

On the Exponential Decay of Laplacian Eigenfunctions in Planar Domains with Branches

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ABSTRACT. We consider the eigenvalue problem for the Laplace operator in a planar domain which can be decomposed into a bounded domain of arbitrary shape and elongated “branches” of variable cross-sectional profiles. When the eigenvalue is smaller than a prescribed threshold, the corresponding eigenfunction decays exponentially along each branch. We prove this behavior for Robin boundary condition and illustrate some related results by numerically computed eigenfunctions.

1. Introduction

The geometrical structure of Laplacian eigenfunctions has been thoroughly investigated (see the review [1] and references therein). When a domain can be seen as a union of two (or many) subdomains with narrow connections, some low-frequency eigenfunctions can be found localized (or trapped) in one subdomain and of small amplitude in other subdomains. Qualitatively, an eigenfunction cannot “squeeze” through a narrow connection when its typical wavelength is larger than the connection width. This qualitative picture has found many rigorous formulations for dumbbell shapes and classical and quantum waveguides [2–12]. Numerical and experimental evidences for localization in irregularly-shaped domains were also reported [13–21].

In a recent paper, we considered the Laplacian eigenvalue problem

$$(1.1) \quad \begin{aligned} \Delta u + \lambda u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for a large class of domains Ω in \mathbb{R}^d ($d = 2, 3, \dots$) which can be decomposed in a “basic” bounded domain V and a branch Q of a variable cross-sectional profile [10]. We proved that if the eigenvalue λ is smaller than the smallest eigenvalue μ among all cross-sections of the branch, then the associated eigenfunction u exponentially decays along that branch:

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3, \dots$) be a bounded domain with a piecewise smooth boundary $\partial\Omega$ and let $Q(z) = \Omega \cap \{x \in \mathbb{R}^d : x_1 = z\}$ be the cross-section of Ω at $x_1 = z \in \mathbb{R}$ by a hyperplane perpendicular to the coordinate axis x_1 (Fig. 1).*

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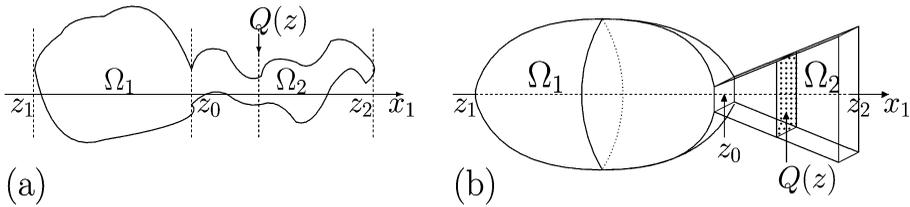


FIGURE 1. Two examples of a bounded domain $\Omega = \Omega_1 \cup \Omega_2$ with a branch Ω_2 of variable cross-sectional profile $Q(z)$. When the eigenvalue λ is smaller than the threshold μ , the associated eigenfunction exponentially decays in the branch Ω_2 and is thus mainly localized in Ω_1 . Note that the branch itself may even be increasing.

Let

$$z_1 = \inf\{z \in \mathbb{R} : Q(z) \neq \emptyset\}, \quad z_2 = \sup\{z \in \mathbb{R} : Q(z) \neq \emptyset\},$$

and we fix some z_0 such that $z_1 < z_0 < z_2$. Let $\mu(z)$ be the first eigenvalue of the Laplace operator in $Q(z)$, with Dirichlet boundary condition on $\partial Q(z)$, and $\mu = \inf_{z \in (z_0, z_2)} \mu(z)$. Let u be a Dirichlet-Laplacian eigenfunction in Ω satisfying

(1.1), and λ the associate eigenvalue. If $\lambda < \mu$, then

$$(1.2) \quad \|u\|_{L_2(Q(z))} \leq \|u\|_{L_2(Q(z_0))} \exp(-\beta \sqrt{\mu - \lambda} (z - z_0)) \quad (z \geq z_0),$$

with $\beta = 1/\sqrt{2}$. Moreover, if $(e_1 \cdot n(x)) \geq 0$ for all $x \in \partial\Omega$ with $x_1 > z_0$, where e_1 is the unit vector $(1, 0, \dots, 0)$ in the direction x_1 , and $n(x)$ is the normal vector at $x \in \partial\Omega$ directed outwards the domain, then the above inequality holds with $\beta = 1$.

In this theorem, a domain Ω is arbitrarily split into two subdomains, a “basic” domain Ω_1 (with $x_1 < z_0$) and a “branch” Ω_2 (with $x_1 > z_0$), by the hyperplane at $x_1 = z_0$ (the coordinate axis x_1 can be replaced by any straight line). Under the condition $\lambda < \mu$, the eigenfunction u exponentially decays along the branch Ω_2 . Note that the choice of the splitting hyperplane (i.e., z_0) determines the threshold μ . Since μ is independent of the basic domain V , one can impose any boundary condition on $\partial\Omega_1$ (that still ensures the self-adjointness of the Laplace operator). In turn, the Dirichlet boundary condition on the boundary of the branch Ω_2 was relevant. Many numerical illustrations for this theorem were given in [10].

REMARK 1.2. It is worth stressing that the sufficient condition $\lambda < \mu$ involves purely spectral information: the eigenvalue λ in the whole domain and the smallest eigenvalue μ over all cross-sections. A simple geometrical condition on the basic domain Ω_1 can be formulated through the inradius ρ of Ω_1 (or Ω), i.e., the radius of the largest inscribed ball B_ρ . Since $B_\rho \subset \Omega$, the first Dirichlet eigenvalue λ is bounded as $\lambda \leq \lambda_1(B_\rho) = j_{\frac{d}{2}-1}^2/\rho^2$, where $j_{\frac{d}{2}-1}$ is the first positive zero of the Bessel function $J_{\frac{d}{2}-1}(z)$. A sufficient geometrical condition for getting exponentially decaying eigenfunction is then

$$(1.3) \quad \rho > j_{\frac{d}{2}-1}/\sqrt{\mu}.$$

For planar domains, the inequality yields $\rho/b > j_0/\pi$ where b is the largest width of the branch, and $j_0 \simeq 2.4048$. This inequality includes only the inradius of Ω_1

(or Ω) and the largest width of the branch, while the length of the branch can be varied arbitrarily. For instance, the localization in the basic domain Ω_1 may hold even when the area of the branch Ω_2 is arbitrarily large, as compared to the area of Ω_1 .

REMARK 1.3. For higher dimensions ($d \geq 3$), the localization may sound even more striking, as the “branch” has to be “narrow” only in one direction (Fig. 1). For instance, if the branch Ω_2 has a constant width b in one direction, then the smallest eigenvalue μ in its cross-sections is greater than π^2/b^2 . If the inradius ρ of Ω_1 is greater than $bj_{\frac{d}{2}-1}/\pi$ then the inequality (1.3) holds, and at least the first eigenfunction is localized in Ω_1 . In the three-dimensional space, $j_{\frac{1}{2}} = \pi$ so that the inradius has to be just greater than b : $\rho > b$. A simple example is a domain decomposed into the unit cube $\Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : 0 < x < 1, 0 < y < 1, 0 < z < 1\}$ and a parallelepiped $\Omega_2 = \{(x, y, z) \in \mathbb{R}^3 : 1 < x < L_x, -L_y < y < L_y, 0 < z < b\}$. When $b < 1/2$, the first eigenfunction is localized in the cube Ω_1 , whatever the lateral spatial sizes L_x and L_y of the “branch” are.

In the remainder of the paper, we extend the above result to the Laplace operator in planar domains with Robin boundary condition. We also provide several numerical illustrations of localized eigenfunctions in planar domains in Sec. 3.

2. Extension for Robin boundary condition

We consider the eigenvalue problem for the Laplace operator in a planar domain Ω with Robin boundary condition on a piecewise smooth boundary $\partial\Omega$:

$$(2.1) \quad \begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + hu &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where h is a nonnegative function, and $\partial/\partial n$ is the normal derivative directed outwards the domain. In that follows, we prove the following

THEOREM 2.1. *Let $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 \subset \mathbb{R}^2$ is a bounded domain, and*

$$(2.2) \quad \Omega_2 = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, y_1(x) < y < y_2(x)\},$$

is a branch of length $a > 0$ and of variable cross-sectional profile which is defined by two functions $y_1, y_2 \in C^1([0, a])$ such that $y_2(a) = y_1(a)$, $y_1'(x) \geq 0$ and $y_2'(x) \leq 0$. Let u and λ be an eigenfunction and eigenvalue of Ω satisfying the eigenvalue problem (2.1), with a nonnegative function h . We define $\mu = \inf_{x_0 < x < a} \mu_1(x)$ where $\mu_1(x)$ is the first eigenvalue of the Laplace operator in the cross-sectional interval $[y_1(x), y_2(x)]$:

$$(2.3) \quad \begin{aligned} v''(y) + \mu_1(x)v(y) &= 0 & (y_1(x) < y < y_2(x)) \\ v'(y) - h_1(x)v(y) &= 0 & (y = y_1(x)) \\ v'(y) + h_2(x)v(y) &= 0 & (y = y_2(x)) \end{aligned}$$

where

$$(2.4) \quad h_i(x) \equiv h(y_i(x))\sqrt{1 + [y_i'(x)]^2} \quad (i = 1, 2).$$

If $\lambda < \mu$, then

$$(2.5) \quad \|u\|_{L_2(\Omega(x))} \leq \|u\|_{L_2(\Omega(x_0))} \exp(-\beta\sqrt{\mu - \lambda}(x - x_0)) \quad (x \geq x_0),$$

where $\beta = 1/\sqrt{2}$ and $\Omega(x_0) = \{(x, y) \in \mathbb{R}^2 : x_0 < x < a, y_1(x) < y < y_2(x)\}$.

PROOF. The proof relies on Maslov’s differential inequality and follows the scheme that we used in [10] for Dirichlet boundary condition. We consider the squared L_2 -norm of the eigenfunction u in the “subbranch” $\Omega(x_0)$:

$$I(x_0) = \int_{\Omega(x_0)} dx dy u^2 = \int_{x_0}^a dx \int_{y_1(x)}^{y_2(x)} dy u^2(x, y)$$

and derive the inequality for its second derivative:

$$(2.6) \quad I''(x_0) \geq 2(\mu - \lambda)I_0(x_0).$$

(i) From the first derivative

$$I'(x_0) = - \int_{y_1(x_0)}^{y_2(x_0)} dy u^2(x_0, y),$$

we obtain

$$\begin{aligned} I''(x_0) &= -2 \int_{y_1(x_0)}^{y_2(x_0)} dy u \frac{\partial u}{\partial x} - y_2'(x_0)u^2(x_0, y(x_0)) + y_1'(x_0)u^2(x_0, y_1(x_0)) \\ &\geq -2 \int_{y_1(x_0)}^{y_2(x_0)} dy u \frac{\partial u}{\partial x}, \end{aligned}$$

where we used the conditions $y_2'(x) \leq 0$ and $y_1'(x) \geq 0$. Taking into account that

$$\begin{aligned} - \int_{y_1(x_0)}^{y_2(x_0)} dy u \frac{\partial u}{\partial x} &= - \int_{y_1(x_0)}^{y_2(x_0)} dy u \frac{\partial u}{\partial x} + \int_{S(x_0)} dS u \frac{\partial u}{\partial n} + \int_{S(x_0)} dS hu^2 \\ &= \int_{\Omega(x_0)} dx dy \operatorname{div}(u \nabla u) + \int_{S(x_0)} dS hu^2 \\ &= \int_{\Omega(x_0)} dx dy (\nabla u, \nabla u) + \int_{\Omega(x_0)} dx dy u \Delta u + \int_{S(x_0)} dS hu^2 \\ &= \int_{\Omega(x_0)} dx dy (\nabla u, \nabla u) - \lambda \int_{\Omega(x_0)} dx dy u^2 + \int_{S(x_0)} dS hu^2 \end{aligned}$$

where

$$S = S_1 \cup S_2, \quad S_i = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, y = y_i(x)\} \quad (i = 1, 2)$$

is the “lateral” boundary of Ω_2 , we obtain

$$\begin{aligned} I''(x_0) &\geq 2 \int_{S(x_0)} dS \, hu^2 + 2 \int_{\Omega(x_0)} dx dy \left(\frac{\partial u}{\partial y} \right)^2 - 2\lambda \int_{\Omega(x_0)} dx dy \, u^2 \\ &= 2 \int_{x_0}^a dx \left\{ h_2(x) \, u^2(x, y_2(x)) + h_1(x) \, u^2(x, y_1(x)) + \int_{y_1(x)}^{y_2(x)} dy \left[\left(\frac{\partial u}{\partial y} \right)^2 - \lambda u^2 \right] \right\}, \end{aligned}$$

where $h_i(x)$ is defined by Eq. (2.4).

According to the Rayleigh principle, the first eigenvalue $\mu_1(x)$ of the eigenvalue problem (2.3) on the cross-sectional interval $[y_1(x), y_2(x)]$ can be written as

$$\mu_1(x) = \inf_{u \in H^1([y_1(x), y_2(x)])} \frac{\left[h_2(x) \, u^2(x, y_2(x)) + h_1(x) \, u^2(x, y_1(x)) \right] + \int_{y_1(x)}^{y_2(x)} dy \left(\frac{\partial u}{\partial y} \right)^2}{\int_{y_1(x)}^{y_2(x)} dy \, u^2(x, y)}, \tag{2.7}$$

from which we get

$$\begin{aligned} I''(x_0) &\geq 2 \int_{x_0}^a dx (\mu_1(x) - \lambda) \int_{y_1(x)}^{y_2(x)} dy \, u^2(x, y) \\ &\geq 2(\mu - \lambda) \int_{x_0}^a dx \int_{y_1(x)}^{y_2(x)} dy \, u^2(x, y) = 2(\mu - \lambda)I(x_0), \end{aligned}$$

where $\mu = \inf_{x_0 < x < a} \mu_1(x)$. That completes the first step.

(ii) We easily check the following relations:

$$I(a) = 0, \quad I'(a) = 0, \quad I(x_0) \neq 0 \quad (0 < x_0 < a), \quad I'(x_0) < 0 \quad (0 < x_0 < a). \tag{2.8}$$

Note that the second relation relies on the assumption that $y_1(a) = y_2(a)$.

(iii) From the inequality (2.6) and relations (2.8), an elementary derivation implies the inequality (2.5). In fact, one multiplies (2.6) by $I'(x_0)$, integrates from x_0 to a , takes the square root and divides by $I(x_0)$ and integrates again from x_0 to x (see [10] for details). \square

The statement of Theorem 2.1 for Robin boundary condition is weaker than that of Theorem 1.1 in several aspects:

- Theorem 2.1 employs an explicit parameterization of the branch through smooth height functions y_1 and y_2 ; in particular, the statement is limited to planar domains.
- The branch has to be non-increasing (conditions $y_1'(x) \geq 0$ and $y_2'(x) \leq 0$) and vanishing at the end (condition $y_1(a) = y_2(a)$).
- The inequality (2.5) characterizes the L_2 -norm of the eigenfunction in the distant part of the branch, $\Omega(x)$, while the inequality (1.2) provided an estimate at the cross-section $Q(x)$.

- The decay rate in Eq. (2.5) involves the coefficient $\beta = 1/\sqrt{2}$ while the inequality (1.2) for non-increasing branches was proved for $\beta = 1$.

These remarks suggest that the statement of theorem 2.1 can be further extended while certain conditions may be relaxed.

We also note that the solution of the eigenvalue problem (2.3) has an explicit form

$$(2.9) \quad v(y) = c_1 \sin(\alpha y) + c_2 \cos(\alpha y),$$

with two constants c_1, c_2 and $\mu_1(x) = \alpha^2$, while the boundary conditions at the endpoints $y = y_1(x)$ and $y = y_2(x)$,

$$c_1[-h_1 \sin(\alpha y_1) + \alpha \cos(\alpha y_1)] + c_2[-h_1 \cos(\alpha y_1) - \alpha \sin(\alpha y_1)] = 0,$$

$$c_1[h_2 \sin(\alpha y_2) + \alpha \cos(\alpha y_2)] + c_2[h_2 \cos(\alpha y_2) - \alpha \sin(\alpha y_2)] = 0,$$

yield a closed equation on α :

$$(2.10) \quad (\alpha^2 + h_1 h_2) \sin \alpha(y_2 - y_1) + \alpha(h_1 + h_2) \cos \alpha(y_2 - y_1) = 0$$

(here $h_{1,2}$ and $y_{1,2}$ depend on x). This equation has infinitely many solutions that can be found numerically. The first positive solution will determine $\mu_1(x)$.

3. Illustrations

In order to illustrate the geometrical structure of Laplacian eigenfunctions, we compute them for several simple domains. For all considered examples, we impose Dirichlet boundary condition for the sake of simplicity.

3.1. Sine-shaped branches. We consider the planar domain Ω composed of a basic domain Ω_1 (square of side L) and a branch Ω_2 of constant profile:

$$(3.1) \quad \Omega = \{(x, y) \in \mathbb{R}^2 : x \in (0, a), y \in (f(x), f(x) + b)\}.$$

For this example, we choose $f(x) = \sin(x)$, fix $b = 1$, $L = 1.54$ and take several values for the length a . Since the inradius of the square Ω_1 is greater than j_0/π , the first eigenvalue λ in these domains is smaller than $\mu = \pi^2$ for any length a so that the first eigenfunction should be localized in Ω_1 and exponentially decay along the branch Ω_2 . This behavior is illustrated on Fig. 2.

3.2. Star-shaped domains. Figure 3 shows the first five Dirichlet-Laplacian eigenfunctions for a “star-shaped” domain which is formed by a disk with many elongated triangles. The inradius of this domain is much greater than the largest width of triangular branches that implies localization of the first eigenfunction. One can see that all the five eigenfunctions are localized in the disk and exponentially decay along the branches.

3.3. Elongated polygons. As we discussed at the beginning, the separation into a basic domain and a branch is conventional. We illustrate this point by showing the exponential decay of the first Dirichlet-Laplacian eigenfunction in elongated polygons for which the ratio between the diameter and the inradius is large enough. We start by considering a right triangle then extend the construction to general elongated polygons.

We consider a rectangle of sides a and b ($a \geq b$) on which a right triangle Ω with legs c and d is constructed as shown on Fig. 4. Note that the triangle is uniquely defined by one leg (e.g., d), while the other leg is $c = ad/(d - b)$. The

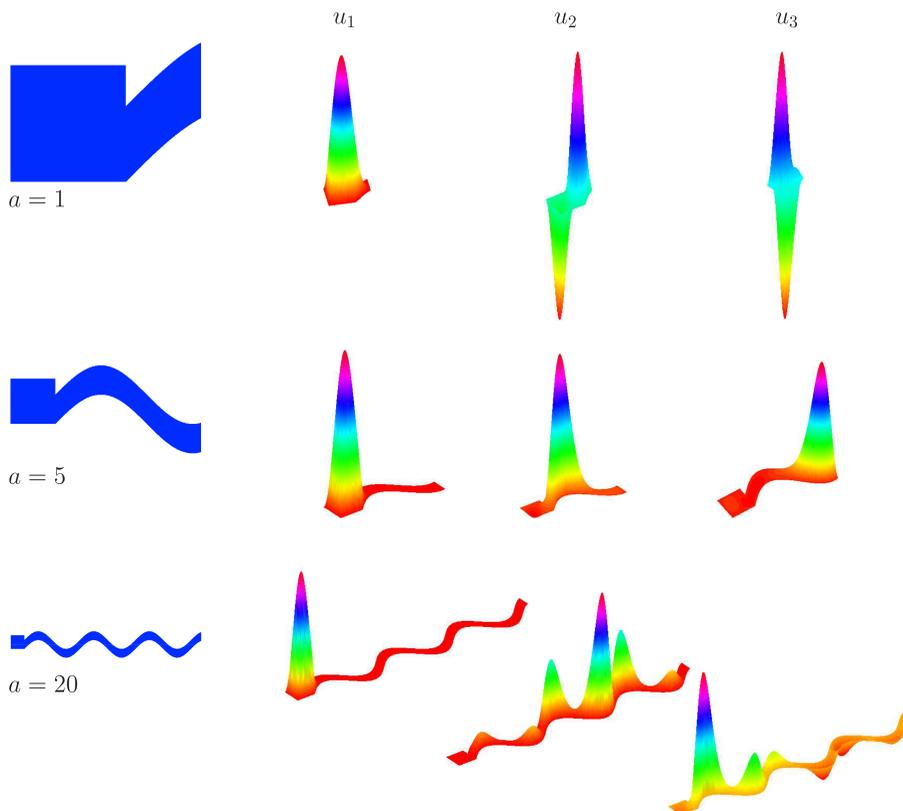


FIGURE 2. First three Dirichlet-Laplacian eigenfunctions for the planar domain Ω with sine-shaped branches, with $b = 1$, $L = 1.54$ and $a = 1$, $a = 5$ and $a = 20$. The first eigenfunction exponentially decays along the branch Ω_2 , while the second and third eigenfunctions do not. The localization occurs in spite of the fact that the area of Ω_1 presents only 7.15% of the total area for the last domain with $a = 20$. Note that the third eigenfunction for $a = 5$ is localized at the end of the branch Ω_2 .

vertical line at $x = a$ splits the triangle Ω into two subdomains: Ω_1 (a trapeze) and Ω_2 (a triangle). For fixed a and b , we are searching for a sufficient condition on d under which the eigenfunction u satisfying the eigenvalue problem (1.1), is localized in Ω_1 and exponentially decays along the subdomain Ω_2 .

LEMMA 3.1. *Let Ω be a right triangle Ω defined by fixed $a, b, d > 0$ ($a \geq b$) (Fig. 4). Let $\zeta \equiv (d/b) - 1$, $P_A(\zeta)$ and $P_B(\zeta)$ be two explicit polynomials defined by Eq. (3.3), and $\zeta_0 \approx 0.0131$ is the zero of $P_A(\zeta)$. If both inequalities*

$$(3.2) \quad \zeta > \zeta_0, \quad \frac{a^2}{b^2} > \frac{P_B(\zeta)}{P_A(\zeta)}$$

are fulfilled, then the first eigenfunction u of the Laplace operator in Ω with Dirichlet boundary condition exponentially decays in Ω_2 .

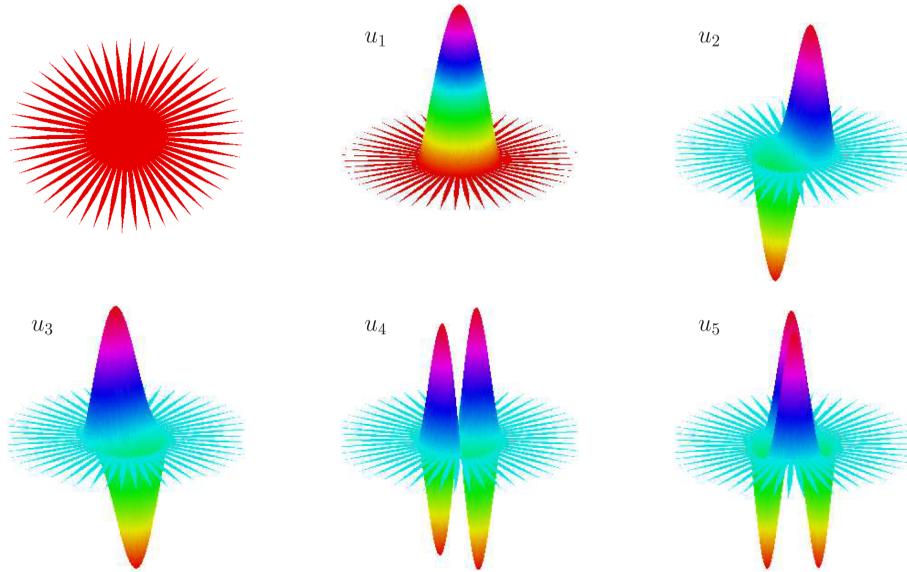


FIGURE 3. Localization of the first five Dirichlet-Laplacian eigenfunctions in a domain Ω with 51 branches.

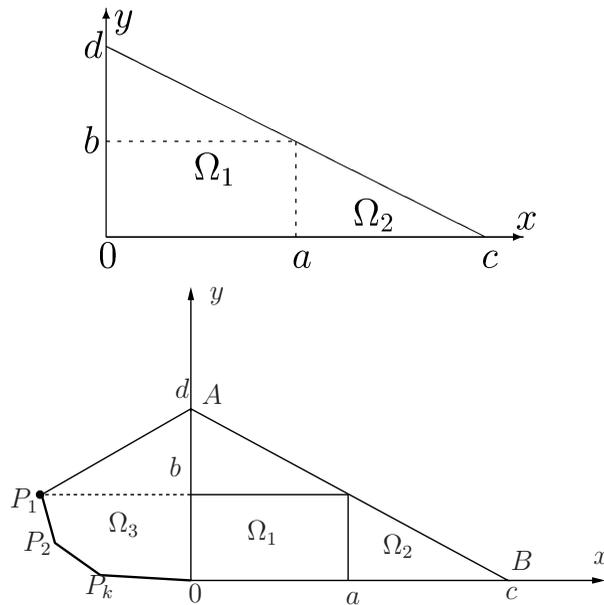


FIGURE 4. **Top:** A right triangle Ω is decomposed into a trapezoid Ω_1 and a right triangle Ω_2 . **Bottom:** An elongated polygon P with n vertices $\{0, B, A, P_1, P_2, \dots, P_k\}$, where $k = n - 3$. Here, Ω_3 is the polygon including $n - 1$ vertices $\{0, A, P_1, P_2, \dots, P_k\}$.

PROOF. The proof of the exponential decay relies on Theorem 1.1. For the “branch” Ω_2 , the largest width is b so that the threshold $\mu = \pi^2/b^2$. Our goal is therefore to find a sufficient geometrical condition to ensure that the first eigenvalue λ_1 is smaller than μ . This condition can be replaced by a weaker condition $\gamma_1 < \pi^2/b^2$ for the first eigenvalue γ_1 of the Laplace operator in the trapeze Ω_1 with Dirichlet boundary condition.

The eigenvalue γ_1 can be found from the Rayleigh’s principle as

$$\gamma_1 = \inf_{v \in H_0^1(\Omega_1)} \gamma(v), \quad \gamma(v) \equiv \frac{(\nabla v, \nabla v)_{L^2(\Omega_1)}}{(v, v)_{L^2(\Omega_1)}}.$$

Taking a trial function

$$v(x, y) = y \left(y - d + \frac{d - b}{a} x \right) \sin(\pi x/a),$$

which satisfies Dirichlet boundary condition on the boundary of Ω_1 , we look for such conditions that $\gamma(v) < \pi^2/b^2$, i.e.,

$$Q(\zeta) \equiv \frac{\pi^2}{b^2} (v, v)_{L^2(\Omega_1)} - (\nabla v, \nabla v)_{L^2(\Omega_1)} > 0.$$

The direct integration yields

$$Q(\zeta) = \frac{b^5}{720\pi^2 a} \left(\kappa^2 P_A(\zeta) - P_B(\zeta) \right),$$

where $\kappa = a/b$, and $P_A(\zeta)$, $P_B(\zeta)$ are two polynomials of the fifth degree:

$$(3.3) \quad P_A(\zeta) \equiv \sum_{j=0}^5 A_j \zeta^j, \quad P_B(\zeta) \equiv \sum_{j=0}^5 B_j \zeta^j,$$

with the explicit coefficients:

$$\begin{aligned} A_5 &= 2\pi^4 - 15\pi^2 + 45 \approx 91.7741 & B_5 &= 2\pi^4 + 15\pi^2 - 45 \approx 297.8622 \\ A_4 &= 6(2\pi^4 - 10\pi^2 + 15) \approx 666.7328 & B_4 &= 6(2\pi^4 + 10\pi^2 - 15) \approx 1671.0854 \\ A_3 &= 30(\pi^4 - 4\pi^2 + 3) \approx 1827.9202 & B_3 &= 30(\pi^4 + 3\pi^2) \approx 3810.5371 \\ A_2 &= 20(2\pi^4 - 9\pi^2 + 9) \approx 2299.8348 & B_2 &= 20(2\pi^4 + 3\pi^2) \approx 4488.5399 \\ A_1 &= 30(\pi^4 - 6\pi^2) \approx 1145.7439 & B_1 &= 30\pi^4 \approx 2922.2727 \\ A_0 &= 12(\pi^4 - 10\pi^2) \approx -15.4434 & B_0 &= 12\pi^4 \approx 1168.9091 \end{aligned}$$

Note that all $B_j > 0$ and $A_j > 0$ except for $A_0 < 0$. From the fact that $A_j < B_j$, one has $Q(\zeta) < 0$ for all $\zeta > 0$ when $\kappa = 1$ (i.e., $a = b$). We have therefore two parameters, ζ and κ , which determine the sign of Q and thus the exponential decay. Since $P_B(\zeta) > 0$ for all $\zeta \geq 0$, the condition $Q(\zeta) > 0$ is equivalent to two inequalities:

$$(3.4) \quad P_A(\zeta) > 0, \quad \kappa^2 > \frac{P_B(\zeta)}{P_A(\zeta)}.$$

One can check that $P_A(\zeta_0) = 0$ at $\zeta_0 \approx 0.0131$ and $P_A(\zeta) > 0$ if and only if $\zeta > \zeta_0$ that completes the proof. □

We remind that this condition is not necessary (as we deal with an estimate for the first eigenvalue). For given a and b (i.e., κ), the above inequalities determine

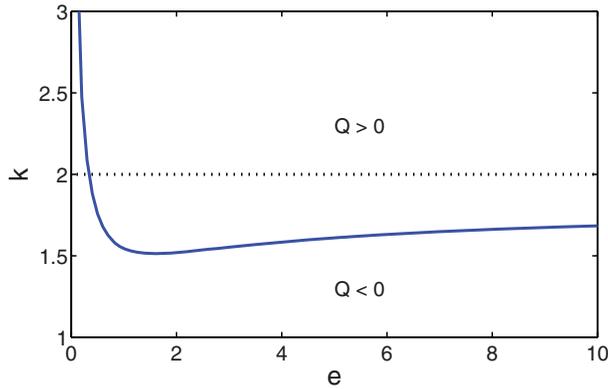


FIGURE 5. The diagram in the space of parameters ζ and κ for positive and negative signs of Q which correspond to localization and non-localization regions. For a given κ (e.g., $\kappa = 2$ shown by horizontal dotted line), one can determine the values of ζ , for which localization occurs.

the values of ζ (and thus the leg d) for which localization occurs. Alternatively, one can express a and b through the legs c and d (and parameter ζ) as

$$a = \frac{c\zeta}{\zeta + 1}, \quad b = \frac{d}{\zeta + 1},$$

from which $\kappa = c\zeta/d$. For given c and d , one can vary ζ to get a family of inclosed rectangles (of sides a and b). The above inequalities can be reformulated as

$$\begin{aligned} \zeta > \zeta_0 &\Leftrightarrow b < \frac{d}{\zeta_0 + 1}, \\ k^2 > \frac{P_B(\zeta)}{P_A(\zeta)} &\Leftrightarrow \frac{c}{d} > \frac{\sqrt{P_B(\zeta)}}{\sqrt{P_A(\zeta)} \zeta} \equiv f(\zeta). \end{aligned}$$

The function $f(\zeta)$ can be checked to be monotonously decreasing so that the last inequality yields

$$(3.5) \quad \zeta > f^{-1}(c/d) \Leftrightarrow b < \frac{d}{f^{-1}(c/d) + 1},$$

where f^{-1} denotes the inverse of the function $f(\zeta)$. This condition determines the choice of the inscribed rectangle (the size b) for a given triangle.

For the “worst” case $c = d$, for which a numerical computation yields $f^{-1}(1) \approx 1.515$, one gets

$$\frac{b}{d} < \frac{1}{2.515} \approx 0.3976.$$

This example shows that one can always inscribe a rectangle in such a way that $\lambda < \pi^2/b^2$. However, the “branch” Ω_2 in which an exponential decay of the eigenfunction is expected, may be small. Figure 6 illustrates these results.

REMARK 3.2. Any enlargement of the subdomain Ω_1 on Fig. 4 further diminishes the eigenvalue γ_1 and thus favors the exponential decay in Ω_2 . In particular,

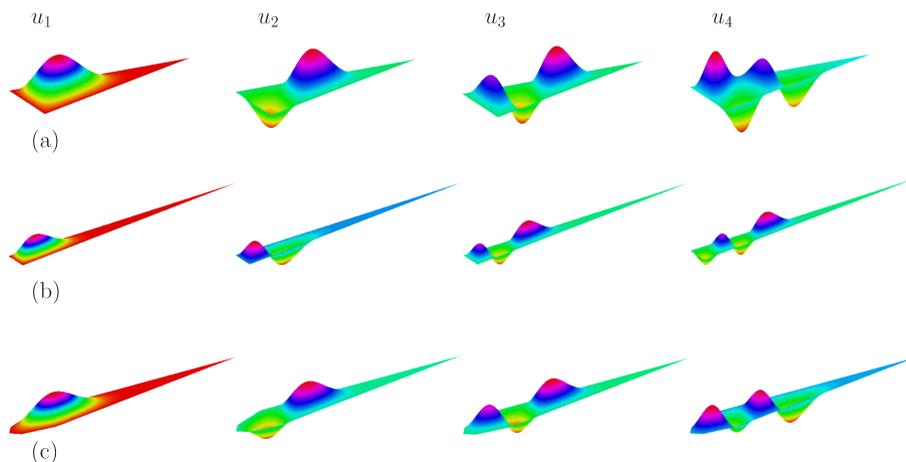


FIGURE 6. Several Dirichlet-Laplacian eigenfunctions in **(a)** the right triangle with $a = 2$, $b = 1$ and $\zeta \approx 0.32$ for which $c = 8.25$ and $d = 1.32$; **(b)** the right triangle with $a = 4$, $b = 1$ and $\zeta \approx 0.08$ for which $c = 61.14$ and $d = 1.07$; and **(c)** elongated hexagon. In all these cases, the first eigenvalue λ_1 is smaller than π^2 , while the associated eigenfunction decays exponentially along the “branch” Ω_2 . The other eigenfunctions are also concentrated in Ω_1 .

for each positive integer n ($n \geq 3$), one can construct elongated polygons of n vertices for which the first Dirichlet-Laplacian eigenfunction is localized in Ω_1 (Fig. 4b). Figure 6c shows first eigenfunctions in elongated hexagons.

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