returns and the reactivity of that estimation. Indeed, the re-normalized returns are rather homoscedastic by construction in the short time scale only (in fact, the re-normalization based on the leverage effect with short relaxation times (λ_s, λ_f) cannot account for long time scale changes of volatility-related due to economic cycles). Since economic uncertainty does not change significantly in a period of 2 months (40 trading days), we set λ_σ to $1/40 = 0.025$. The related sample length leads to a noise of 4% in the annualized volatility, which is on the order of 20%.

Figure 2 shows that $\tilde{\sigma}_I^2$ yields a stable output on the short time scale since the variation of the index $\Delta I(t+1)$ normalized by the index level $L(t+1)$ is rather homoscedastic by construction. The reactive volatility model is then obtained through this stable output but using a reactive re-normalization factor. The reactivity of the model therefore comes from the re-normalization factor $L(t+1)/I(t+1)$ that will adjust to every price move in an instantaneous way. The reactive volatility model $\sigma_I(t)$ for the systematic risk which is governed by the panic effect (the stock index case) is defined as

†We set $\phi = 1/0.3 \approx 3.3$, which corresponds to a maximum stock index daily variation of $\pm 30\%$, or a maximum drawdown on the order of 30% over $1/\lambda_s \approx 40$ days. For example, during the worst American stock market crash on 19 October 1987, the S&P 500 declined by 22% while the VIX climbed to 150%.

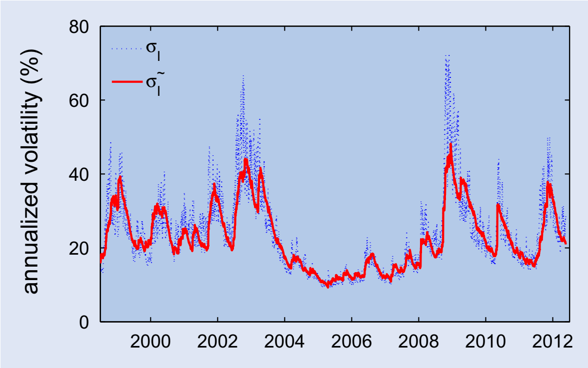

Figure 2. The measures of index volatility, $\sigma_I(t)$, from the reactive volatility model (dashed blue line) and the re-normalized volatility $\tilde{\sigma}_I(t)$ (solid red line), for the period from 22/06/1998 to 18/05/2012. The re-normalized volatility is less volatile in the short term, but is able to capture all the long-term moves.

$$\sigma_I(t+1) = \tilde{\sigma}_I(t+1)\frac{L(t+1)}{I(t+1)}. \tag{7}$$

Using the Taylor expansion (5), we show that the normalization factor depends mainly on the slow and fast EMAs of the stock index price. The reactive volatility is therefore obtained from the following approximation:

$$\sigma_I(t+1) \approx \tilde{\sigma}_I(t+1)\frac{L_s(t+1)}{I(t+1)}\left(1+\ell\frac{L_f(t+1)-I(t+1)}{L_f(t+1)}\right). \tag{8}$$

2.2. *Volatility term structure and the empirical test against the V2X index*

We test the reactivity of equation (7) against the V2X implied volatility. Because the V2X index represents the implied volatilities of the Eurostoxx 50 index with a maturity T of one month, one also needs to consider the volatility term structure of equation (7), which is defined as

$$\sigma_T^2(t) = \frac{1}{T}\left\langle\left(\int_t^{T+t}\sigma_I^2(t')\,\mathrm{d}t'\right)\right\rangle, \tag{9}$$

where $\langle\ldots\rangle$ is the expectation (average) over all possible trajectories of $\sigma_I(t')$ between t and $t+T$.

In order to approximate $\sigma_T^2(t)$, one could have implemented a numerical solution based on Monte Carlo simulation (as described in the caption to figure 1). An alternative is to introduce an empirical recipe based on a two-factor model with the following two 'long-term' volatilities, $\sigma_{Is}(t)$ (the slow factor) and $\sigma_{If}(t)$ (the fast factor). In this approximation, we assume the instantaneous volatility $\sigma_I(t)$ to mean-revert towards the fast 'long-term' volatility $\sigma_{If}(t)$, which, in turn, mean-reverts towards the slow 'long-term' volatility $\sigma_{Is}(t)$. The fast and slow 'long-term' volatilities are defined as

$$\sigma_{Is}(t) = \sigma_I(t)\frac{I(t)}{L_s(t)}, \qquad \sigma_{If}(t) = \sigma_I(t)\frac{I(t)}{L_f(t)}. \tag{10}$$

We split the squared volatility $\sigma_I^2(t)$ into three 'components':

$$\sigma_I^2(t) = \left(\sigma_I^2(t)-\sigma_{If}^2(t)\right)+\left(\sigma_{If}^2(t)-\sigma_{Is}^2(t)\right)+\sigma_{Is}^2(t). \tag{11}$$

The empirical observation from figure 3 confirms, without pretending to any rigor, that $\left(\sigma_I^2(t)-\sigma_{If}^2(t)\right)$ and $\left(\sigma_{If}^2(t)-\sigma_{Is}^2(t)\right)$ can be seen as two mean-reverting processes with two relaxation rates λ_s and λ_f, while $\sigma_{Is}^2(t)$ varies much slower than the other processes. We therefore assume that $\left(\sigma_I^2(t)-\sigma_{If}^2(t)\right)$ and $\left(\sigma_{If}^2(t)-\sigma_{Is}^2(t)\right)$ can be approximated by Ornstein–Uhlenbeck processes with two relaxation rates λ_s and λ_f, while $\sigma_{Is}^2(t)$ follows Brownian motion. These last assumptions (current in the literature, e.g. Hull and White (1987) and Heston (1993)) enable the reactive volatility estimator with the term structure σ_T to be estimated as

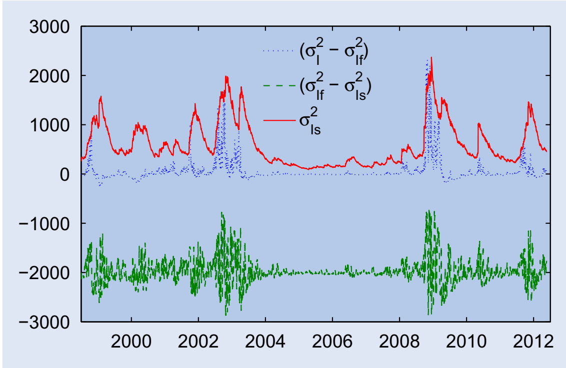

Figure 3. Empirical behavior of three 'components' in equation (11): $\left(\sigma_I^2(t)-\sigma_{If}^2(t)\right)$ (dotted blue line), $\left(\sigma_{If}^2(t)-\sigma_{Is}^2(t)\right)$ (dashed green line, which is shifted by -2000 along the vertical axis), and $\sigma_{Is}^2(t)$ (solid red line). The first two components can be seen as mean-reverting processes with different relaxation rates, while the last varies much slower than the others.

$$\sigma_T^2(t) \approx \left(\sigma_I^2(t)-\sigma_{If}^2(t)\right)\frac{(1-\mathrm{e}^{-\lambda_f T})}{\lambda_f T} + \left(\sigma_{If}^2(t)-\sigma_{Is}^2(t)\right)\frac{(1-\mathrm{e}^{-\lambda_s T})}{\lambda_s T} + \sigma_{Is}^2(t). \tag{12}$$

This relation can alternatively be seen as an empirical definition of $\sigma_T(t)$ or as a practical recipe for its computation.

Figure 4 compares the implied volatility V2X for Eurostoxx 50 with four different volatility estimators: (i) a standard estimator with an exponential moving average of squared returns:

$$\sigma_{I,\mathrm{SD}}^2(t+1) = (1-\lambda_\sigma)\sigma_{I,\mathrm{SD}}^2(t) + \lambda_\sigma\left(\frac{\Delta I(t+1)}{I(t)}\right)^2, \tag{13}$$

with the same value of the weighted parameter $\lambda_\sigma = 1/40$; (ii) a GARCH estimator, $\sigma_{\mathrm{GARCH}}(t)$, which is often considered to be the gold standard; (iii) the reactive volatility estimator $\sigma_I(t)$ from equation (7) without a term structure; and (iv) the reactive volatility estimator $\sigma_T(t)$ from equation (12) with a term structure. One can see that both the standard and GARCH estimators have much lower correlations with V2X than both reactive estimators. Additionally, the term structure model for the reactive volatility estimator, as expected, improves the slope of the linear regression. Indeed, the slope becomes much closer to 1. The R^2 from the linear regression is relatively high (around 0.45). We can therefore consider the model to be nearly as reactive as the volatility index.

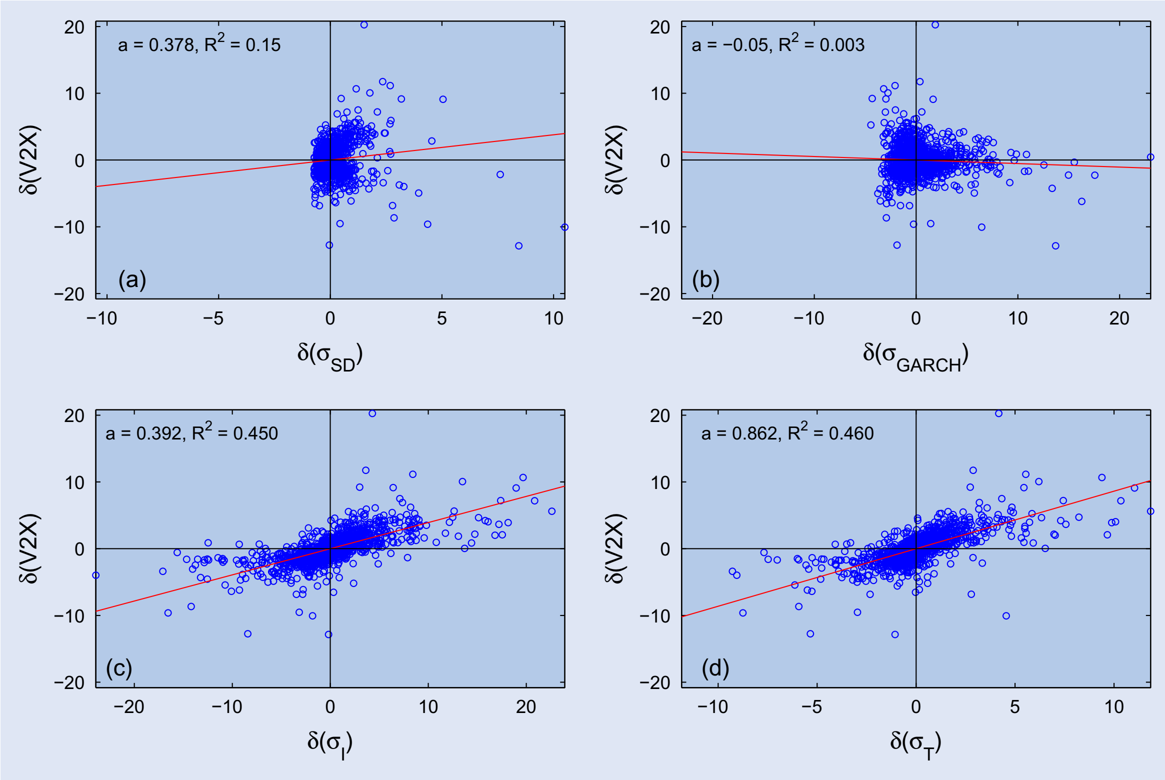

Figure 4. Correlations between increments $\delta(\mathrm{V2X})$ of the Eurostoxx 50 implied volatility index V2X (vertical axis) and increments of four different volatility estimators (horizontal axis): (a) a standard estimator based on the exponential moving average of the squared returns ($\lambda_\sigma = 1/40$); (b) a standard GARCH estimator ($\omega = 0.0000014$, $\alpha = 0.1064523$, $\beta = 0.8966662$); (c) the reactive volatility estimator without a term structure; and (d) the reactive volatility estimator with a term structure. First, the increments in volatility for both the standard volatility and GARCH estimators are much more strongly skewed towards positive values. Second, the correlations between the V2X and both reactive volatility estimators are much higher. Finally, accounting for the term volatility structure increases the slope of the linear regression from 0.392 to 0.862, with the latter value being close to 1.

2.3. Model for combining panic and retarded effects: the case of a single stock

For a single stock, the reactive volatility model relies on an equation similar to equation (7) used for the stock index:

$$\sigma_i(t+1) = \tilde{\sigma}_i(t+1)\frac{L_i(t+1)}{P_i(t+1)}, \tag{14}$$

with $L_i(t+1)$ and $\tilde{\sigma}_i(t+1)$ obtained through equations similar to equations (4) and (6), respectively. The only difference comes from $\hat{L}_\mathrm{s}(t+1)$ in equation (4), which now applies to the single stock price $P_i(t)$ instead of the index price $I(t)$. Using the Taylor expansion (5), we obtain

$$\sigma_i(t+1) \approx \tilde{\sigma}_i(t+1)\frac{L_{i,\mathrm{s}}(t+1)}{P_i(t+1)}\left(1 + \ell\frac{L_\mathrm{f}(t+1) - I(t+1)}{L_\mathrm{f}(t+1)}\right). \tag{15}$$

This formula combines the slow EMA of the single stock price, on one hand, and the fast EMA and the current price of the stock index, on the other. As a consequence, it adequately extends the leverage effect from the specific risk case (already accounted for in the retarded volatility model) to the systematic case, which captures the panic effect. In what follows, the reactive volatility estimator $\sigma_i(t)$ from equation (14) is tested and used to identify precursors and replicas around extreme events.

Finally, although the formula used to determine the level $L_i(t)$ for the single stock case looks similar to that used to determine the level $L(t)$ of the stock index, the leverage effect is different from a practical point of view. Indeed, the leverage effect for the stock index depends only on the historical stock index price and is dominated by the fast panic effect; in turn, the leverage effect for a single stock depends on both the single stock price and the stock index price, and it is dominated by the slow retarded effect. However, as soon as the systematic risk becomes dominant as compared with the specific risk (that happens when the market is highly stressed), the leverage effect in the single stock also starts to be dominated by the panic effect.

Our approach manages to combine the panic effect model introduced in section 2.1 with the retarded volatility model of Bouchaud *et al.* (2001). This combination allows us to model the leverage effect for both the systematic and specific risks that occur in the single stock case. In that case, the level $L(t)$ is that of the specific risk obtained with the retarded volatility model modulated by the panic effect in equation (5). The panic effect is correlated to the relative excess of the systematic risk as compared with the fast moving average.

The model manages to adjust its estimation of the volatility in an instantaneous way at each single stock or index price variation. Moreover, the adjustment is highly correlated to variations of the volatility index. One can expect that the model will manage to capture, in a reactive way, not only the risk of the stock index, but also the systematic risk of any stock.

3. Empirical test of the reactive volatility model around extreme events

3.1. Data analysis

The database consists of daily price series for the 470 most liquid European stocks from January 1, 2000 to April 4, 2012. Let $P_i(t)$ denote the closing price of the ith stock on day t. From each price series, an array of arithmetic returns, $R_i(t) = [P_i(t) - P_i(t-1)]/[P_i(t-1)]$, is constructed. An extreme event is said to occur when the absolute arithmetic return $|R_i(t)|$ is three times larger than the reactive volatility estimator $\sigma_i(t-1)$:

$$|R_i(t)| > 3\sigma_i(t-1). \tag{16}$$

For each stock, we search for extreme events in the array $\{R_i(t)\}$. At each occurrence of an extreme event, we record the subsequence $\{r(-\Delta), r(-\Delta+1), \ldots, r(0), r(1), \ldots, r(\Delta)\}$ of normalized returns before and after the extreme event (at day t),

$$r(k) = \frac{R_i(t+k)}{\sigma_i(t+k-1)}, \quad k = -\Delta, \ldots, \Delta, \tag{17}$$

where Δ is a fixed subsequence length with $\Delta = 9$ trading days.† The subsequence $\{r(k)\}$ characterizes the behavior of a stock before and after an extreme event, which is identified by the magnitude of $r(0)$ (we drop the index i because the extreme events will be analysed for all stocks together). Repeating this procedure for each stock generates a database of 10,213 extreme events. Note that if two (or more) extreme events occur within Δ days, only the first is retained, while any later events are ignored. It is worth emphasizing that the above definition of an extreme event is purely conventional, as is the choice for the threshold $3\sigma_i(t-1)$. Note also that if stock returns were Gaussian, the number of extreme events would be much smaller than what we observe because the probability of a Gaussian return larger than 3σ is 0.0027.

A closer look at the statistical properties of stock returns around extreme events requires distinguishing the systematic risk from the specific risk. The systematic risk mainly affects the stock indices (or even every stock during stress conditions), while the specific risk mainly affect single stocks. More precisely, the database records are split into four groups that are denoted 'systematic positive' (SyP), 'systematic negative' (SyN), 'specific positive' (SpP) and 'specific negative' (SpN). The division into positive and negative groups is determined by the sign of the extreme normalized return $r(0)$. The division into systematic and specific groups is decided by the condition on the Eurostoxx index $I(t)$ at day t of an extreme return: if $|\Delta I(t)|$ exceeds 3%, the extreme event is categorized as systematic, otherwise as specific. Both systematic groups (SyP and SyN) contain the records of extreme returns from individual stocks that are affected by large stock index returns, representing extreme events for the entire market. The specific groups (SpS and SpN) contain the records of extreme returns that are specific to individual stocks. The database of extreme events contains 903 systematic positive records, 1046 systematic negative records, 5135 specific positive records and 3129 specific negative records.

To qualitatively separate the possible sources of deviation (accuracy of the reactive volatility estimator and return correlations), a second database of 'non-extreme' events is constructed with the same structure, in which dates t are chosen randomly (without the selective condition (16)). In this case, no strong correlations between successive returns are expected, and the reactive volatility estimator is expected to accurately capture the fluctuations of the stock price.

3.2. Empirical results

The behavior of the reactive volatility model around extreme events is characterized by the following function:

$$q_k = \sqrt{\langle r^2(k)\rangle} - 1, \quad k = -\Delta, \ldots, \Delta, \tag{18}$$

where the arithmetic average $\langle \ldots \rangle$ is taken over all records in the chosen group (SyP, SyN, SpP, SpN). For the idealized case in which the returns are uncorrelated and the reactive volatility estimator $\sigma_i(t)$ is exact, the average $\langle r^2(k)\rangle$ of normalized returns should be equal to 1, so that q_k would be 0, except for $k = 0$. In other words, the smallness of the deviations of q_k from 0 characterizes the accuracy of the volatility estimator.

Table 1 summarizes the average characteristics obtained for both estimators. The level of the deviations q_k is averaged over 9 days before and after an extreme event, as illustrated in figures 5 and 6. For all groups of extreme events, this level is significantly higher for the standard volatility estimator than for the reactive volatility model because the latter captures the panic effect for the systematic groups and the retarded effect for the specific groups. One can conclude that the reactive volatility model is more robust near extreme events, and its recovery after a shock is faster.

Figure 5 shows q_k for the four groups. Full circles represent q_k from the database of extreme events, while the solid line is used as a reference level from the database of random events (extreme or not). The solid line is close to 0, which indicates that the reactive volatility model is accurate for ordinary days (without extreme events). For comparison, figure 6 shows the quantities q_k computed using equation (18) by replacing $\sigma_i(t)$ in equation (17) by an empirical measure of volatility, $\sigma_{i,\mathrm{SD}}(t)$, which is obtained from a standard volatility estimator with an exponential moving average:

$$\sigma^2_{i,\mathrm{SD}}(t+1) = (1-\lambda_\sigma)\sigma^2_{i,\mathrm{SD}}(t) + \lambda_\sigma[R_i(t+1)]^2, \tag{19}$$

with the same value of the weighting parameter $\lambda_\sigma = 1/40$.

Let us now take a closer look at the results of figure 5. For systematic groups, there are significant deviations of q_k from 0 before and after an extreme event. In other words, the normalized returns before and after an extreme event are significantly larger than typical normalized returns. Prior to an extreme event, the normalized returns increase very slowly and become significantly larger than typical normalized return 30 days before the event. The relaxation time after the extreme event is, on the contrary, much faster.

In the systematic positive group, the observation of relatively strong precursors may be partly explained by political

†The length is fixed to nine trading days after having considered up to ±100 days; empirically, after 9 days, the marginal gain in precision can be neglected.

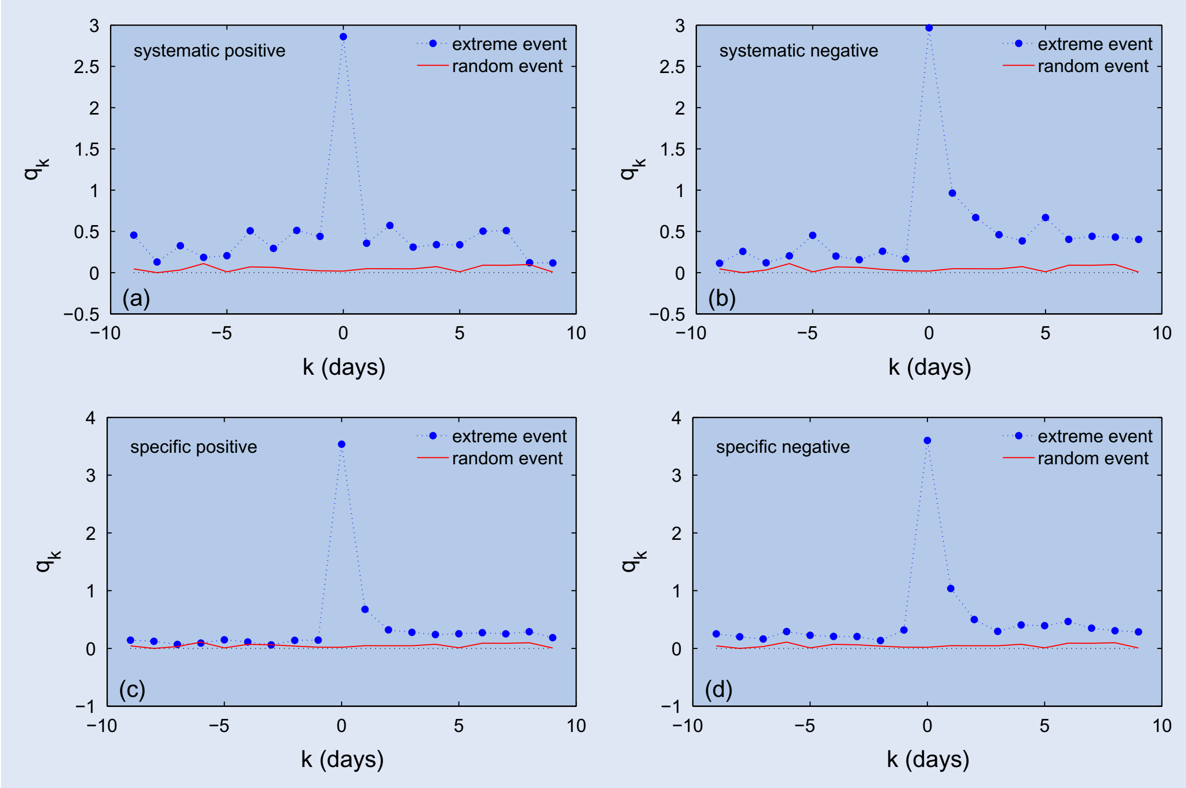

Figure 5. Distribution of q_k around an extreme event for the reactive volatility model. Extreme returns are split into four groups: (a) SyP, (b) SyN, (c) SpP and (d) SpP. A stock return is termed extreme when it exceeds threefold the empirical volatility $\sigma_i(t-1)$ from the reactive volatility model. The selected interval is ± 9 days.

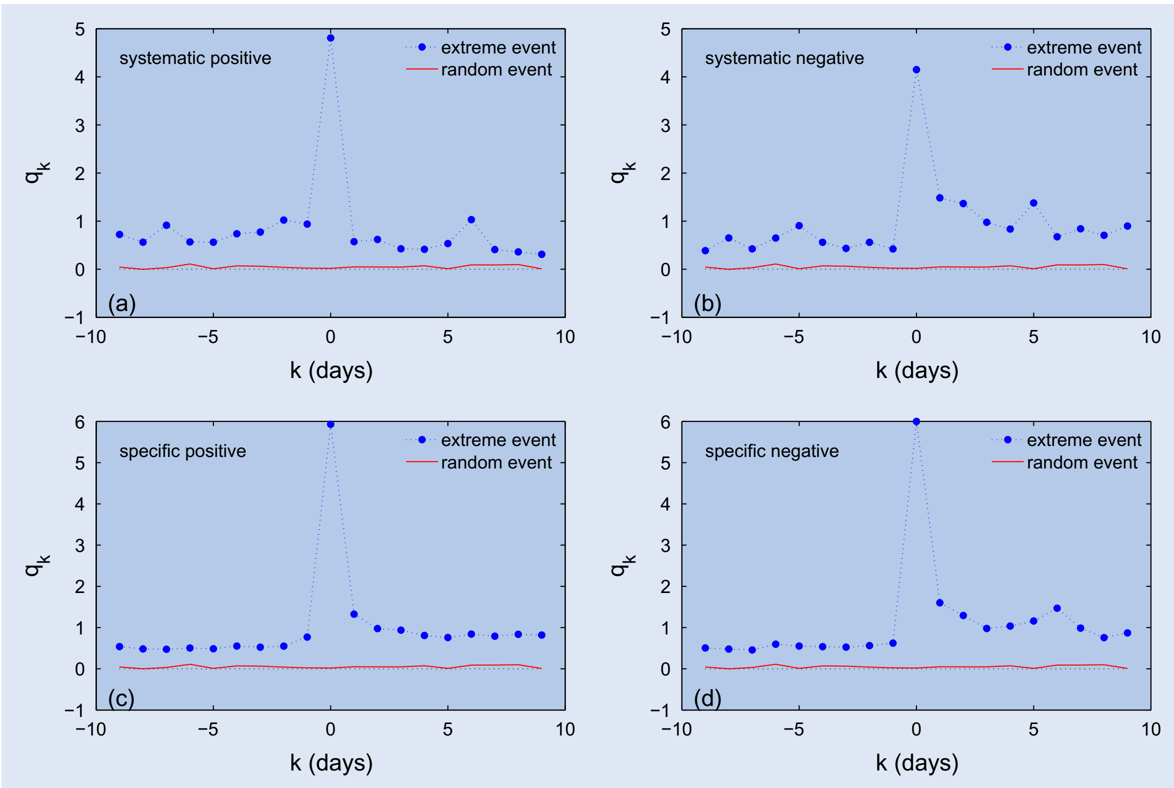

Figure 6. Distribution of q_k around an extreme event for a standard volatility estimator. Extreme returns are split into four groups: (a) SyP, (b) SyN, (c) SpP and (d) SpN. A stock return is termed extreme when it exceeds threefold the empirical volatility $\sigma_{i,\mathrm{SD}}(t-1)$ from a standard volatility estimator. The selected interval is ± 9 days.

Table 1. The level of the deviations q_k is averaged over 9 days before and after an extreme event. For all groups of extreme events, this level is significantly higher for a standard volatility estimator than for the reactive volatility model because the latter captures the panic effect for the systematic groups and the retarded effect for the specific groups. Note also that the recovery after an extreme event takes, on average, 2.95 days for the reactive volatility model, as opposed to 5.29 days for a standard estimator (both times are estimated from an exponential fit of q_k after an extreme event; see figure 5). As a consequence, the reactive volatility model is more robust in response to extreme events.

Group	Before		After	
	Standard	Reactive	Standard	Reactive
Systematic positive	0.76	0.34	0.52	0.35
Systematic negative	0.55	0.21	1.02	0.54
Specific positive	0.54	0.11	0.90	0.31
Specific negative	0.54	0.22	1.13	0.45

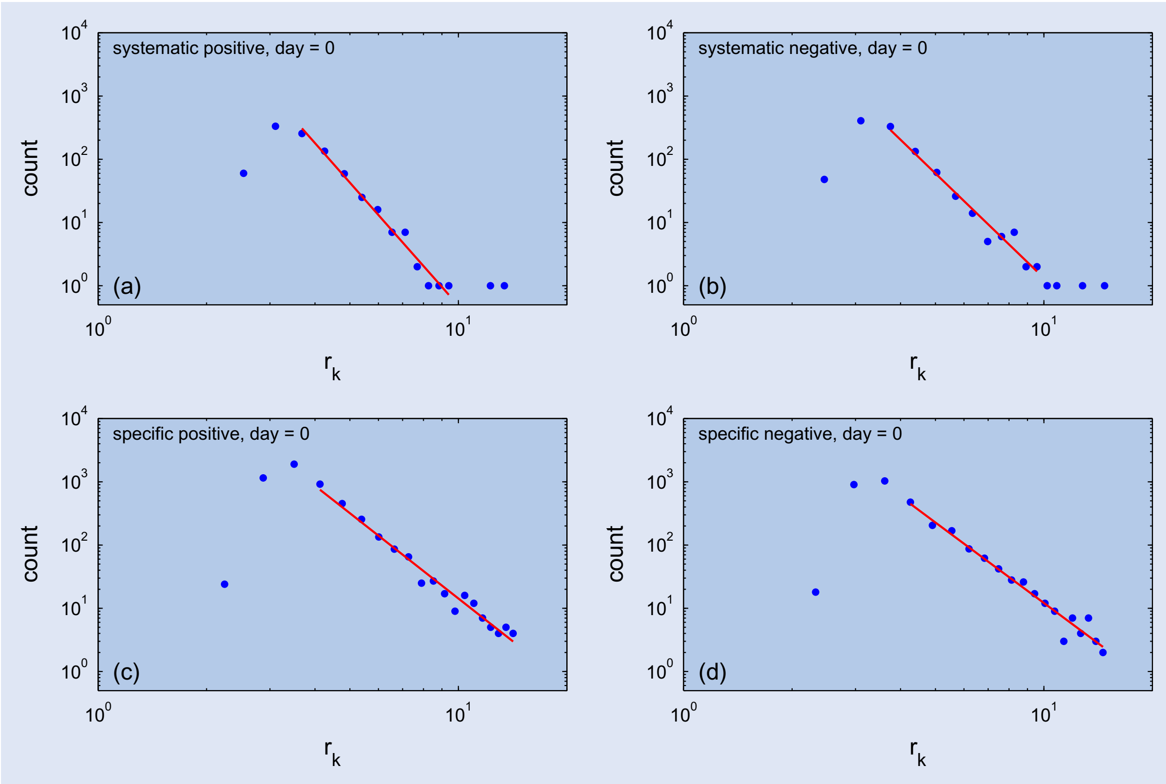

Figure 7. Asymptotic behavior of the probability density $p(z)$ of extreme returns normalized with the reactive volatility model for the four groups: (a) SyP, (b) SyN, (c) SpP and (d) SpN. In all cases, power-law decays $p(z) \propto z^{-1-\alpha}$ are observed, with exponents 5.5, 4.5, 3.5 and 3.2 for the systematic positive, systematic negative, specific positive and specific negative groups, respectively. For the systematic groups, the exponent is greater than 4, indicating the existence of kurtosis, while for the specific groups, the exponent is less than 4.

or monetary decisions made in reaction to earlier market instabilities.

For the systematic negative group, the observation of excitement before an extreme event is less intuitive. These variations could come from the possibility that some investors anticipate the release of very bad economic news earlier than others. After an extreme event, large re-normalized returns are naturally expected as the market relaxes after this event. This means that investors should be worried about replicas after the initial extreme event and that most models consistently underestimate risk.

Figures 5(c) and 5(d) for both specific groups indicate less significant deviations of q_k from 0 before an extreme event.

In the specific positive group, the observation of an extreme positive return can be caused by the announcement of corporate decisions (mergers or acquisitions). Because these corporate decisions remain strictly confidential and are difficult to anticipate, one expects to observe very weak precursors and small values of q_k for negative k. After the day of the announcement, the stock returns may experience a rapid relaxation (one or two days) towards the normal level of small q_k for positive k.

For the specific negative group, the observation of an extreme event can be caused by the announcement of corporate decisions related to bad economic results for the company (profit warning, bankruptcy, or downgrade). This kind of news is partly anticipated by the market through rumors. One can therefore identify stronger precursors of extreme events. Because the situation of the company remains tenuous, stronger replicas are also identified.

The mechanisms of these precursors in every case seem plausible, especially since the volatility forecasts are not particularly low when these relatively large events occur. Indeed, we compared the whole sample and the conditional sample defined by the days prior to any extreme event until 9 days. The average of the volatilities forecasted by our volatility model is 31% in the conditional sample and remains close to that estimated for the whole sample (33%). We also find similar results for the standard deviations of returns. We manage to identify, *statistically*, the presence of precursors of extreme events in each case, but the precursors remain too weak to be detected one by one to forecast the occurrence of an extreme event. Indeed, the predictive power is low and the model cannot be used to forecast any crash or extreme event. For example, if a warning signal emerges each time the realized volatility exceeds twice the level predicted by the reactive volatility model for a period of 9 days, 96.68% of the extreme events will be missed, while 56.8% will be false warning signals.

Similar plots in figure 6 for a standard volatility estimator display a less reactive measure around extreme events.

Figure 7 shows the asymptotic behavior of the probability density $p(z)$ of extreme normalized returns for the four groups. In all cases, power-law decays $z^{-1-\alpha}$ are observed, with the exponent α equal to 5.5, 4.5, 3.5 and 3.2 for the SyP, SyN, SpP and SpN groups, respectively. For the systematic groups, the exponent is greater than 4, indicating the existence of kurtosis. The re-normalization of returns with the reactive volatility model, instead of using the standard EMA volatility estimate, managed to increase the exponent from 3 to 5, which means that this model is able to capture most of the extreme events. For the specific groups, the exponent remains around 3. Most of these extreme risks come from very specific news (for example, a takeover offer), and most of the time, when the price jumps, the volatility does not change.

4. Conclusion

We have developed a new volatility model, easy to implement, that includes a leverage effect whose return–volatility correlation function fits empirical observations. In addition, the model is able to capture both the panic effect induced by the systematic risk and the retarded effect induced by the specific risk. The model is shown to be as reactive as the implied volatility, which is an improvement over other models. To test the robustness of the reactive volatility model near extreme events, an empirical study is performed on the 470 most liquid European stocks from January 1, 2000 to April 4, 2012. The reactive volatility model is used to re-normalize daily returns, from which extreme events are identified and split into four groups: systematic positive, systematic negative, specific positive and specific negative. Our results suggest that market shocks are better assimilated into the reactive volatility model. Moreover, the model identifies, statistically, the presence of precursors and replicas. The model captures much of the extreme systematic risk and a significant part of the extreme specific risk. Future research will include an application of the reactive volatility model to estimate the market beta, the aggregation of risk and the VaR of a Long/Short portfolio.

References

Aït-Sahalia, Y., Fan, J. and Li, Y., The leverage effect puzzle: Disentangling sources of bias at high frequency. Working paper, Princeton University, 2013.

Allez, R. and Bouchaud, J.P., Individual and collective stock dynamics: Intra-day seasonalities. *New J. Phys.*, 2011, **13**, 025010.

Ang, A. and Bekaert, G., International asset allocation with regime shifts. *Rev. Financ. Stud.*, 2002, **11**, 1137–1187.

Bekaert, G. and Wu, G., Asymmetric volatility and risk in equity markets. *Rev. Financ. Stud.*, 2000, **13**, 1–42.

Black, F., Studies in stock price volatility changes. In *Proceedings of the Business and Economic Statistics Section*, pp. 177–181, 1976 (American Statistical Association).

Bollerslev, T., Litvinova, J. and Tauchen, G., Leverage and volatility feedback effects in high-frequency data. *J. Financ. Econometr.*, 2006, **4**, 353–384.

Bouchaud, J.-P. and Potters M., *Theory of Finantial Risks: From Statistical Physics to Risk Management*, 2000 (Cambridge University Press: Cambridge).

Bouchaud, J.-P. and Potters, M., More stylized facts of financial markets: Leverage effect and downside correlations. *Physica A*, 2001, **299**, 60–70.

Bouchaud, J.-P., Matacz, A. and Potters, M., Leverage effect in financial markets: The retarded volatility model. *Phys. Rev. Lett.*, 2001, **87**, 1–4.

Campbell, J.Y. and Hentchel, L., No news is good news: An asymmetric model of changing volatility in stock returns. *J. Financ. Econ.*, 1992, **31**, 281–318.

Carr, P. and Wu, L., Leverage effect, volatility feedback, and self-exciting market disruptions. Working paper, New York University, 2009.

Cevdet, A., Gallmeyer, M. and Hollifield, B., Financial leverage and the leverage effect - A market and firm analysis. Working paper, Carnegie Mellon University, 2007.

Christie, A., The stochastic behavior of common stock variances - Value, leverage, and interest rate effects. *J. Financ. Econ. Theory*, 1982, **10**, 407–432.

Cont, R., Empirical properties of asset returns: Stylized facts and statistical issues. *Quant. Finance*, 2001, **1**, 223–236.

Cox, J., Notes on option pricing I: Constant elasticity of diffusions. Working paper, Stanford University, 1975.

Davis, R. and Mikosch, T., Extremogram: A correlogram for extreme events. *Bernoulli*, 2009, **15**, 977–1009.

Ding, Z., Granger, C. and Engle, R., A long memory property of stock market returns and a new model. *J. Empir. Finance*, 1993, **1**, 83–106.

Engle, R. and Ng, V., Measuring and testing the impact of news in volatility. *J. Finance*, 1993, **43**, 1749–1778.

Filimonov, V. and Sornette, D., Self-excited multifractal dynamics. *Europhys. Lett.*, 2011, **94**, 46003.

Gabaix, X., Gopikrishnan, P., Plerou, V. and Stanley, E.H., A theory of power law distributions in financial market fluctuations. *Nature*, 2003, **423**, 267–270.

Glosten, L.R., Jagannanthan, R. and Runkle, D., Relationship between the expected value and the volatility of the nominal excess return on stocks. *J. Finance*, 1993, **48**, 1779–1802.

Hagan, P., Kumar, D., Lesniewski, A.S. and Woodward, D.E., Managing smile risk. Bear Stearns, Working paper, 2000.

Harvey, A.C. and Shephard, N., The estimation of an asymmetric stochastic volatility model for asset returns. *J. Business Econ. Statist.*, 1996, **14**, 429–434.

Heston, S., A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.*, 1993, **6**, 327–343.

Hull, J. and White, A., The pricing of options on assets with stochastic volatilities. *J. Finance*, 1987, **42**, 281–300.

Longin, F. and Solnik, B., Extreme correlation of international equity markets. *J. Finance*, 2001, **56**, 649–676.

Nelson, D., Conditional heteroskedasticity in asset pricing: A new approach. *Econometrica*, 1991, **59**, 347–370.

Perello, J. and Masoliver, J., Random diffusion and leverage effect in financial markets. *Phys. Rev. E*, 2003, **67**, 037102.
Qiu, T., Zheng, B., Ren, F. and Trimper, S., Return-volatility correlation in financial dynamics. *Phys. Rev. E*, 2006, **73**, 0651031(4).
Reigneron, P.A., Allez, R. and Bouchaud, J-P., Principal regression analysis and the index leverage effect. *Physica A*, 2011, **390**, 3026–3035.
Shen, J. and Zheng, B., On return-volatility correlation in financial dynamics. *Europhys. Lett.*, 2009, **88**, 28003.
Veraart, A.E.D. and Veraart, A.M., Stochastic volatility and stochastic leverage. Working paper, Aarhus University, 2009.
Zakoïan, J.M., Threshold heteroskedastic models. *J. Econ. Dynam. Control*, 1994, **18**, 931–955.