MODE MATCHING METHODS FOR SPECTRAL AND SCATTERING PROBLEMS

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Summary

We present several applications of mode matching methods in spectral and scattering problems. First, we consider the eigenvalue problem for the Dirichlet Laplacian in a finite cylindrical domain that is split into two subdomains by a 'perforated' barrier. Using rather elementary methods, we prove that the first eigenfunction is localized in the larger subdomain, that is, its L_2 norm in the smaller subdomain can be made arbitrarily small by setting the diameter of the 'holes' in the barrier small enough. This result extends the well-known localization of Laplacian eigenfunctions in dumbbell domains. We also discuss an extension to noncylindrical domains with radial symmetry. Second, we study a scattering problem in an infinite cylindrical domain with two identical perforated barriers. If the holes are small, there exists a low frequency at which an incident wave is almost fully transmitted through both barriers. This result is counterintuitive as a single barrier with the same holes would fully reflect incident waves with low frequencies.

1. Introduction

Mode matching is a classical powerful method for the analysis of spectral and scattering problems (1–6). The main idea of the method consists in decomposing a domain into 'basic' subdomains, in which the underlying spectral or scattering problem can be solved explicitly, and then matching the analytical solutions at 'junctions' between subdomains. This matching leads to functional equations at the junctions and thus reduces the dimensionality of the problem. Such a dimensionality reduction is similar, to some extent, to that in the potential theory when searching for a solution of the Laplace equation in the bulk is reduced to finding an appropriate 'charge density' on the boundary. Although the resulting integral or functional equations are in general more difficult to handle than that of the original problem, they are useful for obtaining analytical estimates and numerical solutions.

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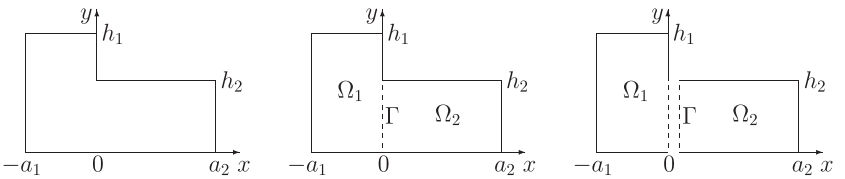

Fig. 1 An L-shaped domain is decomposed into two rectangular subdomains Ω_1 and Ω_2, in which explicit solutions are found and then matched at the inner interface Γ

To illustrate the main idea, let us consider the eigenvalue problem for the Dirichlet Laplacian in a planar L-shape domain (Fig. 1):

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u_{|\partial\Omega} = 0. \tag{1.1}$$

This domain can be naturally decomposed into two rectangular subdomains $\Omega_1 = (-a_1, 0) \times (0, h_1)$ and $\Omega_2 = (0, a_2) \times (0, h_2)$. Without loss of generality, we assume $h_1 \geqslant h_2$. For each of these subdomains, one can explicitly write a general solution of the Helmholtz equation in (1.1) due to a separation of variables in perpendicular directions x and y. For instance, one has in the rectangle Ω_1

$$u_1(x, y) = \sum_{n=1}^{\infty} c_{n,1} \sin(\pi n y/h_1) \sinh(\gamma_{n,1}(a_1 + x)) \qquad (x, y) \in \Omega_1, \tag{1.2}$$

where $\gamma_{n,1} = \sqrt{\pi^2 n^2/h_1^2 - \lambda}$ ensures that each term satisfies (1.1). The sine functions are chosen to fulfill the Dirichlet boundary condition on the horizontal edges of Ω_1, while $\sinh(\gamma_{n,1}(a_1 + x))$ vanishes on the vertical edge at $x = -a_1$. The coefficients $c_{n,1}$ are fixed by the restriction $u_{1|\Gamma}$ of u_1 on the matching region $\Gamma = (0, h_2)$ at $x = 0$. In fact, multiplying (1.2) by $\sin(\pi k y/h_1)$ and integrating over y from 0 to h_1, the coefficients $c_{n,1}$ are expressed through $u_{1|\Gamma}$, and thus

$$u_1(x, y) = \frac{2}{h_1} \sum_{n=1}^{\infty} \big(u_{1|\Gamma}, \sin(\pi n y/h_1)\big)_{L_2(\Gamma)} \sin(\pi n y/h_1) \frac{\sinh(\gamma_{n,1}(a_1 + x))}{\sinh(\gamma_{n,1} a_1)}, \tag{1.3}$$

where $(\cdot, \cdot)_{L_2(\Gamma)}$ is the scalar product in the $L_2(\Gamma)$ space. Here we also used the Dirichlet boundary condition on the vertical edge $\{0\} \times [h_2, h_1]$ to replace the scalar product in $L_2(0, h_1)$ by that in $L_2(0, h_2) = L_2(\Gamma)$. One can see that the solution of (1.1) in the subdomain Ω_1 is fully determined by λ and $u_{1|\Gamma}$, which are yet unknown at this stage.

Similarly, one can write explicitly the solution u_2 in Ω_2:

$$u_2(x, y) = \frac{2}{h_2} \sum_{n=1}^{\infty} \big(u_{2|\Gamma}, \sin(\pi n y/h_2)\big)_{L_2(\Gamma)} \sin(\pi n y/h_2) \frac{\sinh(\gamma_{n,2}(a_2 - x))}{\sinh(\gamma_{n,2} a_2)}, \tag{1.4}$$

where $\gamma_{n,2} = \sqrt{\pi^2 n^2/h_2^2 - \lambda}$. Since the solution u is analytic in the whole domain Ω, u and its derivative should be continuous at the matching region Γ:

$$u_{1|\Gamma} = u_{2|\Gamma} = u_{|\Gamma}, \qquad \left.\frac{\partial u_1}{\partial x}\right|_{\Gamma} = \left.\frac{\partial u_2}{\partial x}\right|_{\Gamma}. \tag{1.5}$$

Due to the explicit form (1.3, 1.4) of the solutions u_1 and u_2, the equality of the derivatives can be written as a functional equation on the yet unknown restriction of the function u on the opening Γ:

$$\begin{aligned}&\frac{2}{h_1}\sum_{n=1}^{\infty}\gamma_{n,1}\coth(\gamma_{n,1}a_1)\left(u_{|\Gamma},\sin(\pi ny/h_1)\right)_{L_2(\Gamma)}\sin(\pi ny/h_1)\\&\quad-\frac{2}{h_2}\sum_{n=1}^{\infty}\gamma_{n,2}\coth(\gamma_{n,2}a_2)\left(u_{|\Gamma},\sin(\pi ny/h_2)\right)_{L_2(\Gamma)}\sin(\pi ny/h_2)=0\qquad y\in\Gamma\,.\end{aligned}\tag{1.6}$$

Multiplying this relation by a function $v \in H^{\frac{1}{2}}(\Gamma)$ and integrating over Γ, one can understand this equation in a weak sense as

$$\mathfrak{a}_\lambda\left(u_{|\Gamma},v\right)=0\qquad v\in H^{\frac{1}{2}}(\Gamma),\tag{1.7}$$

where $\mathfrak{a}_\lambda\left(u_{|\Gamma},v\right)$ is the sesquilinear form associated with (1.6), see rigorous definitions in section 2.1. The original eigenvalue problem for the Laplace operator in the L-shaped domain Ω is thus reduced to a generalized eigenvalue problem (1.7), with the advantage of the reduced dimensionality, from a planar domain to an interval. We have earlier applied this technique to investigate trapped modes in finite quantum waveguides (**7**).

In this article, we illustrate how mode matching methods can be used for investigating spectral and scattering properties in various domains, in particular, for deriving estimates for eigenvalues and eigenfunctions. More precisely, we address three problems:

(i) In section 2, we consider a finite cylinder $\Omega_0 = [-a_1, a_2] \times \mathcal{S} \subset \mathbb{R}^{d+1}$ of a bounded connected cross-section $\mathcal{S} \subset \mathbb{R}^d$ with a piecewise smooth boundary $\partial\mathcal{S}$ (Fig. 2). The cylinder is split into two subdomains by a 'perforated' barrier $\mathcal{B} \subset \mathcal{S}$ at $x = 0$, that is, we consider the domain $\Omega = \Omega_0\backslash(\{0\} \times \mathcal{B})$. If $\mathcal{B} = \mathcal{S}$, the barrier separates Ω_0 into two disconnected subdomains, in which case the spectral analysis can be done separately for each subdomain. When $\mathcal{B} \neq \mathcal{S}$, an opening region $\Gamma = \mathcal{S}\backslash\mathcal{B}$ ('holes' in the barrier) connects two subdomains. When the diameter of the opening region, $\mathrm{diam}\{\Gamma\}$, is small enough, we show that the first Dirichlet eigenfunction u is 'localized' in a larger subdomain, that is, the L_2-norm of u in the smaller subdomain vanishes as $\mathrm{diam}\{\Gamma\} \to 0$. This

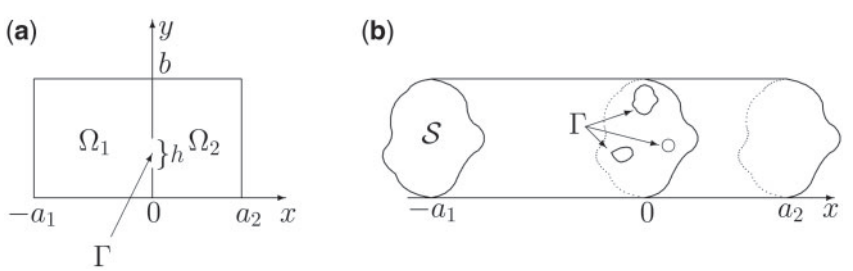

Fig. 2 Illustration of cylindrical domains separated by a 'perforated' barrier. **(a)** Two-dimensional case of a rectangle $\Omega_0 = (-a_1, a_2) \times (0, b)$ separated by a vertical segment with a 'hole' Γ of width h. **(b)** Three-dimensional case of a cylinder $\Omega_0 = (-a_1, a_2) \times \mathcal{S}$ of arbitrary cross-section $\mathcal{S}$ separated by a barrier at $x = 0$ with 'holes' Γ

localization phenomenon resembles the asymptotic behavior of Dirichlet eigenfunctions in dumbbell domains, that is, when two subdomains are connected by a narrow 'channel' (see the review (**8**) and references therein). When the width of the channel vanishes, the eigenfunctions become localized in either of subdomains (see some numerical illustrations in (**8**–**10**)). We emphasize, however, that most of formerly used asymptotic techniques (for example, see (**11**–**26**)) would fail in the case without channel. In fact, these former studies dealt with the width-over-length ratio of the channel as a small parameter that is not applicable in our situation as the channel length is zero. Although the problem without a channel may look simpler than the dumbbell problem, we are not aware of former rigorous proofs of localization in this case. In the two-dimensional case, we prove even stronger result: when the opening region Γ is composed of a finite number of intervals, the localization is controlled by the diameter of the largest interval, irrespective of their mutual arrangment. In other words, the first eigenfunction is localized even if the barrier is almost totally perforated (that is, the total length of intervals forming the barrier is arbitrarily small). A similar behavior can be observed for other Dirichlet eigenfunctions, under stronger assumptions. The practical relevance of geometrically localized eigenmodes and their physical applications were discussed in (**27**–**29**). We note that more advanced methods such as monotone convergence of quadratic forms (**30**–**32**), Brownian motion techniques (**33**), or pointwise controlled estimates (**34**) could potentially be applied to derive similar or even stronger localization results. However, the mode matching method employs the most basic concepts of mathematical physics such as separation of variables and some elementary estimates that are accessible to a broad readership. For this reason, the localization problem is convenient for our illustrative and pedagogic discussion of mode matching methods.

(ii) In section 3, we consider a scattering problem in an infinite cylinder $\Omega_0 = \mathbb{R} \times \mathcal{S} \subset \mathbb{R}^{d+1}$ of a bounded cross-section $\mathcal{S} \subset \mathbb{R}^d$ with a piecewise smooth boundary $\partial\mathcal{S}$. The wave propagation in such waveguides and related problems have been thoroughly investigated (see (**8**, **35**–**45**) and references therein). If the cylinder is blocked by a single barrier $\mathcal{B} \subset \mathcal{S}$ with small holes, an incident wave is fully reflected if its frequency is not high enough for a wave to 'squeeze' through small holes. Intuitively, one might think that putting two identical barriers (Fig. 3) would enhance this blocking effect. We prove that, if the holes in the barriers are small enough, there exists a frequency close to the smallest eigenvalue in the half-domain between two barriers, at which the incident wave is almost fully transmitted through both barriers. This counterintuitive result may have some acoustic applications.

(iii) In section 4, we return to the eigenvalue problem for the Dirichlet Laplacian and show an application of the mode matching method to noncylindrical domains. As an example, we consider the union of a disk of radius R_1 and a part of a circular sector of angle ϕ_1 between two circles of radii R_1 and R_2 (Fig. 4). Under the condition that the sector is thin and long, we prove the existence of an eigenfunction which is localized in the sector and negligible inside the disk. In particular, we establish the inequalities between geometric parameters R_1, R_2 and ϕ_1 to ensure the localization.

2. Barrier-induced localization in a finite cylinder

2.1 *Formulation and preliminaries*

Let $\Omega_0 = (-a_1, a_2) \times \mathcal{S} \subset \mathbb{R}^{d+1}$ be a finite cylinder of a bounded connected cross-section $\mathcal{S} \subset \mathbb{R}^d$ with a piecewise smooth boundary $\partial\mathcal{S}$ (Fig. 2). For an open set $\Gamma \subset \mathcal{S}$, let $\mathcal{B} = \mathcal{S}\backslash\Gamma$ be a barrier, perforated by Γ, that divides the domain Ω_0 into two cylindrical subdomains Ω_1 and Ω_2, connected through Γ. We define then $\Omega = \Omega_0\backslash(\{0\} \times (\mathcal{S}\backslash\Gamma))$ as the cylinder Ω_0 without a barrier at $x = 0$ of the

Fig. 3 **(a)** An infinite cylinder with two identical barriers at distance $2a$. **(b)** The half of the above domain, that is, a semi-infinite cylinder with a single barrier at $x = 0$ and Neumann boundary condition at $x = a$. Although this schematic illustration is shown in two dimensions, the results are valid for a general cylinder $\mathbb{R} \times \mathcal{S}$ of arbitrary bounded cross-section $\mathcal{S} \subset \mathbb{R}^n$ with a piecewise smooth boundary $\partial\mathcal{S}$ and arbitrary opening $\Gamma \subset \mathcal{S}$

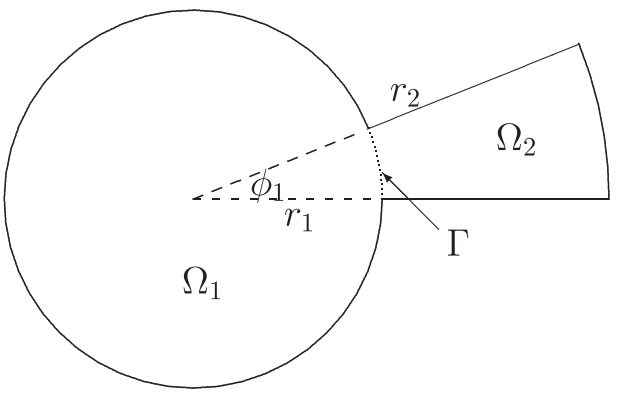

Fig. 4 The domain Ω is decomposed into a disk Ω_1 of radius R_1, and a part of a circular sector Ω_2 of angle ϕ_1 between two circles of radii R_1 and R_2. Two domains are connected through an opening Γ (an arc $(0, \phi_1)$ on the circle of radius R_1)

cross-sectional shape $\mathcal{B}$. For instance, in two dimensions, one can take $\mathcal{S} = (0, b)$ and $\Gamma = (h_1, h_2)$ (with $0 \leqslant h_1 < h_2 \leqslant b$) so that Ω is a rectangle without two vertical segments $(0, h_1)$ and (h_2, b), which has an opening Γ (a 'door') of length $h = h_2 - h_1$, as shown in Fig. 2a. We also denote by $\mathcal{S}_x = \{x\} \times \mathcal{S}$ the cross-section at x.

We consider the Dirichlet eigenvalue problem in Ω

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u_{|\partial\Omega} = 0. \tag{2.1}$$

We denote by ν_n and $\psi_n(\boldsymbol{y})$ the Dirichlet eigenvalues and $L_2(\mathcal{S})$-normalized eigenfunctions of the Laplace operator $\Delta_\perp$ in the cross-section $\mathcal{S}$:

$$-\Delta_\perp \psi_n(\boldsymbol{y}) = \nu_n \psi_n(\boldsymbol{y}) \quad (\boldsymbol{y} \in \mathcal{S}), \qquad \psi_n(\boldsymbol{y}) = 0 \quad (\boldsymbol{y} \in \partial\mathcal{S}), \tag{2.2}$$

where the eigenvalues are ordered:

$$0 < \nu_1 < \nu_2 \leqslant \nu_3 \leqslant \ldots \nearrow +\infty. \tag{2.3}$$

A general solution of (2.1) in each subdomain reads

$$u_1(x,\boldsymbol{y}) \equiv u_{|\Omega_1} = \sum_{n=1}^{\infty} c_{n,1}\ \psi_n(\boldsymbol{y})\ \sinh(\gamma_n(a_1+x)) \qquad (-a_1 < x < 0),$$

$$u_2(x,\boldsymbol{y}) \equiv u_{|\Omega_2} = \sum_{n=1}^{\infty} c_{n,2}\ \psi_n(\boldsymbol{y})\ \sinh(\gamma_n(a_2-x)) \qquad (0 < x < a_2),$$

where

$$\gamma_n = \sqrt{\nu_n - \lambda} \tag{2.4}$$

can be either positive, or purely imaginary (in general, there can be a finite number of purely imaginary γ_n and infinitely many real γ_n). The coefficients $c_{n,1}$ and $c_{n,2}$ are determined by multiplying $u_{1,2}(0,\boldsymbol{y})$ at the matching cross-section $x=0$ by $\psi_n(\boldsymbol{y})$ and integrating over $\mathcal{S}$, from which

$$\begin{aligned} u_1(x,\boldsymbol{y}) &= \sum_{n=1}^{\infty} b_n\ \psi_n(\boldsymbol{y})\ \frac{\sinh(\gamma_n(a_1+x))}{\sinh(\gamma_n a_1)} \qquad (-a_1 < x < 0),\\ u_2(x,\boldsymbol{y}) &= \sum_{n=1}^{\infty} b_n\ \psi_n(\boldsymbol{y})\ \frac{\sinh(\gamma_n(a_2-x))}{\sinh(\gamma_n a_2)} \qquad (0 < x < a_2), \end{aligned} \tag{2.5}$$

where

$$b_n \equiv \big(u_{|\mathcal{S}},\, \psi_n\big)_{L_2(\mathcal{S})} = \big(u_{|\Gamma},\, \psi_n\big)_{L_2(\Gamma)}, \tag{2.6}$$

with the conventional scalar product in $L_2(\Gamma)$:

$$\big(u,\, v\big)_{L_2(\Gamma)} = \int_{\Gamma} d\boldsymbol{y}\ u(\boldsymbol{y})\ v(\boldsymbol{y}) \tag{2.7}$$

(since all considered operators are self-adjoint, we do not use complex conjugate). In the second identity in (2.6), we used the Dirichlet boundary condition on the barrier $\mathcal{S}\backslash\Gamma$. We see that the eigenfunction u in the whole domain Ω is fully determined by its restriction $u_{|\Gamma}$ to the opening Γ.

Since the eigenfunction is analytic inside Ω, the derivatives of u_1 and u_2 with respect to x should match on the opening Γ:

$$\sum_{n=1}^{\infty} b_n\gamma_n\big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big]\psi_n(\boldsymbol{y}) = 0 \quad (\boldsymbol{y} \in \Gamma). \tag{2.8}$$

Multiplying this relation by a function $v \in H^{\frac{1}{2}}(\Gamma)$ and integrating over Γ, one can introduce the associated sesquilinear form $\mathfrak{a}_\lambda(u,v)$:

$$\mathfrak{a}_\lambda\big(u,\, v\big) = \sum_{n=1}^{\infty} \gamma_n\big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big]\big(u,\, \psi_n\big)_{L_2(\Gamma)}\big(v,\, \psi_n\big)_{L_2(\Gamma)}, \tag{2.9}$$

where the Hilbert space $H^{\frac{1}{2}}(\Gamma)$ is defined with the help of eigenvalues ν_n and eigenfunctions $\psi_n(\boldsymbol{y})$ of the Dirichlet Laplacian $\Delta_\perp$ in the cross-section $\mathcal{S}$ as

$$H^{\frac{1}{2}}(\Gamma) = \left\{ v \in L_2(\Gamma) \ : \ \sum_{n=1}^{\infty} \sqrt{\nu_n}\, \big(v, \psi_n\big)^2_{L_2(\Gamma)} < +\infty \right\}. \tag{2.10}$$

Note that this space equipped with the conventional scalar product:

$$\big(u, v\big)_{H^{\frac{1}{2}}(\Gamma)} = \big(u, v\big)_{L_2(\Gamma)} + \sum_{n=1}^{\infty} \sqrt{\nu_n}\, \big(u, \psi_n\big)_{L_2(\Gamma)} \big(v, \psi_n\big)_{L_2(\Gamma)}. \tag{2.11}$$

Using the sesquilinear form $\mathfrak{a}_\lambda(u, v)$, one can understand the matching condition (2.8) in the weak sense as an equation on λ and $u_{|\Gamma} \in H^{\frac{1}{2}}(\Gamma)$:

$$\mathfrak{a}_\lambda\big(u_{|\Gamma}, v\big) = 0 \qquad \forall\, v \in H^{\frac{1}{2}}(\Gamma) \tag{2.12}$$

(since this is a standard technique, we refer to textbooks (**46–51**) for details). Once a pair $\{\lambda, u_{|\Gamma}\} \in \mathbb{R} \times H^{\frac{1}{2}}(\Gamma)$ satisfying this equation for any $v \in H^{\frac{1}{2}}(\Gamma)$ is found, it fully determines the eigenfunction $u \in H^1(\Omega)$ of the original eigenvalue problem (2.1) in the whole domain Ω through (2.5). In turn, this implies that $u \in C^\infty(\Omega)$. In other words, the mode matching method allows one to reduce the original eigenvalue problem in the whole domain $\Omega \subset \mathbb{R}^{d+1}$ to an equivalent problem (2.12) on the opening $\Gamma \subset \mathbb{R}^d$, thus reducing the dimensionality of the problem. More importantly, the reduced problem allows one to derive various estimates on eigenvalues and eigenfunctions. In fact, setting $v = u_\Gamma$ yields the dispersion relation

$$\mathfrak{a}_\lambda\big(u_{|\Gamma}, u_{|\Gamma}\big) = \sum_{n=1}^{\infty} b_n^2 \gamma_n \big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big] = 0 \tag{2.13}$$

(with b_n given by (2.6)), from which estimates on the eigenvalue λ can be derived (see below). In turn, writing the squared L_2-norm of the eigenfunction in an arbitrary cross-section $\mathcal{S}_x$,

$$I(x) \equiv ||u||^2_{L_2(\mathcal{S}_x)} = \int_{\mathcal{S}} d\boldsymbol{y}\, |u(x, \boldsymbol{y})|^2, \tag{2.14}$$

one gets

$$I(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} b_n^2 \frac{\sinh^2(\gamma_n(a_1 + x))}{\sinh^2(\gamma_n a_1)} & (-a_1 < x < 0), \\ \displaystyle\sum_{n=1}^{\infty} b_n^2 \frac{\sinh^2(\gamma_n(a_2 - x))}{\sinh^2(\gamma_n a_2)} & (0 < x < a_2), \end{cases} \tag{2.15}$$

and thus one can control the behavior of the eigenfunction u.

When the opening Γ shrinks, the domain Ω is split into two disjoint subdomains Ω_1 and Ω_2. It is natural to expect that each eigenvalue of the Dirichlet Laplacian in Ω converges to an eigenvalue of

the Dirichlet Laplacian either in Ω_1, or in Ω_2. This statement will be proved in section 2.2 and 2.3. The behavior of eigenfunctions is more subtle. If an eigenvalue in Ω converges to a limit which is an eigenvalue in both Ω_1 and Ω_2, the limiting eigenfunction is expected to 'live' in both subdomains. In turn, when the limit is an eigenvalue of only one subdomain, one can expect that the limiting eigenfunction will be localized in that subdomain. In other words, for any integer N, one expects the existence of a nonempty opening Γ small enough that at least N eigenfunctions of the Dirichlet Laplacian in Ω are localized in one subdomain (that is, the L_2-norm of these eigenfunctions in the other subdomain is smaller than a chosen $\varepsilon > 0$). Using the mode matching method, we prove a weaker form of these yet conjectural statements. We estimate the L_2 norm of an eigenfunction u in arbitrary cross-sections of Ω_1 and Ω_2 and show that the ratio of these norms can be made arbitrarily large under certain conditions. The explicit geometric conditions are obtained for the first eigenfunction in section 2.4, while discussion about other eigenfunctions is given in section 2.5. Some numerical illustrations are provided in Appendix A.

2.2 *Behavior of eigenvalues for the case $d > 1$*

We aim at showing that an eigenvalue of the Dirichlet Laplacian in Ω is close to an eigenvalue of the Dirichlet Laplacian either in Ω_1, or in Ω_2, for a small enough opening Γ. In this subsection, we consider the case $d > 1$, which turns out to be simpler and allows for more general statements. The planar case (with $d = 1$) will be treated separately in section 2.3.

The proof relies on the general classical Lemma 2.1.

LEMMA 2.1. *Let A be a self-adjoint operator with a discrete spectrum, and there exist constants $\varepsilon > 0$ and $\mu \in \mathbb{R}$ and a function v from the domain $\mathcal{D}_A$ of the operator A such that*

$$\|Av - \mu v\|_{L_2} < \varepsilon \|v\|_{L_2}. \tag{2.16}$$

Then there exists an eigenvalue λ of A such that

$$|\lambda - \mu| < \varepsilon. \tag{2.17}$$

An elementary proof is reported in Appendix B for completeness.

The lemma states that if one finds an approximate eigenpair μ and v of the operator A, then there exists its true eigenvalue λ close to μ. Since we aim at proving the localization of an eigenfunction in one subdomain, we expect that an appropriately extended eigenpair in this subdomain can serve as μ and v for the whole domain.

THEOREM 2.2. *Let a_1 and a_2 be strictly positive nonequal real numbers, $\mathcal{S}$ be a bounded domain in $\mathbb{R}^d$ ($d > 1$) with a piecewise smooth boundary $\partial\mathcal{S}$, Γ be a nonempty subset of $\mathcal{S}$, and*

$$\Omega = \big((-a_1, a_2) \times \mathcal{S}\big) \backslash \big(\{0\} \times (\mathcal{S} \backslash \Gamma)\big), \quad \Omega_1 = (-a_1, 0) \times \mathcal{S}, \quad \Omega_2 = (0, a_2) \times \mathcal{S}. \tag{2.18}$$

Let μ be any eigenvalue of the Dirichlet Laplacian in the subdomain Ω_2. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any opening Γ with $\operatorname{diam}\{\Gamma\} < \delta$, there exists an eigenvalue λ of the Dirichlet Laplacian in Ω such that $|\lambda - \mu| < \varepsilon$. The same statement holds for the subdomain Ω_1.

PROOF. We will prove the statement for the subdomain Ω_2. The Dirichlet eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω_2 are

$$\begin{aligned} \mu_{n,k} &= \pi^2 k^2/a_2^2 + \nu_n \\ v_{n,k} &= \sqrt{2/a_2}\,\sin(\pi k x/a_2)\,\psi_n(\boldsymbol{y}) \end{aligned} \qquad (n, k = 1, 2, \ldots), \tag{2.19}$$

where ν_n and $\psi_n(\boldsymbol{y})$ are the Dirichlet eigenpairs in the cross-section $\mathcal{S}$, see (2.2).

Let $\mu = \mu_{n,k}$ and $v = v_{n,k}$ for some n, k. Let 2δ be the diameter of the opening Γ, and $\boldsymbol{y}_\Gamma \in \mathcal{S}$ be the center of a ball $B_{(0,\boldsymbol{y}_\Gamma)}(\delta) \subset \mathbb{R}^{d+1}$ of radius δ that encloses Γ. We introduce a cut-off function $\bar\eta$ defined on Ω as

$$\bar\eta(x, \boldsymbol{y}) = \mathcal{I}_{\Omega_2}(x, \boldsymbol{y})\,\eta\big(|(x, \boldsymbol{y}) - (0, \boldsymbol{y}_\Gamma)|/\delta\big), \tag{2.20}$$

where $\mathcal{I}_{\Omega_2}(x, \boldsymbol{y})$ is the indicator function of Ω_2 (which is equal to 1 inside Ω_2 and 0 otherwise), and $\eta(r)$ is a function from $C^\infty(\mathbb{R}_+)$, which is 0 for $r < 1$ and 1 for $r > 2$. In other words, $\bar\eta(x, \boldsymbol{y})$ is zero outside Ω_2 and in a δ-vicinity of the opening Γ, it changes smoothly to 1 in a thin spherical shell of width δ, and it is equal to 1 in the remaining part of Ω_2.

According to Lemma 2.1, it is sufficient to check that

$$\|\Delta(\bar\eta\, v) + \mu\, \bar\eta\, v\|_{L_2(\Omega)} < \varepsilon \|\bar\eta\, v\|_{L_2(\Omega)}. \tag{2.21}$$

We have

$$\begin{aligned} \|\Delta(\bar\eta\, v) + \mu\, \bar\eta\, v\|^2_{L_2(\Omega)} &= \|\bar\eta\, \underbrace{(\Delta v + \mu\, v)}_{=0} + 2(\nabla\bar\eta \cdot \nabla v) + v\Delta\bar\eta\|^2_{L_2(\Omega)} \\ &\leqslant 2\|2(\nabla\bar\eta \cdot \nabla v)\|^2_{L_2(\Omega)} + 2\|v\,\Delta\bar\eta\|^2_{L_2(\Omega)}, \end{aligned} \tag{2.22}$$

where $(\nabla\bar\eta \cdot \nabla v)$ is the scalar product between two vectors in $\mathbb{R}^{d+1}$. Due to the cut-off function $\bar\eta$, one only needs to integrate over a part of the spherical shell around the point $(0, \boldsymbol{y}_\Gamma)$:

$$Q_\delta = \{(x, \boldsymbol{y}) \in \Omega_2 \;:\; \delta < |(x, \boldsymbol{y}) - (0, \boldsymbol{y}_\Gamma)| < 2\delta\}. \tag{2.23}$$

It is therefore convenient to introduce the spherical coordinates in $\mathbb{R}^{d+1}$ centered at $(0, \boldsymbol{y}_\Gamma)$, with the North pole directed along the positive x-axis.

For the first term in (2.22), we have

$$\|(\nabla v \cdot \nabla\bar\eta)\|^2_{L_2(\Omega)} = \int_{Q_\delta} dx\, d\boldsymbol{y}\, |\big(\nabla v \cdot \nabla\bar\eta\big)|^2 = \int_{Q_\delta} dx\, d\boldsymbol{y} \left(\frac{\partial v}{\partial r}\,\frac{d\bar\eta(r/\delta)}{dr}\right)^2, \tag{2.24}$$

because the function $\bar\eta$ varies only along the radial direction. Since v is an analytic function inside Ω_2, $\partial v/\partial r$ is bounded over Ω_2, so that

$$\|(\nabla v \cdot \nabla\bar\eta)\|^2_{L_2(\Omega)} \leqslant \max_{(x,\boldsymbol{y})\in\Omega_2} \left\{\left(\frac{\partial v}{\partial r}\right)^2\right\} \int_\delta^{2\delta} dr\, r^d \left(\frac{d\eta(r/\delta)}{dr}\right)^2 \int_{x>0} d\Theta_d, \tag{2.25}$$

where $d\Theta_d$ includes all angular coordinates taken over the half of the sphere ($x > 0$). Changing the integration variable, one gets

$$\|(\nabla v \cdot \nabla\bar\eta)\|^2_{L_2(\Omega)} \leqslant C_1 \delta^{d-1}, \tag{2.26}$$

with a constant

$$C_1 = \max_{(x,\boldsymbol{y})\in\Omega_2}\left\{\left(\frac{\partial v}{\partial r}\right)^2\right\}\int_1^2 dr\, r^d\left(\frac{d\eta(r)}{dr}\right)^2\int_{x>0} d\Theta_d. \tag{2.27}$$

For the second term in (2.22), we have

$$\begin{aligned}\|v\,\Delta\bar\eta\|^2_{L_2(\Omega)} &= \int_{Q_\delta} dx\,d\boldsymbol{y}\,(2/a_2)\,\sin^2(\pi n x/a_2)\,\psi_k^2(\boldsymbol{y})\,(\Delta\bar\eta)^2\\ &\leqslant (2/a_2)\max_{\boldsymbol{y}\in\mathcal{S}}\{\psi_k^2(\boldsymbol{y})\}\int_{Q_\delta} dx\,d\boldsymbol{y}\,(\pi n x/a_2)^2\,(\Delta\bar\eta)^2,\end{aligned} \tag{2.28}$$

where we used the explicit form of the eigenfunction v from (2.19), the inequality $\sin x < x$ for $x > 0$, and the boundedness of $\psi_k(\boldsymbol{y})$ over a bounded cross-section $\mathcal{S}$. Denoting $C_2 = (2/a_2)(\pi n/a_2)^2\max_{\boldsymbol{y}\in\mathcal{S}}\{\psi_k^2(\boldsymbol{y})\}$, we get

$$\begin{aligned}\|v\,\Delta\bar\eta\|^2_{L_2(\Omega)} &\leqslant C_2\int_\delta^{2\delta} dr\,r^d\int_{x>0} d\Theta_d\,r^2\cos^2\theta\left(\frac{1}{r^d}\frac{d}{dr}r^d\frac{d}{dr}\eta(r/\delta)\right)^2\\ &\leqslant C_2\delta^{d-1}\int_1^2 dr\,r^{d+2}\left(\frac{1}{r^d}\frac{d}{dr}r^d\frac{d}{dr}\eta(r)\right)^2\int_{x>0} d\Theta_d,\end{aligned} \tag{2.29}$$

where θ is the azimuthal angle from the x-axis. The remaining integral is just a constant which depends on the particular choice of the cut-off function $\eta(r)$. We get thus

$$\|v\,\Delta\bar\eta\|^2_{L_2(\Omega)} \leqslant C_2'\delta^{d-1}, \tag{2.30}$$

with a new constant C_2'.

Combining inequalities (2.22, 2.26, 2.30), we obtain

$$\|\Delta(\bar\eta\,v) + \mu\,\bar\eta\,v\|^2_{L_2(\Omega)} \leqslant C\,\delta^{d-1}, \tag{2.31}$$

with a new constant C.

On the other hand,

$$\begin{aligned}\|\bar\eta\,v\|^2_{L_2(\Omega)} &= \int_{\Omega_2} dx\,d\boldsymbol{y}\,\bar\eta^2\,v^2 = 1 - \int_{B_{2\delta}} dx\,d\boldsymbol{y}\,(1-\bar\eta^2)\,v^2\\ &\geqslant 1 - \max_{\Omega_2}\{v^2\}\int_0^{2\delta} dr\,r^d\int_{x>0} d\Theta_d\,\eta^2(r/\delta) = 1 - C_0\delta^{d+1},\end{aligned} \tag{2.32}$$

where C_0 is a constant, $B_{2\delta} = \{(x,\boldsymbol{y})\in\Omega_2\ :\ |(x,\boldsymbol{y}) - (0,\boldsymbol{y}_\Gamma)| < 2\delta\}$ is a half-ball of radius 2δ centered at $(0,\boldsymbol{y}_\Gamma)$, and we used the $L_2(\Omega_2)$ normalization of v and its boundedness. In other words, for δ small enough, the left-hand side of (2.32) can be bounded from below by a strictly positive constant. Recalling that 2δ is the diameter of Γ, we conclude that inequality (2.21) holds with $\varepsilon = C'(\mathrm{diam}\{\Gamma\})^{d-1}$, with some $C' > 0$. If the diameter is small enough, Lemma 2.1 implies the existence of an eigenvalue λ of the Dirichlet Laplacian in Ω close to μ, which is an eigenvalue in Ω_2 that completes the proof. ■

REMARK 2.3. In this proof, the diameter 2δ of the opening Γ plays the role of a small parameter, whereas the geometric structure of Γ does not matter. If Γ is the union of a finite number of disjoint 'holes', the above estimates can be improved by constructing cut-off functions around each 'hole', if the holes are well separated from each other. In this way, the diameter of Γ can be replaced by diameters of each 'hole'. Note also that the proof is not applicable to a narrow but elongated opening (for example, $\Gamma = (0, h) \times (0, 1/2)$ inside the square cross-section $\mathcal{S} = (0, 1) \times (0, 1)$ with small h): even if the Lebesgue measure of the opening can be arbitrarily small, its diameter can remain large. We expect that the theorem might be extended to such situations but finer estimates are needed.

2.3 *Behavior of the first eigenvalue for the case $d = 1$*

The above proof is not applicable in the planar case (with $d = 1$). For this reason, we provide another proof which is based on the variational analysis of the modified eigenvalue problem with the sesquilinear form $\mathfrak{a}_\lambda(u, v)$. This proof also serves us as an illustration of advantages of mode matching methods. For the sake of simplicity, we only focus on the behavior of the first (smallest) eigenvalue.

Without loss of generality, we assume that

$$a_1 \geqslant a_2, \tag{2.33}$$

that is, the subdomain Ω_1 is larger than or equal to Ω_2.

LEMMA 2.4 (DOMAIN MONOTONICITY) *The domain monotonicity for the Dirichlet Laplacian implies the following inequalities for the first eigenvalue λ_1 of the Dirichlet Laplacian in Ω:*

$$\nu_1 + \frac{\pi^2}{(a_1 + a_2)^2} < \lambda_1 < \nu_1 + \frac{\pi^2}{a_1^2}, \tag{2.34}$$

where $\nu_1 + \pi^2/(a_1 + a_2)^2$ is the smallest eigenvalue in $(-a_1, a_2) \times \mathcal{S}$, $\nu_1 + \pi^2/a_1^2$ is the smallest eigenvalue in $(-a_1, 0) \times \mathcal{S}$, and ν_1 is the smallest eigenvalue in $\mathcal{S}$.

This lemma implies that $\gamma_1 = \sqrt{\nu_1 - \lambda_1}$ is purely imaginary. To ensure that the other γ_n with $n \geqslant 2$ are positive, we assume that

$$a_1 \geqslant \frac{\pi}{\sqrt{\nu_2 - \nu_1}}. \tag{2.35}$$

THEOREM 2.5. *Let $a_1 \geqslant 1/\sqrt{3}$ and $0 < a_2 \leqslant a_1$ be two real numbers, $\mathcal{S} = (0, 1)$, Γ be a nonempty subset of $\mathcal{S}$, and*

$$\Omega = \big((-a_1, a_2) \times \mathcal{S}\big) \backslash \big(\{0\} \times (\mathcal{S} \backslash \Gamma)\big), \quad \Omega_1 = (-a_1, 0) \times \mathcal{S}, \quad \Omega_2 = (0, a_2) \times \mathcal{S}. \tag{2.36}$$

Let $\mu = \pi^2 + \pi^2/a_1^2$ be the smallest eigenvalue of the Dirichlet Laplacian in the (larger) rectangle Ω_1. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any opening Γ with $\operatorname{diam}\{\Gamma\} = h < \delta$, the first eigenvalue λ_1 of the Dirichlet Laplacian in Ω satisfies $|\lambda_1 - \mu| < \varepsilon$.

PROOF. In a nutshell, the idea of the proof is to study the smallest eigenvalue $\eta_1(\lambda)$ of an auxiliary eigenvalue problem, which depends on λ as a free parameter such that $\eta_1(\lambda)$ vanishes when $\lambda = \lambda_1$ is a Dirichlet Laplacian eigenvalue in Ω. To prove the existence of a zero λ_1 of the continuous function

$\eta_1(\lambda)$, we demonstrate that (i) $\eta_1(\lambda)$ is negative when λ is approaching μ for a fixed opening Γ, and (ii) $\eta_1(\lambda)$ becomes positive when the opening Γ shrinks (with a fixed λ). The second statement means that for any small enough $\varepsilon > 0$, there exists $\delta > 0$ such that for any Γ with $\mathrm{diam}\{\Gamma\} < \delta$, one has $\eta_1(\mu - \varepsilon) > 0$. In turn, the first statement implies that, for a fixed $\mathrm{diam}\{\Gamma\}$ (smaller than δ), there exists λ' such that $\mu - \varepsilon < \lambda' < \mu$ and $\eta_1(\lambda') < 0$. The continuity of $\eta_1(\lambda)$ implies the existence of λ_1, lying between $\mu - \varepsilon$ and λ', such that $\eta_1(\lambda_1) = 0$. As a consequence, λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω such that $|\lambda_1 - \mu| < \varepsilon$.

As discussed earlier, the eigenfunction u in the whole domain is fully determined by its restriction on the opening Γ that obeys (2.12). In order to derive estimates on the eigenvalue λ, we introduce an auxiliary eigenvalue problem for the sesquilinear form $\mathfrak{a}_\lambda(u, v)$

$$\mathfrak{a}_\lambda\big(u, v\big) = \eta(\lambda)\big(u, v\big)_{L_2(\Gamma)} \qquad \forall\, v \in H^{\frac{1}{2}}(\Gamma), \tag{2.37}$$

where λ is considered as a free parameter. Each eigenpair $\{\eta(\lambda), u\}$ of this problem also depends on λ. The value of λ, at which $\eta(\lambda)$ vanishes (and thus (2.37) reduces to (2.12)), is an eigenvalue of the original problem for the Dirichlet Laplacian in Ω.

The smallest eigenvalue can then be written as

$$\eta_1(\lambda) = \inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma) \\ v \neq 0}} \left\{ \frac{F(v)}{||v||^2_{L_2(\Gamma)}} \right\}, \tag{2.38}$$

with

$$F(v) = \sum_{n=1}^{\infty} \gamma_n \big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big]\big(v, \psi_n\big)^2_{L_2(\Gamma)}. \tag{2.39}$$

If $\eta_1(\lambda_c) = 0$ at some λ_c, then λ_c is an eigenvalue of the Dirichlet Laplacian in Ω. By construction, we focus on $\lambda_c < \mu$ so that λ_c is the first eigenvalue λ_1, according to (2.34). We will prove that the zero λ_c of $\eta_1(\lambda)$ converges to μ as the opening Γ shrinks.

First, we rewrite the functional $F(v)$ under the assumptions (2.33, 2.35) ensuring that γ_1 is purely imaginary while γ_n with $n \geqslant 2$ are positive:

$$\begin{aligned} F(v) = {} & |\gamma_1|\big[\mathrm{ctan}(|\gamma_1|a_1) + \mathrm{ctan}(|\gamma_1|a_2)\big]\big(v, \psi_1\big)^2_{L_2(\Gamma)} \\ & + \sum_{n=2}^{\infty} \gamma_n \big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big]\big(v, \psi_n\big)^2_{L_2(\Gamma)}. \end{aligned} \tag{2.40}$$

On one hand, an upper bound reads

$$\begin{aligned} & \sum_{n=2}^{\infty} \gamma_n \big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big]\big(v, \psi_n\big)^2_{L_2(\Gamma)} \\ & \leqslant C_1 \sum_{n=2}^{\infty} \sqrt{\nu_n}\big(v, \psi_n\big)^2_{L_2(\Gamma)} \leqslant C_2 \|v\|^2_{H^{\frac{1}{2}}(\Gamma)}, \end{aligned} \tag{2.41}$$

because $\gamma_n = \sqrt{\nu_n - \lambda} \leqslant C_1' \sqrt{\nu_n}$ for all $n \geqslant 2$, where $C_1' > 0$ is a constant, and $\nu_n = \pi^2 n^2$ are the Dirichlet Laplacian eigenvalues in $\mathcal{S} = (0, 1)$. As a consequence,

$$\eta_1(\lambda) \leqslant \frac{F(v)}{\|v\|^2_{L_2(\Gamma)}} \leqslant |\gamma_1|\big[\mathrm{ctan}(|\gamma_1|a_1) + \mathrm{ctan}(|\gamma_1|a_2)\big]\frac{\big(v, \psi_1\big)^2_{L_2(\Gamma)}}{\|v\|^2_{L_2(\Gamma)}} + C_2 \frac{\|v\|^2_{H^{\frac{1}{2}}(\Gamma)}}{\|v\|^2_{L_2(\Gamma)}}, \tag{2.42}$$

where v can be any smooth function from $H^{\frac{1}{2}}(\Gamma)$ that vanishes at $\partial\Gamma$. Choosing a test function v, which is not orthogonal to ψ_1, so that $\big(v, \psi_1\big)_{L_2(\Gamma)} \neq 0$, one concludes that the limit $\mathrm{ctan}(|\gamma_1|a_1) \to -\infty$ as $|\gamma_1|a_1 \to \pi$ implies *negative* values of $\eta_1(\lambda)$ for λ approaching μ.

On the other hand, for any fixed λ, we will show that $\eta_1(\lambda)$ becomes positive as $\mathrm{diam}\{\Gamma\} \to 0$. For this purpose, we write

$$\begin{aligned}\eta_1(\lambda) &= \inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma)\\ v \neq 0}} \left\{ \beta \frac{\big(v, \psi_1\big)^2_{L_2(\Gamma)}}{||v||^2_{L_2(\Gamma)}} + \frac{F_1(v)}{||v||^2_{L_2(\Gamma)}} \right\},\\ &\geqslant \inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma)\\ v \neq 0}} \left\{ \beta \frac{\big(v, \psi_1\big)^2_{L_2(\Gamma)}}{||v||^2_{L_2(\Gamma)}} \right\} + \inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma)\\ v \neq 0}} \left\{ \frac{F_1(v)}{||v||^2_{L_2(\Gamma)}} \right\},\end{aligned} \tag{2.43}$$

with

$$\beta = |\gamma_1|\big[\mathrm{ctan}(|\gamma_1|a_1) + \mathrm{ctan}(|\gamma_1|a_2) - \coth(|\gamma_1|a_1) - \coth(|\gamma_1|a_2)\big] \tag{2.44}$$

and

$$F_1(v) = \sum_{n=1}^{\infty} |\gamma_n|\big[\coth(|\gamma_n|a_1) + \coth(|\gamma_n|a_2)\big]\big(v, \psi_n\big)^2_{L_2(\Gamma)}. \tag{2.45}$$

If $\beta \geqslant 0$, the first infimum in (2.43) is bounded from below by 0. If $\beta < 0$, the first infimum is bounded from below as

$$\inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma)\\ v \neq 0}} \left\{ \beta \frac{\big(v, \psi_1\big)^2_{L_2(\Gamma)}}{||v||^2_{L_2(\Gamma)}} \right\} = -|\beta| \sup_{\substack{v \in H^{\frac{1}{2}}(\Gamma)\\ v \neq 0}} \left\{ \frac{\big(v, \psi_1\big)^2_{L_2(\Gamma)}}{||v||^2_{L_2(\Gamma)}} \right\} \geqslant -|\beta|, \tag{2.46}$$

because

$$0 \leqslant \frac{\big(v, \psi_n\big)^2_{L_2(\Gamma)}}{||v||^2_{L_2(\Gamma)}} \leqslant ||\psi_n||^2_{L_2(\Gamma)} \leqslant ||\psi_n||^2_{L_2(\mathcal{S})} = 1 \qquad (\forall\, n \geqslant 1). \tag{2.47}$$

We conclude that

$$\eta_1(\lambda) \geqslant \min\{\beta, 0\} + \inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma)\\ v \neq 0}} \left\{ \frac{F_1(v)}{||v||^2_{L_2(\Gamma)}} \right\}. \tag{2.48}$$

Since β does not depend on Γ, it remains to check that the second term diverges as $\mathrm{diam}\{\Gamma\} \to 0$ that would ensure positive values for $\eta_1(\lambda)$. Since

$$F_1(v) \geqslant CF_0(v), \qquad F_0(v) = \sum_{n=1}^{\infty} \sqrt{\nu_n}\, \big(v, \psi_n\big)^2_{L_2(\Gamma)} \tag{2.49}$$

for some constant $C > 0$, it is enough to prove that

$$\inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma) \\ v \neq 0}} \left\{ \frac{F_0(v)}{||v||_{L_2(\Gamma)}} \right\} \xrightarrow[\mathrm{diam}\{\Gamma\} \to 0]{} +\infty. \tag{2.50}$$

This is proved in Lemma 2.6. When the opening Γ shrinks, $\eta_1(\lambda)$ becomes *positive*.

Since $\eta_1(\lambda)$ is a continuous function of λ (see Lemma 2.7), there should exist a value λ_c at which $\eta_1(\lambda_c) = 0$. This is the smallest eigenvalue of the Dirichlet Laplacian in Ω, which is close to μ. This completes the proof of the theorem. ■

LEMMA 2.6. *For a nonempty set* $\Gamma \subset (0, 1)$*, we have*

$$I(\Gamma) \equiv \inf_{\substack{v \in H^{\frac{1}{2}}(\Gamma) \\ v \neq 0}} \left\{ \frac{\sum_{n=1}^{\infty} n\big(v, \sin(\pi n y)\big)^2_{L_2(\Gamma)}}{\big(v, v\big)_{L_2(\Gamma)}} \right\} \xrightarrow[\mathrm{diam}\{\Gamma\} \to 0]{} +\infty. \tag{2.51}$$

PROOF. The minimax principle implies that $I(\Gamma) \geqslant I(\Gamma')$ for any Γ' such that $\Gamma \subset \Gamma'$. When $h = \mathrm{diam}\{\Gamma\}$ is small, one can choose $\Gamma' = (p/q, (p+1)/q)$, with two integers p and q. For instance, one can set $q = \lfloor 1/h \rfloor$, where $\lfloor 1/h \rfloor$ is the integer part of $1/h$ (the largest integer that is less than or equal to $1/h$). Since the removal of a subsequence of positive terms does not increase the sum,

$$\sum_{n=1}^{\infty} n\big(v, \sin(\pi n y)\big)^2_{L_2(\Gamma')} \geqslant \sum_{k=1}^{\infty} n_k\big(v, \sin(\pi n_k y)\big)^2_{L_2(\Gamma')}, \tag{2.52}$$

one gets by setting $n_k = kq$:

$$\frac{\sum_{n=1}^{\infty} n\big(v, \sin(\pi n y)\big)^2_{L_2(\Gamma')}}{\big(v, v\big)_{L_2(\Gamma')}} \geqslant q \frac{\sum_{k=1}^{\infty} k\big(v, \sin(\pi k q y)\big)^2_{L_2(\Gamma')}}{\big(v, v\big)_{L_2(\Gamma')}}. \tag{2.53}$$

Since the sine functions $\sin(\pi k q y)$ form a complete basis of $L_2(\Gamma')$, the infimum of the right-hand side can be easily computed and is equal to q, from which

$$I(\Gamma) \geqslant I(\Gamma') \geqslant q = \lfloor 1/h \rfloor \geqslant \frac{1}{2h} \xrightarrow[h \to 0]{} +\infty \tag{2.54}$$

that completes the proof. ■

LEMMA 2.7. *$\eta_1(\lambda)$ from (2.38) is a continuous function of λ for $\lambda \in (\nu_1, \nu_1 + \pi^2/a_1^2)$.*

PROOF. Let us denote the coefficients in (2.39) as

$$\beta_n(\lambda) = \begin{cases} |\gamma_1|\big[\operatorname{ctan}(|\gamma_1|a_1) + \operatorname{ctan}(|\gamma_1|a_2)\big] & (n = 1), \\ \gamma_n\big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big] & (n \geqslant 2). \end{cases} \tag{2.55}$$

We recall that $|\gamma_1| = \sqrt{\lambda - \nu_1}$ and $\gamma_n = \sqrt{\nu_n - \lambda}$ for $n \geqslant 2$. Under the assumptions (2.33, 2.35), one can easily check that all $\beta_n(\lambda)$ are continuous functions of λ when $\lambda \in (\nu_1, \nu_1 + \pi^2/a_1^2)$. In addition, all $\beta_n(\lambda)$ with $n \geqslant 2$ have an upper bound uniformly on λ

$$\beta_n(\lambda) \leqslant \sqrt{\nu_n - \lambda}\,\big(\coth(\gamma_2 a_1) + \coth(\gamma_2 a_2)\big) \leqslant C\sqrt{\nu_n}, \tag{2.56}$$

where

$$C = \coth(\sqrt{\nu_2 - \nu_1}\, a_1) + \coth(\sqrt{\nu_2 - \nu_1}\, a_2) \tag{2.57}$$

(here we replaced λ by its minimal value ν_2). Under these conditions, it was shown in (**52**) that $\eta_1(\lambda)$ is a continuous function of λ that completes the proof. ■

Theorem 2.5 claims that when the diameter of the opening set Γ is small, the first eigenvalue in the perforated rectangle Ω is close to the first eigenvalue in the larger subset Ω_1. This result is similar to the conclusion for the classic problem of dumbbell shapes in which two subsets are connected by a thin long channel. However, former asymptotic results on dumbbell shapes relied on the width-to-length ratio of the channel as a small parameter, which is not applicable in our situation.

The proof of Theorem 2.5 can be extended to demonstrate a much stronger result.

THEOREM 2.8. *Let $a_1 \geqslant 1/\sqrt{3}$ and $0 < a_2 \leqslant a_1$ be two real numbers, $\mathcal{S} = (0, 1)$, Γ is the union of a finite number of intervals, $\Gamma = \bigcup_{i=1}^N (c_i, d_i)$, with*

$$0 \leqslant c_1 < d_1 < \cdots < c_i < d_i < \cdots < c_N < d_N \leqslant 1, \tag{2.58}$$

and

$$\Omega = \big((-a_1, a_2) \times \mathcal{S}\big)\backslash\big(\{0\} \times (\mathcal{S}\backslash\Gamma)\big), \quad \Omega_1 = (-a_1, 0) \times \mathcal{S}, \quad \Omega_2 = (0, a_2) \times \mathcal{S}. \tag{2.59}$$

Let $\mu = \pi^2 + \pi^2/a_1^2$ be the smallest eigenvalue of the Dirichlet Laplacian in the (larger) rectangle Ω_1. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for an opening Γ with $\max\limits_{1\leqslant i\leqslant N}\{(d_i - c_i)\} = h < \delta$, there exists an eigenvalue λ of the Dirichlet Laplacian in Ω such that $|\lambda - \mu| < \varepsilon$.

The proof of this theorem remains the same as that of Theorem 2.5, the only change concerns Lemma 2.6, which can be replaced by Lemma 2.9.

LEMMA 2.9. *Let $\Gamma_i = (c_i, d_i)$ and $\Gamma = \bigcup_{i=1}^N \Gamma_i$, where c_i and d_i satisfy (2.58). Setting $h_i = \operatorname{diam}\{\Gamma_i\} = d_i - c_i$ and $h = \max\limits_{1\leqslant i\leqslant N}\{h_i\}$, we have*

$$I(\Gamma) \equiv \inf_{\substack{v\in H^{\frac{1}{2}}(\Gamma)\\ v\neq 0}} \left\{ \frac{\sum\limits_{n=1}^{\infty} n\big(v, \sin(\pi n y)\big)^2_{L_2(\Gamma)}}{\big(v, v\big)_{L_2(\Gamma)}} \right\} \xrightarrow[h\to 0]{} +\infty. \tag{2.60}$$

PROOF. As Γ is a union of disjoint intervals, a function v from $H^{\frac{1}{2}}(\Gamma)$ can be replaced by a set of functions v_i from $H^{\frac{1}{2}}(\Gamma_i)$, that is,

$$I(\Gamma) = \inf_{\substack{v_1 \in H^{\frac{1}{2}}(\Gamma_1), \cdots, v_N \in H^{\frac{1}{2}}(\Gamma_N) \\ \sum_i (v_i, v_i)_{L_2(\Gamma_i)} \neq 0}} \left\{ \frac{\sum_{i=1}^{N} \sum_{n=1}^{\infty} n\big(v_i, \sin(\pi n y)\big)^2_{L_2(\Gamma_i)}}{\sum_{i=1}^{N} \big(v_i, v_i\big)_{L_2(\Gamma_i)}} \right\}. \tag{2.61}$$

One can treat the elements v_i independently. When h is small enough, each subinterval Γ_i can be increased to Γ_i' in such a way that there exists a positive integer q_i such that the functions $\sin(\pi k q_i y)$ form a complete basis in $L^2(\Gamma_i')$. For instance, one can set $\Gamma_i' = (p_i/q_i, (p_i+1)/q_i)$ with $q_i = \lfloor 1/h_i \rfloor$ and p_i being respectively an integer part of $1/h_i$ and an appropriate integer 'shift'. Since we treat v_i independently, the subintervals Γ_i' can be overlapping.

In each sum, we select a subsequence of terms with $n_k = q_i k$ $(k = 1, 2, \ldots)$ and remove other terms. The increase of the domain and the removal of a subsequence of positive terms can only decrease the infimum so that

$$I(\Gamma) \geqslant \inf_{\substack{v_1 \in H^{\frac{1}{2}}(\Gamma_1'), \cdots, v_N \in H^{\frac{1}{2}}(\Gamma_N') \\ \sum_i (v_i, v_i)_{L_2(\Gamma_i')} \neq 0}} \left\{ \frac{\sum_{i=1}^{N} \sum_{k=1}^{\infty} q_i k\big(v_i, \sin(\pi q_i k y)\big)^2_{L_2(\Gamma_i')}}{\sum_{i=1}^{N} \big(v_i, v_i\big)_{L_2(\Gamma_i')}} \right\}. \tag{2.62}$$

Since $\sin(\pi k q_i y)$ is a complete basis in $L^2(\Gamma_i')$, one has

$$\sum_{k=1}^{\infty} k\big(v_i, \sin(\pi q_i k y)\big)^2_{L_2(\Gamma_i')} \geqslant \sum_{k=1}^{\infty} \big(v_i, \sin(\pi q_i k y)\big)^2_{L_2(\Gamma_i')} = \big(v_i, v_i\big)_{L_2(\Gamma_i')}. \tag{2.63}$$

Using $q_i \geqslant q = \min_i\{q_i\}$, one obtains that the expression in large curved parentheses is bounded from below by q so that

$$I(\Gamma) \geqslant q. \tag{2.64}$$

Since the minimal q_i corresponds to the maximal subinterval, one has $q = \lfloor 1/h \rfloor$, and thus

$$I(\Gamma) \geqslant \lfloor 1/h \rfloor \geqslant \frac{1}{2h} \xrightarrow[h \to 0]{} +\infty \tag{2.65}$$

that completes the proof. ■

REMARK 2.10. The Theorem 2.8 shows that the control parameter for the localization (see below) is not the diameter of the whole opening set Γ (which can be comparable to the diameter of the rectangle) but the diameter of its largest component. Moreover, the distances between successive opening subintervals Γ_i can be arbitrarily small. This is a much stronger result that the former Theorems 2.2 and 2.5. In fact, these distances do not play any role in the above proof. In other words, the barrier can be formed by arbitrarily short segments but the eigenfunction is still localized in a larger subregion of the domain (see the next subsection). This localization behavior can be related

to the known probabilistic results that a Brownian motion has a very small probability to cross such a perforated barrier with absorbing segments (**33**). It also resembles the numerically observed localization of thin plate vibrations induced by one clamped point and governed by a bi-Laplacian operator (**53**).

2.4 *First eigenfunction*

Our first goal is to show that the first eigenfunction u (with the smallest eigenvalue λ) is 'localized' in the larger subdomain when the opening Γ is small enough. By localization we understand that the L_2-norm of the eigenfunction in the smaller domain vanishes as the opening shrinks, that is, $\operatorname{diam}\{\Gamma\} \to 0$.

We will obtain a stronger result by estimating the ratio of L_2-norms of the eigenfunction in two arbitrary cross-sections $\mathcal{S}_{x_1}$ and $\mathcal{S}_{x_2}$ on two sides of the barrier (that is, with $x_1 \in (-a_1, 0)$ and $x_2 \in (0, a_2)$) and showing its divergence as the opening shrinks. We also discuss the rate of divergence.

Now we formulate the main Theorem 2.11.

THEOREM 2.11. *Let $\mathcal{S}$ be a bounded domain in $\mathbb{R}^d$ ($d > 1$) with a piecewise smooth boundary $\partial\mathcal{S}$, ν_n are the ordered Dirichlet Laplacian eigenvalues in $\mathcal{S}$, Γ be a nonempty subset of $\mathcal{S}$, and*

$$\Omega = \big((-a_1, a_2) \times \mathcal{S}\big)\backslash\big(\{0\} \times (\mathcal{S}\backslash\Gamma)\big), \quad \Omega_1 = (-a_1, 0) \times \mathcal{S}, \quad \Omega_2 = (0, a_2) \times \mathcal{S}, \tag{2.66}$$

with two constants a_1 and a_2 such that

$$a_1 \geqslant \frac{\pi}{\sqrt{\nu_2 - \nu_1}}, \qquad 0 < a_2 < a_1 . \tag{2.67}$$

Let u be the first Dirichlet Laplacian eigenfunction (with the smallest eigenvalue λ) in Ω. Then the ratio of squared L_2-norms of u in two arbitrary cross-sections $\mathcal{S}_{x_1}$ (for any $x_1 \in (-a_1, 0)$) and $\mathcal{S}_{x_2}$ (for any $x_2 \in (0, a_2)$) on two sides of the barrier diverges as the opening Γ shrinks. More precisely,

$$\frac{I(x_1)}{I(x_2)} \geqslant C\frac{\sin^2(|\gamma_1|(a_1 + x_1))}{\sin(|\gamma_1|a_1)} \xrightarrow[\operatorname{diam}\{\Gamma\}\to 0]{} +\infty, \tag{2.68}$$

where $C > 0$ is a constant, and $I(x)$ is defined in (2.14).

PROOF. According to Theorem 2.5, $\lambda \to \nu_1 + \pi^2/a_1^2$ as $\operatorname{diam}\{\Gamma\} \to 0$. We denote then

$$\lambda = \nu_1 + \pi^2/a_1^2 - \varepsilon, \tag{2.69}$$

with a small parameter ε that vanishes as $\operatorname{diam}\{\Gamma\} \to 0$. As a consequence, γ_1 is purely imaginary, and $|\gamma_1|a_1 \simeq \pi - a_1^2\varepsilon/(2\pi)$. In turn, the assumption (2.67) implies that all γ_n with $n = 2, 3, \ldots$ are positive.

We get then

$$\begin{aligned} I(x_1) &= b_1^2\frac{\sin^2(|\gamma_1|(a_1 + x_1))}{\sin^2(|\gamma_1|a_1)} + \sum_{n=2}^{\infty} b_n^2\frac{\sinh^2(\gamma_n(a_1 + x_1))}{\sinh^2(\gamma_n a_1)} \\ &\geqslant b_1^2\frac{\sin^2(|\gamma_1|(a_1 + x_1))}{\sin^2(|\gamma_1|a_1)}, \end{aligned} \tag{2.70}$$

whereas

$$I(x_2) = b_1^2 \frac{\sin^2(|\gamma_1|(a_2 - x_2))}{\sin^2(|\gamma_1|a_2)} + \sum_{n=2}^{\infty} b_n^2 \frac{\sinh^2(\gamma_n(a_2 - x_2))}{\sinh^2(\gamma_n a_2)}$$
$$\leqslant b_1^2 \frac{\sin^2(|\gamma_1|(a_2 - x_2))}{\sin^2(|\gamma_1|a_2)} + \sum_{n=2}^{\infty} b_n^2, \tag{2.71}$$

because $\sinh(x)$ monotonously grows. The last sum can be estimated by rewriting the dispersion relation (2.13):

$$-b_1^2|\gamma_1|\big[\mathrm{ctan}(|\gamma_1|a_1) + \mathrm{ctan}(|\gamma_1|a_2)\big] = \sum_{n=2}^{\infty} b_n^2 \gamma_n \big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big]$$
$$\geqslant \gamma_2 \big[\coth(\gamma_2 a_1) + \coth(\gamma_2 a_2)\big] \sum_{n=2}^{\infty} b_n^2, \tag{2.72}$$

because $x\coth(x)$ monotonously grows. As a consequence, one gets

$$I(x_2) \leqslant b_1^2 \left[\frac{1}{\sin^2(|\gamma_1|a_2)} - \frac{|\gamma_1|\big[\mathrm{ctan}(|\gamma_1|a_1) + \mathrm{ctan}(|\gamma_1|a_2)\big]}{\gamma_2\big[\coth(\gamma_2 a_1) + \coth(\gamma_2 a_2)\big]} \right], \tag{2.73}$$

where we used $\sin^2(|\gamma_1|(a_2 - x_2)) \leqslant 1$. We conclude that

$$\frac{I(x_1)}{I(x_2)} \geqslant C_\lambda \frac{\sin^2(|\gamma_1|(a_1 + x_1))}{\sin(|\gamma_1|a_1)}, \tag{2.74}$$

where

$$C_\lambda = \left[\frac{\sin(|\gamma_1|a_1)}{\sin^2(|\gamma_1|a_2)} - \frac{|\gamma_1|\big[\cos(|\gamma_1|a_1) + \sin(|\gamma_1|a_1)\mathrm{ctan}(|\gamma_1|a_2)\big]}{\gamma_2\big[\coth(\gamma_2 a_1) + \coth(\gamma_2 a_2)\big]} \right]^{-1}. \tag{2.75}$$

Since $|\gamma_1|a_1 \to \pi$ as $\mathrm{diam}\{\Gamma\} \to 0$, the denominator $\sin(|\gamma_1|a_1)$ in (2.74) diverges, while C_λ in (2.75) converges to a strictly positive constant (because $a_2 < a_1$) that completes the proof of the theorem. ■

REMARK 2.12. In the two-dimensional case, when Γ is the union of disjoint intervals, the statement $\mathrm{diam}\{\Gamma\} \to 0$ can be relaxed to a weaker statement that the diameter of the largest component of Γ vanishes. In this case, one uses Theorem 2.8 instead of Theorem 2.5 in the beginning of the proof of Theorem 2.11.

REMARK 2.13. If two subdomains Ω_1 and Ω_2 are equal, that is, $a_1 = a_2$, the first eigenfunction cannot be localized due to the reflection symmetry: $u(x, \boldsymbol{y}) = u(-x, \boldsymbol{y})$. While the estimate (2.74) on the ratio of two squared L_2 norms remains valid, the constant C_λ in (2.75) vanishes as $|\gamma_1|a_1 \to \pi$ because $\sin(|\gamma_1|a_2) \to 0$.

REMARK 2.14. The assumption (2.67) was used to ensure that γ_n with $n \geqslant 2$ were positive while the corresponding modes were exponentially decaying. We checked numerically that this assumption is not a necessary condition for localization (see Appendix A).

REMARK 2.15. The above formulation of the mode matching method can be extended to two cylinders of different cross-sections $\mathcal{S}^1$ and $\mathcal{S}^2$ connected through an opening set Γ: $\Omega = \big(((-a_1, 0) \times \mathcal{S}^1) \cup ((0, a_2) \times \mathcal{S}^2)\big) \backslash \big(\{0\} \times (\mathcal{S} \backslash \Gamma)\big)$, where $S = S^1 \cap S^2$. For instance, the eigenfunction representation (2.5) would read as

$$u_1(x, \boldsymbol{y}) = \sum_{n=1}^{\infty} \big(u_{|\Gamma}, \psi_n^1\big)_{L_2(\Gamma)} \, \psi_n^1(\boldsymbol{y}) \, \frac{\sinh(\gamma_n^1(a_1 + x))}{\sinh(\gamma_n^1 a_1)} \quad (-a_1 < x < 0), \tag{2.76}$$

$$u_2(x, \boldsymbol{y}) = \sum_{n=1}^{\infty} \big(u_{|\Gamma}, \psi_n^2\big)_{L_2(\Gamma)} \, \psi_n^2(\boldsymbol{y}) \, \frac{\sinh(\gamma_n^2(a_2 - x))}{\sinh(\gamma_n^2 a_2)} \quad (0 < x < a_2), \tag{2.77}$$

with $\gamma_n^{1,2} = \sqrt{\nu_n^{1,2} - \lambda}$, where $\nu_n^{1,2}$ and $\psi_n^{1,2}$ are the eigenvalues and eigenfunctions of $\Delta_\perp$ in cross-sections $\mathcal{S}^1$ and $\mathcal{S}^2$. For instance, the dispersion relation reads

$$\sum_{n=1}^{\infty} \Big[\big(u_{|\Gamma}, \psi_n^1\big)^2_{L_2(\Gamma)} \gamma_n^1 \coth(\gamma_n^1 a_1) + \big(u_{|\Gamma}, \psi_n^2\big)^2_{L_2(\Gamma)} \gamma_n^2 \coth(\gamma_n^2 a_2) \Big] = 0. \tag{2.78}$$

The remaining analysis would be similar although statements about localization would be more subtle. In particular, the localization of the first eigenfunction does not necessarily occur in the subdomain with larger Lebesgue measure (**54**).

2.5 *Higher-order eigenfunctions*

Similar arguments can be applied to show localization of other eigenfunctions. When the eigenvalue λ is progressively increased, there are more and more purely imaginary γ_n and thus more and more oscillating terms in the representation of an eigenfunction. These oscillations start to interfere with each other, and localization is progressively reduced. When the characteristic wavelength $1/\sqrt{\lambda}$ becomes comparable to the size of the opening, no localization is expected. From these qualitative arguments, it is clear that proving localization for higher-order eigenfunctions becomes more challenging while some additional constraints are expected to appear. To illustrate these difficulties, we consider an eigenfunction of the Dirichlet Laplacian for which $|\gamma_2| a_1 \to \pi$, that is,

$$\lambda = \nu_2 + \pi^2/a_1^2 + \varepsilon, \tag{2.79}$$

with $\varepsilon \to 0$.

As earlier for the first eigenfunction, we estimate the squared L_2 norm of this eigenfunction in two arbitrary cross-sections $\mathcal{S}_{x_1}$ and $\mathcal{S}_{x_2}$. From (2.15), we have

$$I(x_1) \leqslant b_2^2 \frac{\sin^2(|\gamma_2|(a_1 + x_1))}{\sin^2(|\gamma_2| a_1)}, \tag{2.80}$$

where we kept only the leading term with $n = 2$ (for which the denominator diverges).

In order to estimate $I(x_2)$, we use the dispersion relation (2.13) to get

$$\begin{aligned}
&- b_2^2|\gamma_2|\big[\operatorname{ctan}(|\gamma_2|a_1) + \operatorname{ctan}(|\gamma_2|a_2)\big] \\
&= b_1^2|\gamma_1|\big[\operatorname{ctan}(|\gamma_1|a_1) + \operatorname{ctan}(|\gamma_1|a_2)\big] + \sum_{n=3}^{\infty} b_n^2\gamma_n\big[\coth(\gamma_n a_1) + \coth(\gamma_n a_2)\big] \\
&\geqslant b_1^2|\gamma_1|\big[\operatorname{ctan}(|\gamma_1|a_1) + \operatorname{ctan}(|\gamma_1|a_2)\big] + \gamma_3\big[\coth(\gamma_3 a_1) + \coth(\gamma_3 a_2)\big]\sum_{n=3}^{\infty} b_n^2 \\
&\geqslant C\left[b_1^2 + \sum_{n=3}^{\infty} b_n^2\right],
\end{aligned} \tag{2.81}$$

where

$$C = \min\Big\{|\gamma_1|\big[\operatorname{ctan}(|\gamma_1|a_1) + \operatorname{ctan}(|\gamma_1|a_2)\big], \gamma_3\big[\coth(\gamma_3 a_1) + \coth(\gamma_3 a_2)\big]\Big\} > 0, \tag{2.82}$$

and we assumed that

$$\operatorname{ctan}(|\gamma_1|a_1) + \operatorname{ctan}(|\gamma_1|a_2) > 0. \tag{2.83}$$

Using (2.81), we get

$$\begin{aligned}
I(x_2) &= b_1^2\frac{\sin^2(|\gamma_1|(a_2 - x_2))}{\sin^2(|\gamma_1|a_2)} + b_2^2\frac{\sin^2(|\gamma_2|(a_2 - x_2))}{\sin^2(|\gamma_2|a_2)} + \sum_{n=3}^{\infty} b_n^2\frac{\sinh^2(\gamma_n(a_2 - x_2))}{\sinh^2(\gamma_n a_2)} \\
&\leqslant \frac{b_1^2}{\sin^2(|\gamma_1|a_2)} + \frac{b_2^2}{\sin^2(|\gamma_2|a_2)} + \sum_{n=3}^{\infty} b_n^2 \\
&\leqslant \frac{b_2^2}{\sin^2(|\gamma_2|a_2)} + \frac{1}{\sin^2(|\gamma_1|a_2)}\left[b_1^2 + \sum_{n=3}^{\infty} b_n^2\right] \\
&\leqslant b_2^2\left[\frac{1}{\sin^2(|\gamma_2|a_2)} - \frac{|\gamma_2|\big[\operatorname{ctan}(|\gamma_2|a_1) + \operatorname{ctan}(|\gamma_2|a_2)\big]}{C\sin^2(|\gamma_1|a_2)}\right].
\end{aligned} \tag{2.84}$$

We finally obtain

$$\frac{I(x_1)}{I(x_2)} \geqslant C_\lambda\frac{\sin^2(|\gamma_2|(a_1 + x_1))}{\sin(|\gamma_2|a_1)}, \tag{2.85}$$

with

$$C_\lambda = \left(\frac{\sin(|\gamma_2|a_1)}{\sin^2(|\gamma_2|a_2)} - \frac{|\gamma_2|\big[\cos(|\gamma_2|a_1) + \sin(|\gamma_2|a_1)\operatorname{ctan}(|\gamma_2|a_2)\big]}{C\sin^2(|\gamma_1|a_2)}\right)^{-1}. \tag{2.86}$$

When $|\gamma_2|a_1 \to \pi$, the denominator in (2.85) diverges while C_λ converges to a strictly positive constant if $\sin(|\gamma_1|a_2)$ does not vanish. This additional constraint can be formulated as

$$\sqrt{a_2^2(\nu_2 - \nu_1)/\pi^2 + a_2^2/a_1^2} \notin \mathbb{N}. \tag{2.87}$$

REMARK 2.16. The additional constraints (2.83, 2.87) were used to ensure that $|\gamma_1|a_2$ does not converge to a multiple of π as $|\gamma_2|a_1 \to \pi$. Given that the lengths a_1 and a_2 can in general be chosen arbitrarily, one can easily construct domains, in which this condition is not satisfied. For instance, setting $|\gamma_1|a_2 = |\gamma_2|a_1 = \pi$, one gets the relation $1/a_2^2 - 1/a_1^2 = (\nu_2 - \nu_1)/\pi^2$ under which the constraint is not fulfilled. At the same time, numerical evidence (see Fig. C.4) suggests that the constraint (2.87) may potentially be relaxed.

REMARK 2.17. One can also consider eigenfunctions for which $|\gamma_n|a_1 \to \pi$. These modes exhibit 'one oscillation' in the lateral direction x and 'multiple oscillations' in the transverse directions $\boldsymbol{y}$. The analysis is very similar but additional constrains may appear. In turn, the analysis would be more involved in a more general situation when $|\gamma_n|a_1 \to \pi k$, with an integer k. Finally, one can investigate the eigenfunctions localized in the smaller domain Ω_2. In this situation, which is technically more subtle, one can get similar estimates on the ratio of squared L_2 norms. Moreover, Theorems 2.2 and 2.5 ensure the existence of an eigenvalue λ, which is close to the first eigenvalue in the smaller domain, $\lambda \to \nu_1 + \pi^2/a_2^2$, implying $|\gamma_1|a_2 \to \pi$. We do not provide rigorous statements for these extensions but some numerical examples are given in Appendix A.

3. Scattering problem with two barriers

Various scattering problems, their solutions and properties are discussed in (**2**). The existence of localized eigenmodes of the Laplace operator is the peculiar feature of scattering problems in locally inhomogeneous waveguides, as first discovered by Rellich (**35**) and then thoroughly investigated by many authors (**36–45**). However, the presence of localized eigenmodes can only lead to nonuniqueness of the solution but does not affect its existence, nor transmission and reflection coefficients (**55**).

We consider a scattering problem for an infinite cylinder $\tilde{\Omega}_0 = \mathbb{R} \times \mathcal{S}$ of arbitrary bounded cross-section $\mathcal{S} \subset \mathbb{R}^d$ with a piecewise smooth boundary $\partial\mathcal{S}$, with two barriers located at $x = 0$ and $x = 2a$. In other words, for a given 'opening' $\Gamma \subset \mathcal{S}$ inside the cross-section $\mathcal{S}$, we consider the domain $\tilde{\Omega} = \tilde{\Omega}_0 \backslash (\{0, 2a\} \times (\mathcal{S} \backslash \Gamma))$ (Fig. 3). As earlier, we denote the cross-section at x by $\mathcal{S}_x = \{x\} \times \mathcal{S}$.

An acoustic wave u with the wave number $\sqrt{\lambda}$ satisfies the Helmholtz equation

$$-\Delta u = \lambda u \quad \text{in } \tilde{\Omega}, \qquad u_{|\partial\tilde{\Omega}} = 0, \tag{3.1}$$

subject to the Dirichlet boundary condition at both the cylinder surface and two barriers. In contrast to spectral problems considered in section 2, the squared wave number, λ, is fixed, and one studies the propagation of such a wave along the waveguide. As earlier, we consider the auxiliary Dirichlet eigenvalue problem (2.2) in the cross-section $\mathcal{S}$ with ordered eigenvalues ν_k, and set $\gamma_n = \sqrt{\nu_n - \lambda}$. The first eigenvalue ν_1 determines the cut-off frequency below which no wave can travel in the waveguide of the cross-section $\mathcal{S}$. Here we focus on the waves near cut-off frequency and we assume that

$$\nu_1 < \lambda < \nu_2. \tag{3.2}$$

As a consequence, γ_1 is purely imaginary while all other γ_n are positive.

If there were no barriers, an incident wave of the form $e^{i|\gamma_1|x}\psi_1(\boldsymbol{y})$ would be a standing wave in the infinite cylinder. The barriers perturb this wave, generating reflected and transmissed waves in the waveguide. In a standard way, the radiation condition for the solution of the Hemlholtz equation

can be written as

$$u(x,\boldsymbol{y}) = \begin{cases} e^{i|\gamma_1|x}\,\psi_1(\boldsymbol{y}) + c_1^r\,e^{-i|\gamma_1|x}\,\psi_1(\boldsymbol{y}) + \displaystyle\sum_{n=2}^{\infty} c_n^r\,e^{\gamma_n x}\,\psi_n(\boldsymbol{y}) & (x<0), \\ c_1^t\,e^{i|\gamma_1|x}\,\psi_1(\boldsymbol{y}) + \displaystyle\sum_{n=2}^{\infty} c_n^t\,e^{-\gamma_n x}\,\psi_n(\boldsymbol{y}) & (x>2a), \end{cases} \tag{3.3}$$

where c_n^r and c_n^t are yet unknown coefficients of the reflected and transmitted waves, respectively. Our scattering problem consists in finding the solution of (3.1) that satisfies the radiation condition (3.3). This problem can be reduced to a Fredholm equation of the second kind, in which the right-hand side is determined by the incident wave in (3.3). This implies that a solution of the scattering problem exists for any $\nu_1 < \lambda < \nu_2$ but its uniqueness may not hold at the frequency of an eventual localized eigenmode, see (**55**).

Due to the reflection symmetry at $x = a$, an infinite cylinder with two barriers can be replaced by a semi-infinite cylinder, $\Omega_0 = (-\infty, a) \times \mathcal{S}$, with a single barrier at $x = 0$ so that $\Omega = \Omega_0 \backslash (\{0\} \times (\mathcal{S}\backslash\Gamma))$. It is easy to show that the solution of the above scattering problem in $\tilde{\Omega}$ can be reduced to the solution of a similar problem in Ω. For this purpose, we consider two problems in Ω:

(i) the Dirichlet problem

$$-\Delta u^D = \lambda u^D \quad \text{in } \Omega, \qquad u^D|_{\partial\Omega\backslash\mathcal{S}_a} = 0, \qquad u^D|_{\mathcal{S}_a} = 0, \tag{3.4}$$

with the radiation condition

$$u^D(x,\boldsymbol{y}) = e^{i|\gamma_1|x}\psi_1(\boldsymbol{y}) + c_1^D e^{-i|\gamma_1|x}\psi_1(\boldsymbol{y}) + \sum_{n=2}^{\infty} c_n^D e^{\gamma_n x}\psi_n(\boldsymbol{y}) \qquad (x<0) \tag{3.5}$$

and unknown reflection coefficients c_n^D;

(ii) the Neumann problem

$$-\Delta u^N = \lambda u^N \quad \text{in } \Omega, \qquad u^N|_{\partial\Omega\backslash\mathcal{S}_a} = 0, \qquad \frac{\partial u^N}{\partial n}|_{\mathcal{S}_a} = 0, \tag{3.6}$$

with the radiation condition

$$u^N(x,\boldsymbol{y}) = e^{i|\gamma_1|x}\psi_1(\boldsymbol{y}) + c_1^N e^{-i|\gamma_1|x}\psi_1(\boldsymbol{y}) + \sum_{n=2}^{\infty} c_n^N e^{\gamma_n x}\psi_n(\boldsymbol{y}) \qquad (x<0) \tag{3.7}$$

and unknown reflection coefficients c_n^N.

Extending these functions to the positive semi-infinite cylinder $(x > a)$ as $u^N(x,\boldsymbol{y}) = u^N(2a-x,\boldsymbol{y})$ and $u^D(x,\boldsymbol{y}) = -u^D(2a-x,\boldsymbol{y})$, we obtain two solutions u^D and u^N in the whole infinite cylinder

satisfying the following radiation conditions for $x > 2a$:

$$u^D(x,\mathbf{y}) = -e^{i|\gamma_1|(2a-x)}\psi_1(\mathbf{y}) - c_1^D e^{-i|\gamma_1|(2a-x)}\psi_1(\mathbf{y}) - \sum_{n=2}^{\infty} c_n^D e^{\gamma_n(2a-x)}\psi_n(\mathbf{y}),$$

$$u^N(x,\mathbf{y}) = e^{i|\gamma_1|(2a-x)}\psi_1(\mathbf{y}) + c_1^N e^{-i|\gamma_1|(2a-x)}\psi_1(\mathbf{y}) + \sum_{n=2}^{\infty} c_n^N e^{\gamma_n(2a-x)}\psi_n(\mathbf{y}).$$

The half sum of two solutions, $u = (u^D + u^N)/2$, satisfies (3.1, 3.3) and is thus equivalent to the original scattering problem, with $c_n^r = (c_n^D + c_n^N)/2$ $(n \geqslant 1)$, $c_1^t = (c_1^N - c_1^D)e^{-i|\gamma_1|2a}/2$, and $c_n^t = (c_n^N - c_n^D)e^{2\gamma_n a}/2$ $(n \geqslant 2)$. In the following, we will prove that $c_1^N = 1$ at the resonance frequency. In turn, c_1^D can be shown to be arbitrary close to -1 at the resonance frequency of the Neumann problem. This will imply that $c_1^r \approx 0$ and thus almost full wave transmission in the waveguide with two barriers.

In the rest of this section, we focus on the scattering problem in the semi-infinite cylinder Ω with a single barrier that splits the domain Ω into two subdomains: Ω_1 (for $x < 0$) and Ω_2 (for $0 < x < a$).

THEOREM 3.1. *For any opening Γ with a small enough Lebesgue measure $|\Gamma|$, there exists the critical wave number $\sqrt{\lambda_c}$ at which the wave is almost fully propagating across two barriers, that is, the reflection coefficient c_1^r can be made arbitrarily close to* 0.

PROOF. The proof consists in three steps: (i) we establish an explicit equation that determines c_1^N for any opening Γ; (ii) we prove the existence of the resonance frequency $\sqrt{\lambda_c}$ at which $c_1^N = 1$ when $|\Gamma|$ is small enough; and (iii) we prove that the reflection coefficient c_1^D of the Dirichlet problem can be made arbitrarily close to -1 by decreasing $|\Gamma|$.

Step 1. Taking the scalar product of (3.7) at $x = 0$ with ψ_1, one finds

$$c_1^N = (u^N_{|\Gamma}, \psi_1)_{L_2(\Gamma)} - 1. \tag{3.8}$$

Similarly, one finds $c_n^N = (u^N_{|\Gamma}, \psi_n)_{L_2(\Gamma)}$ so that the wave in the first domain Ω_1 is

$$u_1^N(x,\mathbf{y}) = e^{i|\gamma_1|x}\psi_1 + e^{-i|\gamma_1|x}\Big[\big(u^N_{|\Gamma}, \psi_1\big)_{L_2(\Gamma)} - 1\Big]\psi_1 + \sum_{n=2}^{\infty} e^{\gamma_n x}\big(u^N_{|\Gamma}, \psi_n\big)_{L_2(\Gamma)}\psi_n. \tag{3.9}$$

In the second domain Ω_2, we search for a solution of the Helmholtz equation (3.6) in the form

$$u_2^N(x,\mathbf{y}) = \tilde{c}_1^N \cos(|\gamma_1|(a-x))\,\psi_1 + \sum_{n=2}^{\infty} \tilde{c}_n^N \cosh(\gamma_n(a-x))\,\psi_n \qquad (0 < x < a), \tag{3.10}$$

where we separated the first oscillating term from the remaining exponentially decaying terms. The unknown coefficients $\tilde{c}_n^N$ are obtained again via the scalar product with ψ_n at $x = 0$:

$$u_2^N(x, \boldsymbol{y}) = \frac{\cos(|\gamma_1|(a-x))}{\cos(|\gamma_1|a)} \big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)} \psi_1 + \sum_{n=2}^{\infty} \frac{\cosh(\gamma_n(a-x))}{\cosh(\gamma_n a)} \big(u_{|\Gamma}^N, \psi_n\big)_{L_2(\Gamma)} \psi_n \qquad (0 < x < a). \tag{3.11}$$

Matching the derivatives u_1' and u_2' with respect to x at the opening Γ, one gets for any $\boldsymbol{y} \in \Gamma$:

$$2i|\gamma_1|\psi_1(\boldsymbol{y}) - i|\gamma_1|\big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)}\psi_1(\boldsymbol{y}) + \sum_{n=2}^{\infty} \gamma_n \big(u_{|\Gamma}^N, \psi_n\big)_{L_2(\Gamma)} \psi_n(\boldsymbol{y})$$
$$= |\gamma_1|\big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)} \tan(|\gamma_1|a)\psi_1(\boldsymbol{y}) - \sum_{n=2}^{\infty} \gamma_n \big(u_{|\Gamma}^N, \psi_n\big)_{L_2(\Gamma)} \tanh(\gamma_n a)\psi_n(\boldsymbol{y}),$$

or, in a shorter form,

$$A u_{|\Gamma}^N - \beta \big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)} \psi_1(\boldsymbol{y}) = -2i|\gamma_1|\psi_1(\boldsymbol{y}) \quad (\boldsymbol{y} \in \Gamma), \tag{3.12}$$

where

$$\beta = i|\gamma_1| + |\gamma_1|\tan(|\gamma_1|a) + 1 + \tanh(|\gamma_1|a), \tag{3.13}$$

and we introduced a positive-definite self-adjoint operator A acting on a function v from $H^{\frac{1}{2}}(\Gamma)$ as

$$Av = \sum_{n=1}^{\infty} \beta_n \big(v, \psi_n\big)_{L_2(\Gamma)} \psi_n(\boldsymbol{y}), \tag{3.14}$$

with

$$\beta_n = \begin{cases} 1 + \tanh(|\gamma_1|a) & (n = 1), \\ \gamma_n\big(1 + \tanh(\gamma_n a)\big) & (n > 1). \end{cases} \tag{3.15}$$

Since the coefficients β_n are strictly positive, the operator A can be inverted to get

$$u_{|\Gamma}^N - \beta \big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)} A^{-1}\psi_1 = -2i|\gamma_1|A^{-1}\psi_1 \quad (\boldsymbol{y} \in \Gamma). \tag{3.16}$$

This is a Fredholm equation of the second kind with an operator of rank one, for which one can easily check the existence and uniqueness of the solution.

Multiplying the relation (3.16) by ψ_1 and integrating over Γ, one finds

$$\big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)} = -\frac{2i|\gamma_1|\big(A^{-1}\psi_1, \psi_1\big)_{L_2(\Gamma)}}{1 - \beta\big(A^{-1}\psi_1, \psi_1\big)_{L_2(\Gamma)}} = \frac{2i|\gamma_1|}{i|\gamma_1| + \eta_N(\lambda)}, \tag{3.17}$$

where

$$\eta_N(\lambda) \equiv |\gamma_1|\tan(|\gamma_1|a) + 1 + \tanh(|\gamma_1|a) - \frac{1}{\big(A^{-1}\psi_1, \psi_1\big)_{L_2(\Gamma)}}. \tag{3.18}$$

If there exists λ_c such that $\eta_N(\lambda_c) = 0$, then $\big(u_{|\Gamma}^N, \psi_1\big)_{L_2(\Gamma)} = 2$ at this λ_c and thus $c_1^N = 1$ due to (3.8).

Step 2. It is easy to show that $\eta_N(\lambda)$ is a continuous function of λ for $\lambda \in (\nu_1, \nu_1 + \pi^2/(4a^2))$. In fact, all $|\gamma_n|$ are continuous for any λ, $\tan(x)$ is continuous on the interval $(0, \pi/2)$ and thus for $x = |\gamma_1|a = \sqrt{\lambda - \nu_1}\, a$. Finally, the continuity of $\left(A^{-1}\psi_1, \psi_1\right)_{L_2(\Gamma)}$ for any λ was shown in (**52**). In what follows, we re-enforce the assumption (3.2) as

$$\lambda \in \Lambda, \qquad \Lambda = \left(\nu_1,\ \min\{\nu_2,\ \nu_1 + \pi^2/(4a^2)\}\right). \tag{3.19}$$

According to Lemma 3.2, for any fixed $\lambda \in \Lambda$, the scalar product $\left(A^{-1}\psi_1, \psi_1\right)_{L_2(\Gamma)}$ can be made arbitrarily small by taking the opening Γ small enough. As a consequence, if the opening Γ is small enough, there exists λ such that $\eta_N(\lambda) < 0$.

Now, fixing Γ, we vary λ in such a way that $|\gamma_1|a \to \pi/2$. Since $\tan(|\gamma_1|a)$ grows up to infinity, the first term in (3.18) becomes dominating, and $\eta_N(\lambda)$ gets positive values. We conclude thus that there exists λ_c at which $\eta_N(\lambda_c) = 0$.

Step 3. The last step consists in proving that the coefficient c_1^D of the Dirichlet problem at the resonance frequency $\sqrt{\lambda_c}$ is arbitrarily close to -1 when the opening is small enough. For this purpose, one can repeat the derivation of the step 1 for the Dirichlet problem that consists in replacing cos, cosh, tan and tanh by sin, sinh, ctan and ctanh, respectively. Skipping the intermediate steps, we get

$$c_1^D = \frac{2i|\gamma_1|}{i|\gamma_1| + \eta_D(\lambda)} - 1, \tag{3.20}$$

where

$$\eta_D(\lambda) \equiv -|\gamma_1|\mathrm{ctan}(|\gamma_1|a) + 1 + \mathrm{ctanh}(|\gamma_1|a) - \frac{1}{\left(A_D^{-1}\psi_1, \psi_1\right)_{L_2(\Gamma)}}, \tag{3.21}$$

and the operator A_D is defined by (3.14), but in the expression (3.15) for β_n, tanh are replaced by ctanh. According to the Step 2, $|\gamma_1|a$ is close to $\pi/2$ at the resonance frequency $\sqrt{\lambda_c}$. While $\tan(|\gamma_1|a)$ was growing up to infinity in this region, $\mathrm{ctan}(|\gamma_1|a)$ is uniformly bounded. As a consequence, when the Lebesgue measure of the opening Γ is small, the last term in (3.21) is dominant. We conclude that when the opening Γ shrinks, $\eta_D \to -\infty$, and therefore c_1^D approaches -1. This completes the proof. ■

LEMMA 3.2. *For any $\lambda \in \Lambda$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for $|\Gamma| < \delta$, one has $\left(A^{-1}\psi_1, \psi_1\right)_{L_2(\Gamma)} < \varepsilon$.*

PROOF. Denoting $\phi = A^{-1}\psi_1$, we can write $A\phi = \psi_1$ as

$$\sum_{n=1}^{\infty} \beta_n \left(\phi, \psi_n\right)_{L_2(\Gamma)} \psi_n = \psi_1 \quad (\boldsymbol{y} \in \Gamma), \tag{3.22}$$

with β_n given by (3.15). Multiplying this relation by ϕ and integrating over Γ yield

$$\sum_{n=1}^{\infty} \beta_n \left(\phi, \psi_n\right)^2_{L_2(\Gamma)} = \left(\phi, \psi_1\right)_{L_2(\Gamma)}. \tag{3.23}$$

Since the coefficients β_n asymptotically grow with n, one gets the following estimate

$$\big(\phi, \psi_1\big)_{L_2(\Gamma)} \geqslant C_\lambda \sum_{n=1}^{\infty} \big(\phi, \psi_n\big)^2_{L_2(\Gamma)} = C_\lambda ||\phi||^2_{L_2(\Gamma)}, \tag{3.24}$$

where

$$C_\lambda = \min_{n \geqslant 1}\{\beta_n\} > 0. \tag{3.25}$$

On the other hand, $\big(\phi, \psi_1\big)_{L_2(\Gamma)} \leqslant ||\phi||_{L_2(\Gamma)}\, ||\psi_1||_{L_2(\Gamma)}$ so that

$$C_\lambda\, ||\phi||_{L_2(\Gamma)} \leqslant ||\psi_1||_{L_2(\Gamma)}. \tag{3.26}$$

Note that

$$||\psi_1||^2_{L_2(\Gamma)} = \int_\Gamma d\boldsymbol{y}\, |\psi_1(\boldsymbol{y})|^2 \leqslant |\Gamma| \max_{\boldsymbol{y}\in\Gamma}\{|\psi_1(\boldsymbol{y})|^2\} \leqslant |\Gamma| \max_{\boldsymbol{y}\in\mathcal{S}}\{|\psi_1(\boldsymbol{y})|^2\}. \tag{3.27}$$

Since the maximum of an eigenfunction ψ_1 is fixed by its normalization in the cross-section $\mathcal{S}$ (and does not depend on Γ), we conclude that $||\psi_1||_{L_2(\Gamma)}$ vanishes as the opening Γ shrinks. Finally, we have

$$0 \leqslant \big(A^{-1}\psi_1, \psi_1\big)_{L_2(\Gamma)} \leqslant ||A^{-1}\psi_1||_{L_2(\Gamma)}\, ||\psi_1||_{L_2(\Gamma)} \leqslant \frac{1}{C_\lambda} ||\psi_1||^2_{L_2(\Gamma)} \xrightarrow[|\Gamma|\to 0]{} 0 \tag{3.28}$$

that completes the proof of the lemma. ■

4. Geometry-induced localization in a noncylindrical domain

4.1 *Preliminaries*

Examples from previous sections relied on the orthogonality of the lateral coordinate x and the transverse coordinates $\boldsymbol{y}$. In this section, we illustrate an application of mode matching methods to another situation admitting the separation of variables.

We consider the planar domain

$$\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2, \tag{4.1}$$

where Ω_1 is the disk of radius R_1, $\Omega_1 = \{0 < r < R_1, 0 \leqslant \phi \leqslant 2\pi\}$, and Ω_2 is a part of a circular sector of angle ϕ_1 between two circles of radii R_1 and R_2: $\Omega_2 = \{R_1 < r < R_2, 0 < \phi < \phi_1\}$ (see Fig. 4). In general, one can consider a disk with several sector-like 'petals'. We focus on the Dirichlet eigenvalue problem

$$-\Delta u = \lambda u, \qquad u|_{\partial\Omega} = 0. \tag{4.2}$$

In polar coordinates (r, ϕ), the separation of variables allows one to write explicit representations $u_1(r, \phi)$ and $u_2(r, \phi)$ of the solution of (4.2) in domains Ω_1 and Ω_2 as

$$\begin{aligned} u_1(r, \phi) = {} & \frac{1}{2\pi} \frac{J_0(\sqrt{\lambda}\, r)}{J_0(\sqrt{\lambda}\, R_1)} \big(u_{|\Gamma}, 1\big)_{L_2(\Gamma)} \\ & + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{J_n(\sqrt{\lambda}\, r)}{J_n(\sqrt{\lambda}\, R_1)} \Big(\big(u_\Gamma, \cos n\phi\big)_{L_2(\Gamma)} \cos n\phi + \big(u_{|\Gamma}, \sin n\phi\big)_{L_2(\Gamma)} \sin n\phi \Big) \end{aligned} \tag{4.3}$$

and

$$u_2(r,\phi) = \frac{2}{\phi_1}\sum_{n=1}^{\infty}\frac{\psi_n(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,R_1)}\big(u_\Gamma, \sin\alpha_n\phi\big)_{L_2(\Gamma)}\sin\alpha_n\phi, \tag{4.4}$$

where

$$\alpha_n = \frac{\pi}{\phi_1}n, \tag{4.5}$$

and $\psi_n(\sqrt{\lambda}\,r)$ are solutions of the Bessel equation satisfying the Dirichlet boundary condition at $r = R_2$, that is, $\psi_n(\sqrt{\lambda}\,R_2) = 0$, which read

$$\psi_n(r) = J_{\alpha_n}(r)\,Y_{\alpha_n}(\sqrt{\lambda}\,R_2) - Y_{\alpha_n}(r)\,J_{\alpha_n}(\sqrt{\lambda}\,R_2), \tag{4.6}$$

and $J_n(z)$ and $Y_n(z)$ are Bessel functions of the first and second kind.

Since the eigenfunction u is analytic in Ω, its radial derivatives match at the opening Γ:

$$\begin{aligned}
&\frac{1}{2\pi}\frac{J_0'(\sqrt{\lambda}\,R_1)}{J_0(\sqrt{\lambda}\,R_1)}\big(u_{|\Gamma}, 1\big)_{L_2(\Gamma)}\\
&\quad+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{J_n'(\sqrt{\lambda}\,R_1)}{J_n(\sqrt{\lambda}\,R_1)}\Big(\big(u_{|\Gamma}, \cos n\phi\big)_{L_2(\Gamma)}\cos n\phi + \big(u_{|\Gamma}, \sin n\phi\big)_{L_2(\Gamma)}\sin n\phi\Big)\\
&= \frac{2}{\phi_1}\sum_{n=1}^{\infty}\frac{\psi_n'(\sqrt{\lambda}\,R_1)}{\psi_n(\sqrt{\lambda}\,R_1)}\big(u_{|\Gamma}, \sin\alpha_n\phi\big)_{L_2(\Gamma)}\sin\alpha_n\phi \qquad (0<\phi<\phi_1).
\end{aligned} \tag{4.7}$$

Multiplying this equation by $u_{|\Gamma}$ and integrating over Γ yield the dispersion relation

$$\begin{aligned}
&\frac{1}{2\pi}\frac{J_0'(\sqrt{\lambda}\,R_1)}{J_0(\sqrt{\lambda}\,R_1)}\big(u_{|\Gamma}, 1\big)^2_{L_2(\Gamma)}\\
&\quad+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{J_n'(\sqrt{\lambda}\,R_1)}{J_n(\sqrt{\lambda}\,R_1)}\Big(\big(u_{|\Gamma}, \cos n\phi\big)^2_{L_2(\Gamma)} + \big(u_{|\Gamma}, \sin n\phi\big)^2_{L_2(\Gamma)}\Big)\\
&= \frac{2}{\phi_1}\sum_{n=1}^{\infty}\frac{\psi_n'(\sqrt{\lambda}\,R_1)}{\psi_n(\sqrt{\lambda}\,R_1)}\big(u_{|\Gamma}, \sin\alpha_n\phi\big)^2_{L_2(\Gamma)}.
\end{aligned} \tag{4.8}$$

Our goal is to show that there exists an eigenfunction u which is localized in the 'petal' Ω_2 and negligible in the disk Ω_1. For this purpose, we consider an auxiliary Dirichlet eigenvalue problem in the sector $\Omega_3 = \{0 < r < R_2,\ 0 < \phi < \phi_1\}$ for which all eigenvalues and eigenfunctions are known explicitly. These eigenfunctions are natural candidates to 'build' localized eigenfunctions in Ω.

We proceed as follows. In section 4.2, we show the existence of an eigenvalue λ of the Dirichlet Laplacian in Ω which is close to the first eigenvalue μ of the Dirichlet Laplacian in the sector Ω_3. In section 4.3, we estimate the L_2-norm of an eigenfunction in Ω. These estimates rely on some technical inequalities on Bessel functions that we prove in Appendix C. These steps reveal restrictions on three geometric parameters of Ω: two radii R_1 and R_2, and the angle ϕ_1. We will show that localization

occurs for thin long 'petals' (that is, large R_2 and small ϕ_1). The radius R_1 of the disk should be small as compared to R_2. In particular, we set

$$R_1 \leqslant \frac{j_1'}{\sqrt{\lambda}}, \tag{4.9}$$

where j_1' is the first zero of $J_1'(z)$.

4.2 *Localization in* Ω_2

In order to prove the existence of an eigenfunction u in Ω which is localized in Ω_2, we consider an auxiliary eigenvalue problem for the Dirichlet Laplacian in the sector $\Omega_3 = \{0 < r < R_2,\ 0 < \phi < \phi_1\}$. For this domain, all eigenvalues and eigenfunctions are known explicitly. In particular, the first eigenvalue μ and the corresponding eigenfunction are

$$\mu = \frac{j_{\alpha_1}^2}{R_2^2}, \qquad v(r,\phi) = C_v J_{\alpha_1}(\sqrt{\mu}\, r)\, \sin(\pi\phi/\phi_1), \tag{4.10}$$

where $\alpha_1 = \pi/\phi_1$, j_{α_1} is the first zero of the Bessel function $J_{\alpha_1}(z)$, $J_{\alpha_1}(j_{\alpha_1}) = 0$, and C_v is the normalization constant to ensure $||v||_{L_2(\Omega_3)} = 1$.

LEMMA 4.1 (OLVER'S ASYMPTOTICS (56)) *If $\nu \gg 1$, then $j_\nu = \nu(1 + c\nu^{-2/3} + O(\nu^{-4/3}))$, with $c = -a_1 2^{-1/3} \approx 1.855757$, where a_1 is the first zero of the Airy function.*

COROLLARY 4.2. *If $\phi_1 \ll 1$, then*

$$\mu = \frac{\alpha_1^2(1+\varepsilon_0)^2}{R_2^2}, \tag{4.11}$$

with $\alpha_1 = \pi/\phi_1$ and $\varepsilon_0 \propto \alpha_1^{-2/3} \ll 1$.

THEOREM 4.3. *If*

$$\phi_1 \ll 1 \quad \text{and} \quad R_1 \ll R_2, \tag{4.12}$$

then there exists an eigenvalue λ of the Dirichlet Laplacian in Ω which is close to the first eigenvalue μ of the Dirichlet Laplacian in Ω_3 from (4.10). In other words, there exists λ such that

$$\lambda = \frac{\alpha_1^2(1+\varepsilon')^2}{R_2^2}, \tag{4.13}$$

with some $\varepsilon' \ll 1$.

PROOF. When ϕ_1 is small but R_2 is large, the eigenfunction v of the Dirichlet Laplacian in Ω_3 is very small for $r < R_1$, that is, in $\Omega_3 \cap \Omega_1$. This function is a natural candidate to prove, using Lemma 2.1, the existence of an eigenvalue λ for which the associated eigenfunction would be localized in Ω_2.

In order to apply Lemma 2.1, we introduce a cut-off function $\bar{\eta}(r,\phi) = \eta(r)$ in Ω, with $\eta(r)$ being an analytic function on $\mathbb{R}_+$ such that $\eta(r) = 0$ for $r < R_1$ and $\eta(r) = 1$ for $r > R_1 + \delta$, for a small

fixed $\delta > 0$. We need to prove that for some small $\varepsilon > 0$

$$\|\Delta(\bar{\eta}\, v) + \mu\, \bar{\eta}\, v\|_{L_2(\Omega)} < \varepsilon \|\bar{\eta}\, v\|_{L_2(\Omega)}. \tag{4.14}$$

First we estimate the left-hand side:

$$\|\Delta(\bar{\eta}\, v) + \mu\, \bar{\eta}\, v\|_{L_2(\Omega)} = \|v\, \Delta\bar{\eta} + 2(\nabla\bar{\eta} \cdot \nabla v)\|_{L_2(Q_\delta)}, \tag{4.15}$$

where we used that $\{\mu, v\}$ is an eigenpair in Ω_3 while $\nabla\bar{\eta}$ is zero everywhere except $Q_\delta = \{R_1 < r < R_1 + \delta,\ 0 < \phi < \phi_1\}$. Since $\bar{\eta}$ is analytic, its derivatives are bounded so that

$$\begin{aligned}\|v\, \Delta\bar{\eta} + 2(\nabla\bar{\eta} \cdot \nabla v)\|^2_{L_2(Q_\delta)} &\leqslant 2\|v\, \Delta\bar{\eta}\|^2_{L_2(Q_\delta)} + 2\|2(\nabla\bar{\eta} \cdot \nabla v)\|^2_{L_2(Q_\delta)} \\ &\leqslant C_1 \int_{R_1}^{R_1+\delta} dr\, r\, [J_{\alpha_1}(\sqrt{\mu}\, r)]^2 + C_2 \int_{R_1}^{R_1+\delta} dr\, r\, [J'_{\alpha_1}(\sqrt{\mu}\, r)]^2,\end{aligned}$$

with some positive constants C_1 and C_2. Since $\sqrt{\mu}\, r = \alpha_1(1+\varepsilon_0)r/R_2 \ll \alpha_1$ for $r \in (R_1, R_1+\delta)$ when $R_2 \gg R_1 + \delta$, one can apply the asymptotic formula for Bessel functions, $J_\nu(z) \simeq (z/2)^\nu / \Gamma(\nu + 1)$, to estimate the first integral:

$$\begin{aligned}C_1 \int_{R_1}^{R_1+\delta} dr\, r\, [J_{\alpha_1}(\sqrt{\mu}\, r)]^2 &\leqslant C_1\, \delta(R_1 + \delta/2)\, \frac{(\frac{1}{2}\sqrt{\mu}(R_1+\delta))^{2\alpha_1}}{\Gamma^2(\alpha_1+1)} \\ &\leqslant C'_1 \alpha_1 \left(\frac{e(1+\varepsilon_1)(R_1+\delta)}{2R_2}\right)^{2\alpha_1},\end{aligned} \tag{4.16}$$

which vanishes exponentially fast when $\alpha_1 = \pi/\phi_1$ grows (here we used the Stirling's formula for the Gamma function). A similar estimate holds for the second integral. We conclude that when $R_2 \gg R_1$ and α_1 is large enough, the left-hand side of (4.14) can be made arbitrarily small.

On the other hand, the norm in the right-hand side of (4.14) is estimated as

$$\begin{aligned}\|\bar{\eta}\, v\|^2_{L_2(\Omega)} = \|\bar{\eta}\, v\|^2_{L_2(\Omega_2)} &= 1 - \|\sqrt{1-\bar{\eta}^2}\, v\|^2_{L_2(\Omega_2)} = 1 - \|\sqrt{1-\bar{\eta}^2}\, v\|^2_{L_2(Q_\delta)} \\ &\geqslant 1 - \delta(R_1 + \delta/2)\, \phi_1 \max_{r\in(R_1,R_1+\delta)} \{J^2_{\alpha_1}(\sqrt{\mu}\, r)\}.\end{aligned} \tag{4.17}$$

As a consequence, this norm can be made arbitrarily close to 1 that completes the proof of the inequality (4.14). According to Lemma 2.1, there exists an eigenvalue λ which is close to μ that can be written in the form (4.13), with a new small parameter ε'. ■

4.3 *Estimate of the norms*

To prove the localization in the subdomain Ω_2, we estimate the ratio of L_2-norms in two arbitrary 'radial cross-sections' of subdomains Ω_1 and Ω_2. More precisely, we consider the squared L_2-norm

of u_1 on a circle of radius r inside Ω_1

$$I_1(r) = \int_0^{2\pi} d\phi\, |u_1(r,\phi)|^2 = \frac{1}{2\pi}\frac{J_0^2(\sqrt{\lambda}\, r)}{J_0^2(\sqrt{\lambda}\, R_1)}\big(u_\Gamma, 1\big)^2_{L_2(\Gamma)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{J_n^2(\sqrt{\lambda}\, r)}{J_n^2(\sqrt{\lambda}\, R_1)}\left(\big(u_{|\Gamma}, \cos n\phi\big)^2_{L_2(\Gamma)} + \big(u_{|\Gamma}, \sin n\phi\big)^2_{L_2(\Gamma)}\right), \tag{4.18}$$

and the squared L_2-norm of u_2 on an arc $(0, \phi_1)$ of radius r in Ω_2

$$I_2(r) = \int_0^{\phi_1} d\phi\, |u_2(r,\phi)|^2 = \frac{2}{\phi_1}\sum_{n=1}^{\infty}\frac{\psi_n^2(\sqrt{\lambda}\, r)}{\psi_n^2(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, \sin\alpha_n\phi\big)^2_{L_2(\Gamma)}. \tag{4.19}$$

We aim at showing that the ratio $I_2(r_2)/I_1(r_1)$ can be made arbitrarily large as $\phi_1 \to 0$.

THEOREM 4.4. *For any $0 < r_1 < R_1 < r_2 < R_2$, the ratio $I_2(r_2)/I_1(r_1)$ is bounded from below as*

$$\frac{I_2(r_2)}{I_1(r_1)} \geqslant \frac{\psi_1^2(\sqrt{\lambda}\, r_2)}{\psi_1^2(\sqrt{\lambda}\, R_1)}\,\frac{J_0^2(\sqrt{\lambda}\, R_1)}{1+\Psi(\sqrt{\lambda}\, R_1)}, \tag{4.20}$$

where

$$\Psi(r) = -\left(\frac{\psi_1'(r)}{\psi_1(r)} - \frac{J_0'(r)}{J_0(r)}\right)\left(\frac{\psi_2'(r)}{\psi_2(r)} - \frac{J_0'(r)}{J_0(r)}\right)^{-1}. \tag{4.21}$$

PROOF. First, we rewrite the dispersion relation (4.8) as

$$\frac{2}{\phi_1}\frac{\psi_1'(\sqrt{\lambda}\, R_1)}{\psi_1(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, \sin\alpha_1\phi\big)^2_{L_2(\Gamma)} - \frac{1}{2\pi}\frac{J_0'(\sqrt{\lambda}\, R_1)}{J_0(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, 1\big)^2_{L_2(\Gamma)} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{J_n'(\sqrt{\lambda}\, R_1)}{J_n(\sqrt{\lambda}\, R_1)}\left(\big(u_{|\Gamma}, \cos n\phi\big)^2_{L_2(\Gamma)} + \big(u_{|\Gamma}, \sin n\phi\big)^2_{L_2(\Gamma)}\right) - \frac{2}{\phi_1}\sum_{n=2}^{\infty}\frac{\psi_n'(\sqrt{\lambda}\, R_1)}{\psi_n(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, \sin\alpha_n\phi\big)^2_{L_2(\Gamma)}. \tag{4.22}$$

The inequality (C.8) ensures that the first term in the right-hand side is positive, from which

$$\frac{2}{\phi_1}\frac{\psi_1'(\sqrt{\lambda}\, R_1)}{\psi_1(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, \sin\alpha_1\phi\big)^2_{L_2(\Gamma)} - \frac{1}{2\pi}\frac{J_0'(\sqrt{\lambda}\, R_1)}{J_0(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, 1\big)^2_{L_2(\Gamma)} \geqslant -\frac{2}{\phi_1}\sum_{n=2}^{\infty}\frac{\psi_n'(\sqrt{\lambda}\, R_1)}{\psi_n(\sqrt{\lambda}\, R_1)}\big(u_{|\Gamma}, \sin\alpha_n\phi\big)^2_{L_2(\Gamma)}. \tag{4.23}$$

Expressions (4.18, 4.19) at $r = R_1$ imply

$$\frac{1}{2\pi}\big(u_{|\Gamma}, 1\big)^2_{L_2(\Gamma)} \leqslant I_1(R_1) = I_2(R_1) = \frac{2}{\phi_1}\sum_{n=1}^{\infty}\big(u_{|\Gamma}, \sin\alpha_n\phi\big)^2_{L_2(\Gamma)}, \tag{4.24}$$

so that

$$-\frac{2}{\phi_1}\sum_{n=2}^{\infty}\frac{\psi_n'(\sqrt{\lambda}R_1)}{\psi_n(\sqrt{\lambda}R_1)}\big(u_{|\Gamma},\sin\alpha_n\phi\big)^2_{L_2(\Gamma)}$$
$$\leqslant\frac{2}{\phi_1}\frac{\psi_1'(\sqrt{\lambda}R_1)}{\psi_1(\sqrt{\lambda}R_1)}\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)}-\frac{J_0'(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}\frac{2}{\phi_1}\sum_{n=1}^{\infty}\big(u_{|\Gamma},\sin\alpha_n\phi\big)^2_{L_2(\Gamma)},$$

where we used the inequality

$$\frac{J_0'(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}=-\sqrt{\lambda}\frac{J_1(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}<0 \tag{4.25}$$

that follows from (C.1).

As a result, we get

$$-\sum_{n=2}^{\infty}\left(\frac{\psi_n'(\sqrt{\lambda}R_1)}{\psi_n(\sqrt{\lambda}R_1)}-\frac{J_0'(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}\right)\big(u_{|\Gamma},\sin\alpha_n\phi\big)^2_{L_2(\Gamma)}$$
$$\leqslant\left(\frac{\psi_1'(\sqrt{\lambda}R_1)}{\psi_1(\sqrt{\lambda}R_1)}-\frac{J_0'(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}\right)\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)}. \tag{4.26}$$

The inequality (C.30) with $n_1=2$ and $n_2=n\geqslant 2$ yields

$$-\frac{\psi_n'(\sqrt{\lambda}R_1)}{\psi_n(\sqrt{\lambda}R_1)}\geqslant-\frac{\psi_2'(\sqrt{\lambda}R_1)}{\psi_2(\sqrt{\lambda}R_1)}, \tag{4.27}$$

from which one deduces

$$-\left(\frac{\psi_2'(\sqrt{\lambda}R_1)}{\psi_2(\sqrt{\lambda}R_1)}-\frac{J_0'(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}\right)\sum_{n=2}^{\infty}\big(u_{|\Gamma},\sin\alpha_n\phi\big)^2_{L_2(\Gamma)}$$
$$\leqslant\left(\frac{\psi_1'(\sqrt{\lambda}R_1)}{\psi_1(\sqrt{\lambda}R_1)}-\frac{J_0'(\sqrt{\lambda}R_1)}{J_0(\sqrt{\lambda}R_1)}\right)\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)}. \tag{4.28}$$

Using the inequality (C.25), we rewrite it as

$$\sum_{n=2}^{\infty}\big(u_{|\Gamma},\sin\alpha_n\phi\big)^2_{L_2(\Gamma)}\leqslant\Psi(\sqrt{\lambda}R_1)\,\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)}, \tag{4.29}$$

where $\Psi(r)$ is defined in (4.21).

With these inequalities, we estimate the squared L_2-norm from (4.18):

$$\begin{aligned}
I_1(r_1) &\leqslant \frac{1}{J_0^2(\sqrt{\lambda}\,R_1)}\frac{1}{2\pi}\big(u_{|\Gamma},1\big)^2_{L_2(\Gamma)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big(\big(u_{|\Gamma},\cos n\phi\big)^2_{L_2(\Gamma)} + \big(u_{|\Gamma},\sin n\phi\big)^2_{L_2(\Gamma)}\Big)\\
&\leqslant \frac{1}{J_0^2(\sqrt{\lambda}\,R_1)} I_1(R_1)\\
&= \frac{1}{J_0^2(\sqrt{\lambda}\,R_1)} I_2(R_1)\\
&= \frac{1}{J_0^2(\sqrt{\lambda}\,R_1)}\frac{2}{\phi_1}\Big[\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)} + \sum_{n=2}^{\infty}\big(u_{|\Gamma},\sin\alpha_n\phi\big)^2_{L_2(\Gamma)}\Big]\\
&\leqslant \frac{1}{J_0^2(\sqrt{\lambda}\,R_1)}\frac{2}{\phi_1}\Big[\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)} + \Psi(\sqrt{\lambda}\,R_1)\,\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)}\Big]\\
&= \frac{1+\Psi(\sqrt{\lambda}\,R_1)}{J_0^2(\sqrt{\lambda}\,R_1)}\frac{2}{\phi_1}\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)},
\end{aligned}$$

where we used the inequality (C.25).

On the other hand, we obtain

$$I_2(r_2) \geqslant \frac{2}{\phi_1}\frac{\psi_1^2(\sqrt{\lambda}\,r_2)}{\psi_1^2(\sqrt{\lambda}\,R_1)}\big(u_{|\Gamma},\sin\alpha_1\phi\big)^2_{L_2(\Gamma)}. \tag{4.30}$$

Combining these inequalities, we conclude for any $0 \leqslant r_1 \leqslant R_1 \leqslant r_2 \leqslant R_2$ that

$$\frac{I_2(r_2)}{I_1(r_1)} \geqslant \frac{\psi_1^2(\sqrt{\lambda}\,r_2)}{\psi_1^2(\sqrt{\lambda}\,R_1)}\,\frac{J_0^2(\sqrt{\lambda}\,R_1)}{1+\Psi(\sqrt{\lambda}\,R_1)} \tag{4.31}$$

that completes the proof of the theorem. ■

LEMMA 4.5. *Under conditions (4.12), if $|\lambda-\mu| < \varepsilon$ for some $\varepsilon > 0$, then $|\psi_1(\sqrt{\lambda}\,R_1)| < C\varepsilon$.*

PROOF. The continuity of Bessel functions implies

$$\lim_{\lambda\to\mu}\psi_1(\sqrt{\lambda}\,R_1) = \psi_1(\sqrt{\mu}\,R_1) = J_{\alpha_1}(\sqrt{\mu}\,R_1)\,Y_{\alpha_1}(\sqrt{\mu}\,R_2), \tag{4.32}$$

where the second term in the definition (4.6) was dropped because μ is the Dirichlet eigenvalue for the sector Ω_3 and thus $J_{\alpha_1}(\sqrt{\mu}\,R_2) = 0$.

When $R_1 \ll R_2$, one can apply the asymptotic formulas for the Bessel functions

$$J_{\alpha_1}(\sqrt{\mu}\,R_1) \simeq \frac{1}{\Gamma(\alpha_1+1)}\left(\frac{\alpha_1(1+\varepsilon_1)R_1}{2R_2}\right)^{\alpha_1}, \tag{4.33}$$

$$Y_{\alpha_1}(\sqrt{\mu}\,R_2) \simeq -\frac{\Gamma(\alpha_1+1)}{\pi}\left(\frac{2}{\alpha_1(1+\varepsilon_1)}\right)^{\alpha_1}, \tag{4.34}$$

from which

$$|\psi_1(\sqrt{\mu}\,R_1)| \simeq \frac{1}{\pi}\left(\frac{R_1}{R_2}\right)^{\alpha_1}. \tag{4.35}$$

When $R_1 \ll R_2$ and α_1 is large, the right-hand side can be made arbitrarily small that completes the proof. ■

COROLLARY 4.6. *Under conditions (4.12), there exists an eigenfunction u of the Dirichlet Laplacian in Ω which is localized in Ω_2. In particular, the ratio $I_2(r_2)/I_1(r_1)$ of squared L_2 norms from (4.20) can be made arbitrarily large.*

PROOF. Let us examine the right-hand side of (4.20). The function $\psi_1(\sqrt{\lambda}\,r_2)$ does not vanish on $R_1 < r_2 < R_2$ except at $r_2 = R_2$. Since $J_0(\sqrt{\lambda}\,R_1)$ is a constant, it remains to show that $\psi_1^2(\sqrt{\lambda}\,R_1)(1+\Psi(\sqrt{\lambda}\,R_1))$ can be made arbitrarily small. On one hand, the absolute value of

$$\psi_1(\sqrt{\lambda}\,R_1)\big(1+\Psi(\sqrt{\lambda}\,R_1)\big) = \psi_1(\sqrt{\lambda}\,R_1) - \left(\psi_1'(\sqrt{\lambda}\,R_1) - \psi_1(\sqrt{\lambda}\,R_1)\frac{J_0'(\sqrt{\lambda}\,R_1)}{J_0(\sqrt{\lambda}\,R_1)}\right) \times \left(\frac{\psi_2'(\sqrt{\lambda}\,R_1)}{\psi_2(\sqrt{\lambda}\,R_1)} - \frac{J_0'(\sqrt{\lambda}\,R_1)}{J_0(\sqrt{\lambda}\,R_1)}\right)^{-1} \tag{4.36}$$

can be bounded from above by a constant. On the other hand, the remaining factor $\psi_1(\sqrt{\lambda}\,R_1)$ can be made arbitrarily small by Lemma 4.5 that completes the proof. ■

Acknowledgements

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APPENDIX A

Numerical illustrations

We illustrate the localization of eigenfunctions by considering a rectangle $(-a_2, a_1) \times (0, 1)$ with a vertical barrier $\{0\} \times (h, 1)$: $\Omega = \big((-a_1, a_2) \times (0, 1)\big)\backslash\big(\{0\} \times (h, 1)\big)$. In other words, two rectangles $(-a_1, 0) \times (0, 1)$ and $(0, a_2) \times (0, 1)$ are connected through an opening $\Gamma = (0, h)$ at $x = 0$ (Fig. 2a). Setting $a_1 = 1$ and $a_2 = 0.8$, we compute several eigenfunctions of the Dirichlet Laplacian by a finite element method in Matlab PDEtools for several values of h. This standard method relies on a weak formulation of the eigenvalue problem and its projection onto a finite-dimensional space of linear basis functions defined on a triangular mesh of the domain Ω. The accuracy of computation is controlled by the maximal mesh size (indicated in figure captions for each considered case).

Figure C.1 shows the first six Dirichlet eigenfunctions for $h = 0.1$. Even though the diameter of the opening Γ is not so very small (just one tenth of the rectangle width), one observes a very strong localization: eigenfunctions u_1, u_3, u_4 are localized in the larger domain Ω_1, whereas u_2, u_5 and u_6 are localized in Ω_2. The corresponding eigenvalues are provided in Table A.1. Even if the opening Γ is increased to the quarter of the rectangle width ($h = 0.25$), the localization of first eigenfunctions is still present (Fig. C.2). However, one can see that the eigenfunctions localized in one subdomain start to penetrate into the other subdomain. This penetration is

Table A.1 The first six Dirichlet eigenvalues in a rectangle with a vertical barrier $\{0\} \times (h, 1)$, $\Omega = \big((-a_1, a_2) \times (0, 1)\big) \backslash \big(\{0\} \times (h, 1)\big)$, with $a_1 = 1$, $a_2 = 0.8$, and five values of h, including the limit cases: $h = 0$ (no opening, two disjoint subdomains) and $h = 1$ (no barrier)

n	$h = 0$	$h = 0.1$	$h = 0.25$	$h = 0.5$	$h = 1$
1	$(1+1)\pi^2 \approx 19.74$	19.79	19.70	18.39	$12.92 \approx \pi^2(1 + 1/1.8^2)$
2	$(1 + 1/0.8^2)\pi^2 \approx 25.29$	25.39	25.22	23.58	$22.05 \approx \pi^2(1 + 4/1.8^2)$
3	$(1+4)\pi^2 \approx 49.35$	49.42	48.77	41.40	$37.29 \approx \pi^2(1 + 9/1.8^2)$
4	$(1+4)\pi^2 \approx 49.35$	49.59	49.49	49.33	$42.52 \approx \pi^2(4 + 1/1.8^2)$
5	$(4 + 1/0.8^2)\pi^2 \approx 54.30$	55.04	54.52	53.32	$51.66 \approx \pi^2(4 + 4/1.8^2)$
6	$(1 + 4/0.8^2)\pi^2 \approx 71.55$	72.01	70.99	62.88	$58.61 \approx \pi^2(1 + 16/1.8^2)$

enhanced for higher-order eigenfunctions. Setting $h = 0.5$ destroys the localization of eigenfunctions, except for u_1 and u_4 (Fig. C.3). Looking at these figures in the backward order, one can observe the progressive emergence of the localization as the opening Γ shrinks. It is remarkable how strong the localization can be even for not too narrow openings.

Finally, Fig. C.4 shows the second eigenfunction for the geometric setting with $a_1 = 1$ and $a_2 = 0.5$. For this choice of a_2, the condition (2.87), which was used to prove the localization of the second eigenfunction in section 2.5, is not satisfied. Nevertheless, the second eigenfunction turns out to be localized in the larger domain. This example suggests that the condition (2.87) may potentially be relaxed.

APPENDIX B

Proof of the classical Lemma 2.1

Here we provide an elementary proof of the classical Lemma 2.1.

PROOF. Let λ_k and ψ_k denote eigenvalues and L_2-normalized eigenfunctions of the operator A forming a complete basis in L_2. Let us decompose v on this basis: $v = \sum_k c_k \psi_k$. One has then

$$\|Av - \mu v\|_{L_2}^2 = \left\| \sum_k (\lambda_k - \mu) c_k \psi_k \right\|_{L_2}^2 = \sum_k (\lambda_k - \mu)^2 c_k^2 \geqslant \min_k \{(\lambda_k - \mu)^2\} \sum_k c_k^2, \tag{B.1}$$

that is,

$$\|Av - \mu v\|_{L_2} \geqslant \min_k \{|\lambda_k - \mu|\} \, \|v\|_{L_2}. \tag{B.2}$$

On the other hand, (2.16) implies that

$$\min_k \{|\lambda_k - \mu|\} < \varepsilon \tag{B.3}$$

that completes the proof. ■

APPENDIX C

Some inequalities involving Bessel functions

In this Appendix, we prove several inequalities involving Bessel functions. The technique of proofs is standard and can be found in classical textbooks (56–58).

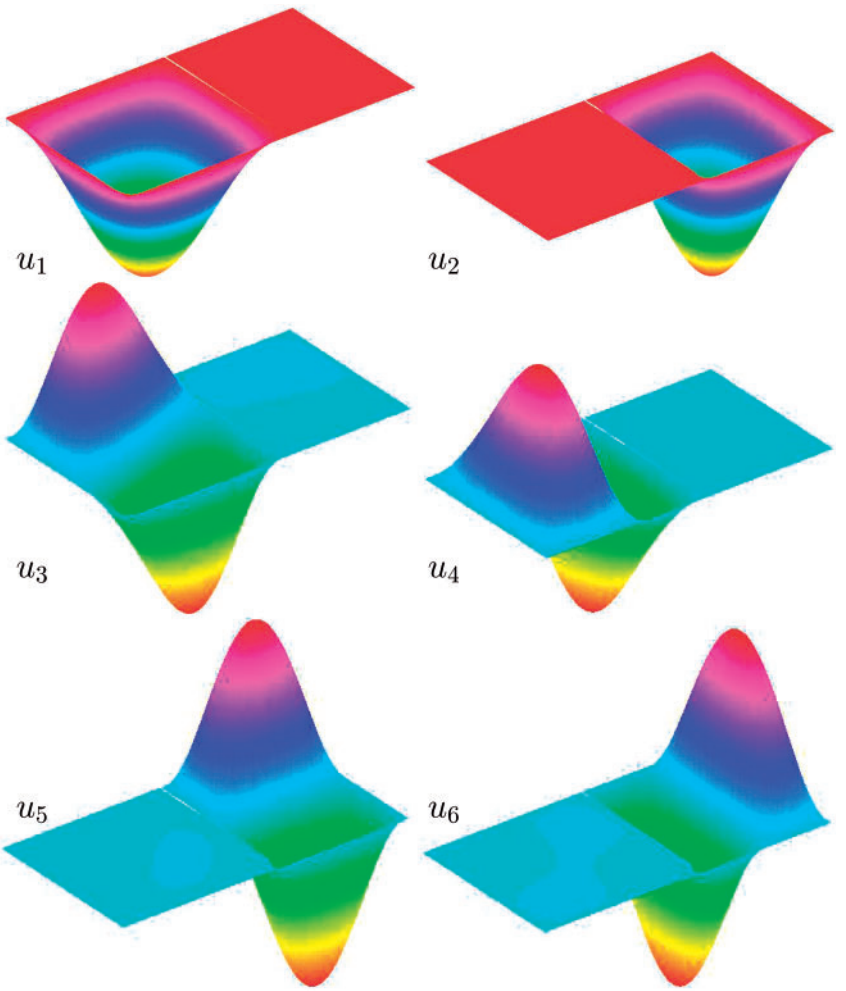

Fig. C.1 The first six Dirichlet eigenfunctions in a rectangle with a vertical barrier $\{0\} \times (h, 1)$, $\Omega = \big((-a_1, a_2) \times (0, 1)\big) \backslash \big(\{0\} \times (h, 1)\big)$, with $a_1 = 1$, $a_2 = 0.8$ and $h = 0.1$. Eigenfunctions u_1, u_3, u_4 are localized in the larger domain Ω_1 while u_2, u_5 and u_6 are localized in Ω_2. The maximal mesh size was $1.75 \cdot 10^{-4}$

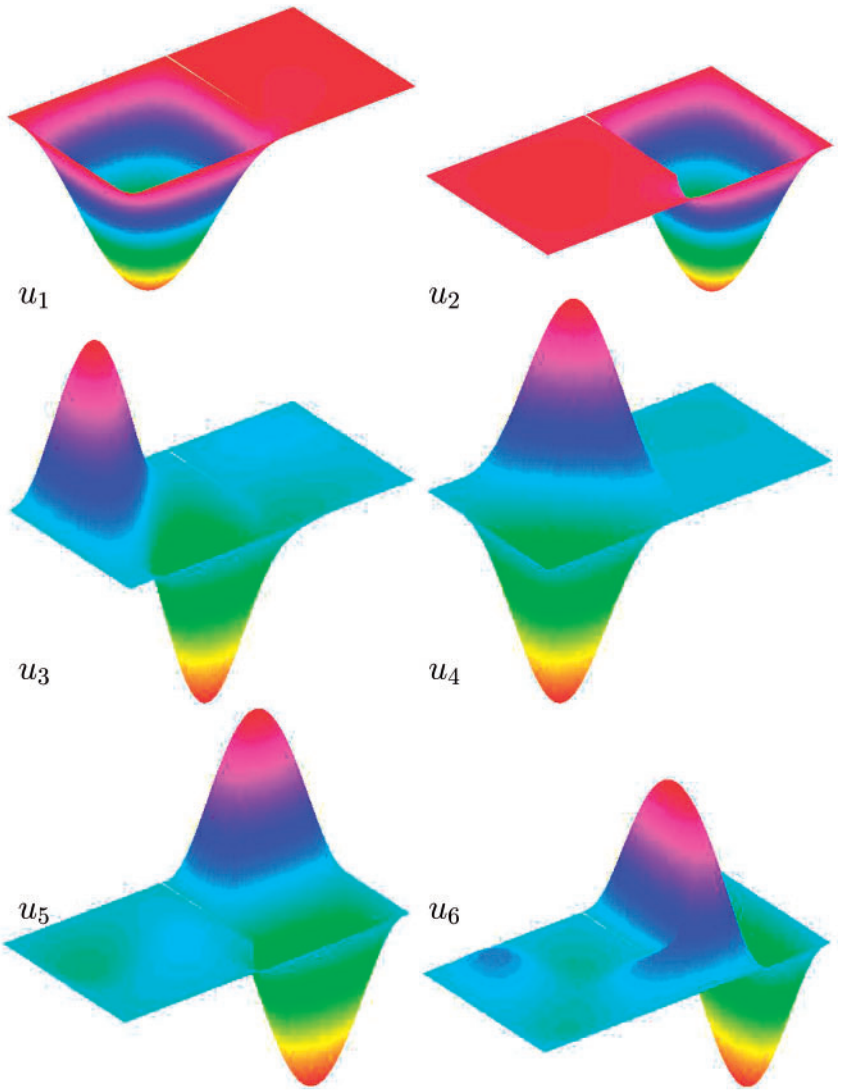

Fig. C.2 The first six Dirichlet eigenfunctions in a rectangle with a vertical barrier $\{0\} \times (h, 1)$, $\Omega = \big((-a_1, a_2) \times (0, 1)\big) \backslash \big(\{0\} \times (h, 1)\big)$, with $a_1 = 1, a_2 = 0.8$ and $h = 0.25$. Eigenfunctions u_1, u_3, u_4 are localized in the larger domain Ω_1 while u_2, u_5 and u_6 are localized in Ω_2. The maximal mesh size was $1.72 \cdot 10^{-4}$

LEMMA C.1. *If $\sqrt{\lambda}\, R_1 \leqslant j_1'$, then for any $\nu \geqslant 0$,*

$$J_\nu(\sqrt{\lambda}\, r) > 0 \qquad \forall\, 0 < r \leqslant R_1, \tag{C.1}$$

where $j_1' \approx 1.8412$ is the first zero of $J_1'(z)$.

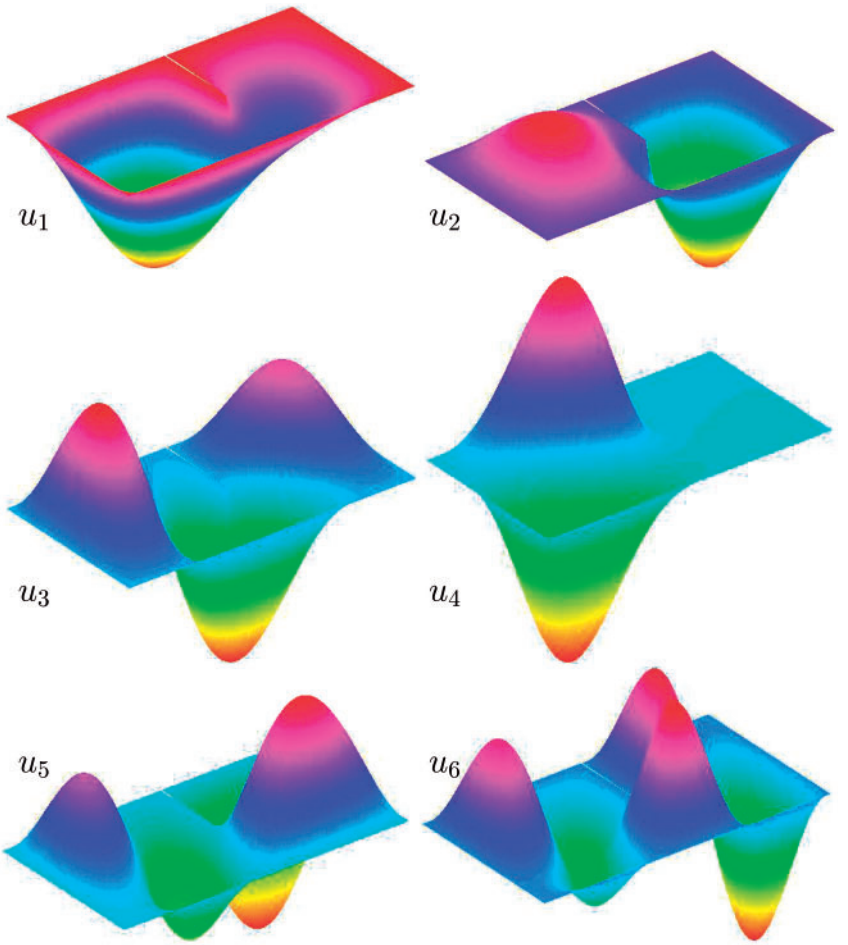

Fig. C.3 The first six Dirichlet eigenfunctions in a rectangle with a vertical barrier $\{0\} \times (h, 1)$, $\Omega = \big((-a_1, a_2) \times (0, 1)\big) \backslash \big(\{0\} \times (h, 1)\big)$, with $a_1 = 1$, $a_2 = 0.8$ and $h = 0.5$. Only eigenfunctions u_1 and u_4 remain localized. The maximal mesh size was $1.69 \cdot 10^{-4}$

PROOF. The known inequalities on the first zeros j_ν of Bessel functions $J_\nu(z)$,

$$j_1' < j_0 \leqslant j_\nu \qquad \forall\, \nu \geqslant 0, \tag{C.2}$$

imply that $J_\nu(\sqrt{\lambda}\, r)$ does not change sign in the interval $(0, R_1)$. In turn, the asymptotic behavior $J_\nu(x) \simeq (x/2)^\nu / \Gamma(\nu + 1)$ as $x \to 0$ ensures the positive sign. ■

LEMMA C.2. *If* $\sqrt{\lambda}\, R_1 \leqslant j_1'$, *then for any* $0 \leqslant \nu_1 < \nu_2$, *one has*

$$\frac{J_{\nu_1}'(\sqrt{\lambda}\, r)}{J_{\nu_1}(\sqrt{\lambda}\, r)} \leqslant \frac{J_{\nu_2}'(\sqrt{\lambda}\, r)}{J_{\nu_2}(\sqrt{\lambda}\, r)} \qquad \forall\, 0 < r \leqslant R_1. \tag{C.3}$$

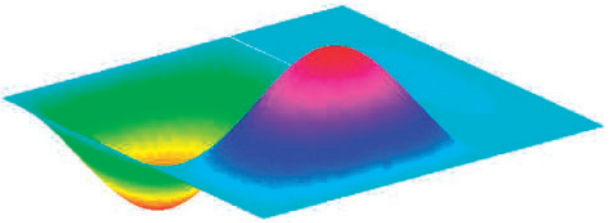

Fig. C.4 The second Dirichlet eigenfunction in a rectangle with a vertical barrier $\{0\} \times (h, 1)$, $\Omega = \big((-a_1, a_2) \times (0, 1)\big) \backslash \big(\{0\} \times (h, 1)\big)$, with $a_1 = 1$, $a_2 = 0.5$ and $h = 0.1$. This eigenfunction is still localized in Ω_1 although the condition (2.87) is not satisfied: $\sqrt{a_2^2(\nu_2 - \nu_1)/\pi^2 + a_2^2/a_1^2} = 1$, given that $\nu_1 = \pi^2$ and $\nu_2 = 4\pi^2$. The maximal mesh size was $5.75 \cdot 10^{-4}$

PROOF. Although the proof is standard, we provide it for completeness.

Writing the Bessel equations for $J_{\nu_1}(\sqrt{\lambda}\, r)$ and $J_{\nu_2}(\sqrt{\lambda}\, r)$,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_{\nu_1}(\sqrt{\lambda}\, r)}{dr}\right) + \lambda J_{\nu_1}(\sqrt{\lambda}\, r) - \frac{\nu_1^2}{r^2}J_{\nu_1}(\sqrt{\lambda} r) = 0,$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_{\nu_2}(\sqrt{\lambda}\, r)}{dr}\right) + \lambda J_{\nu_2}(\sqrt{\lambda}\, r) - \frac{\nu_2^2}{r^2}J_{\nu_2}(\sqrt{\lambda} r) = 0,$$

multiplying the first one by $rJ_{\nu_2}(\sqrt{\lambda}\, r)$, the second one by $rJ_{\nu_1}(\sqrt{\lambda}\, r)$, and subtracting one from the other, one gets

$$\frac{d}{dr}r\left(J_{\nu_1}(\sqrt{\lambda}\, r)\frac{dJ_{\nu_2}(\sqrt{\lambda}\, r)}{dr} - J_{\nu_2}(\sqrt{\lambda}\, r)\frac{dJ_{\nu_1}(\sqrt{\lambda}\, r)}{dr}\right) + \frac{\nu_1^2 - \nu_2^2}{r}J_{\nu_1}(\sqrt{\lambda}\, r)J_{\nu_2}(\sqrt{\lambda}\, r) = 0. \tag{C.4}$$

The integration from 0 to r yields

$$\sqrt{\lambda} r\left(J_{\nu_1}(\sqrt{\lambda}\, r)J'_{\nu_2}(\sqrt{\lambda}\, r) - J_{\nu_2}(\sqrt{\lambda}\, r)J'_{\nu_1}(\sqrt{\lambda}\, r)\right) = \int_0^r \frac{\nu_2^2 - \nu_1^2}{r'}J_{\nu_1}(\sqrt{\lambda}\, r')J_{\nu_2}(\sqrt{\lambda}\, r')dr', \tag{C.5}$$

and the integral is strictly positive due to (C.1). ■

LEMMA C.3. *If $\sqrt{\lambda} R_1 \leqslant j'_1$, then for any $\nu \geqslant 1$, one has*

$$J'_\nu(\sqrt{\lambda}\, r) \geqslant 0 \qquad \forall\, 0 < r < R_1, \tag{C.6}$$

$$J_\nu(\sqrt{\lambda}\, r) \leqslant J_\nu(\sqrt{\lambda} R_1) \qquad \forall\, 0 < r \leqslant R_1, \tag{C.7}$$

$$\frac{J'_\nu(\sqrt{\lambda}\, r)}{J_\nu(\sqrt{\lambda}\, r)} \geqslant 0 \qquad \forall\, 0 < r \leqslant R_1. \tag{C.8}$$

PROOF. The first inequality (C.6) is a direct consequence of Lemma C.1, Lemma C.2, and the inequality $J_1'(\sqrt{\lambda}\, r) \geqslant 0$ which is fulfilled for $0 < r < R_1$. The second inequality (C.7) follows from the first one. The third inequality (C.8) is a consequence of the first one and Lemma C.1. ■

Now we turn to the function $\psi_n(\sqrt{\lambda}\, r)$ defined by (4.6) which satisfies the Bessel equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_n(\sqrt{\lambda}\, r)}{dr}\right) + \lambda\psi_n(\sqrt{\lambda}\, r) - \frac{\alpha_n^2}{r^2}\psi_n(\sqrt{\lambda}\, r) = 0. \tag{C.9}$$

LEMMA C.4. *For any $0 < a < b$, one has*

$$\int_a^b dr\, r\, \psi_n^2(\sqrt{\lambda}\, r) = \frac{1}{2\lambda}\left((r\psi_n'(\sqrt{\lambda}\, r))^2 + (\lambda r^2 - \alpha_n^2)\psi_n^2(\sqrt{\lambda}\, r)\right)\Big|_{r=a}^{r=b}. \tag{C.10}$$

PROOF. The proof is obtained by multiplying the Bessel equation (C.9) by $r^2\psi_n'$ and integrating from a to b. ■

LEMMA C.5. *If $\alpha_n^2/R_2^2 > \lambda$, then*

$$-\frac{\psi_n'(\sqrt{\lambda}\, r)}{\psi_n(\sqrt{\lambda}\, r)} \geqslant \frac{r\, b(r)\, b(R_2)}{\lambda^{3/2} + \sqrt{\lambda^3 + r^2\, b(r)\, b^2(R_2)}} \qquad \forall\, 0 < r < R_2, \tag{C.11}$$

where $b(r) = \alpha_n^2/r^2 - \lambda$.

PROOF. We introduce a new function $v(z) = \psi_n(\sqrt{\lambda}\, r)$ by changing the variable $z = \ln(r/\alpha_n)$, with $r \in (0, R_2)$. Starting from the Bessel equation for $\psi_n(\sqrt{\lambda}\, r)$, it is easy to check that the new function v satisfies the equation

$$v''(z) = (1 - \lambda e^{2z})\alpha_n^2 v(z), \tag{C.12}$$

where prime denotes the derivative with respect to z. One has then

$$\begin{aligned}(v^2)'' &= 2vv'' + 2(v')^2 \geqslant 2vv'' = 2\alpha_n^2(1 - \lambda e^{2z})v^2 \\ &= 2\alpha_n^2\left(1 - \lambda(r/\alpha_n)^2\right)v^2 \geqslant 2\alpha_n^2\left(1 - \lambda(R_2/\alpha_n)^2\right)v^2 \\ &= 2R_2^2\, b(R_2)\, v^2,\end{aligned} \tag{C.13}$$

where we used $r < R_2$, and $b(R_2) = \alpha_n^2/R_2^2 - \lambda > 0$. Integration of this inequality from z to $z_2 = \ln(R_2/\alpha_n)$ yields

$$-2v'v = -(v^2)' = \int_z^{z_2} dz'(v^2)'' \geqslant 2R_2^2\, b(R_2)\int_z^{z_2} dz' v^2(z'), \tag{C.14}$$

where $(v^2)'_{z=z_2} = 2v'(z_2)v(z_2) = 0$ due to the boundary condition $\psi_n(\sqrt{\lambda}\, R_2) = 0$. One gets then

$$-\frac{v'(z)}{v(z)} \geqslant \frac{R_2^2\, b(R_2)}{v^2(z)}\int_z^{z_2} dz' v^2(z'). \tag{C.15}$$

Since

$$\frac{d}{dr}\psi_n(\sqrt{\lambda}\, r) = \frac{dv}{dr} = \frac{dz}{dr}\frac{dv}{dz} = \frac{1}{r}v', \tag{C.16}$$

one obtains

$$-\frac{\frac{d}{dr}\psi_n(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,r)} \geqslant \frac{R_2^2\,b(R_2)}{r\psi_n^2(\sqrt{\lambda}\,r)}\int_r^{R_2}\frac{dr'}{r'}\psi_n^2(\sqrt{\lambda}\,r'). \tag{C.17}$$

We conclude that

$$-\frac{\psi_n'(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,r)} > 0. \tag{C.18}$$

One can further improve the lower bound. The integral in the right-hand side of (C.17) can be estimated as

$$\begin{aligned}\int_r^{R_2}\frac{dr'}{r'}\psi_n^2(\sqrt{\lambda}\,r') &= \int_r^{R_2}\frac{dr'}{r'^2}r'\psi_n^2(\sqrt{\lambda}\,r') \geqslant \frac{1}{R_2^2}\int_r^{R_2}dr'\,r'\,\psi_n^2(\sqrt{\lambda}\,r')\\ &= \frac{1}{R_2^2}\frac{1}{2\lambda}\left((r'\psi_n'(\sqrt{\lambda}\,r'))^2 + (\lambda r'^2-\alpha_n^2)\psi_n^2(\sqrt{\lambda}\,r')\right)\Big|_{r'=r}^{r'=R_2}\\ &= \frac{1}{2\lambda R_2^2}\left((R_2\psi_n'(\sqrt{\lambda}\,R_2))^2 - (r\psi_n'(\sqrt{\lambda}\,r))^2 + (\alpha_n^2-\lambda r^2)\psi_n^2(\sqrt{\lambda}\,r)\right),\end{aligned}$$

where we used (C.10) and the condition $\psi_n(\sqrt{\lambda}\,R_2)=0$. We get then

$$-\frac{\psi_n'(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,r)} \geqslant \frac{b(R_2)\,r}{2\lambda^{3/2}}\left(-\left(\frac{\psi_n'(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,r)}\right)^2 + b(r)\right), \tag{C.19}$$

where we dropped the positive term $(R_2\psi_n'(\sqrt{\lambda}\,R_2))^2$. Denoting $w=-\frac{\psi_n'(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,r)}$ and $a=rb(R_2)/(2\lambda^{3/2})$, the above inequality can be written as

$$aw^2+w-ab(r)\geqslant 0. \tag{C.20}$$

Since $a>0$ and $w>0$, this inequality implies

$$-\frac{\psi_n'(\sqrt{\lambda}\,r)}{\psi_n(\sqrt{\lambda}\,r)} \geqslant \frac{2ab(r)}{1+\sqrt{1+4a^2b(r)}} \qquad \forall\, 0<r<R_2, \tag{C.21}$$

which is equivalent to (C.11). ■

COROLLARY C.6. *If $\alpha_n^2/R_2^2>\lambda$, then $\psi_n(\sqrt{\lambda}\,r)$ is a positive monotonously decreasing function on the interval $(0,R_2)$:*

$$\psi_n(\sqrt{\lambda}\,r)\geqslant 0, \qquad \psi_n'(\sqrt{\lambda}\,r)\leqslant 0, \qquad \forall\, 0<r\leqslant R_2. \tag{C.22}$$

PROOF. The inequality (C.18) implies that $\psi_n(\sqrt{\lambda}\,r)$ is either positive monotonously decreasing or negative monotonously increasing on the interval $(0,R_2)$. We compute then

$$\psi_n'(\sqrt{\lambda}\,R_2) = J_{\alpha_n}'(\sqrt{\lambda}\,R_2)Y_{\alpha_n}(\sqrt{\lambda}\,R_2) - Y_{\alpha_n}'(\sqrt{\lambda}\,R_2)J_{\alpha_n}(\sqrt{\lambda}\,R_2) = -\frac{2}{\pi\sqrt{\lambda}\,R_2} < 0, \tag{C.23}$$

where we used the Wronskian for Bessel functions. As a consequence, ψ_n' is negative in a vicinity of R_2 and thus on the whole interval. ■

COROLLARY C.7. *If λ is fixed by (4.13), then there exists α_1 large enough such that for any $n \geqslant 2$,*

$$\psi_n(\sqrt{\lambda}\, r) \geqslant 0, \qquad \psi_n'(\sqrt{\lambda}\, r) \leqslant 0, \qquad \forall\, 0 < r \leqslant R_2, \tag{C.24}$$

and

$$-\frac{\psi_n'(\sqrt{\lambda}\, R_1)}{\psi_n(\sqrt{\lambda}\, R_1)} \geqslant -\frac{J_0'(\sqrt{\lambda}\, R_1)}{J_0(\sqrt{\lambda}\, R_1)}. \tag{C.25}$$

PROOF. When λ is given by (4.13), one has

$$b(R_2) = \frac{n^2\alpha_1^2}{R_2^2} - \lambda = c_2\lambda, \qquad c_2 = \frac{n^2}{(1+\varepsilon_1)^2} - 1 > 0 \tag{C.26}$$

and

$$b(R_1) = \frac{n^2\alpha_1^2}{R_1^2} - \lambda = (R_2^2 c_1 - 1)\lambda, \qquad c_1 = \frac{n^2}{R_1^2(1+\varepsilon_1)^2} > 0 \tag{C.27}$$

for any $n \geqslant 2$ and α_1 large enough (that makes ε_1 small enough). As a consequence, Corollary C.6 implies (C.24), while Lemma C.5 implies

$$-\frac{\psi_n'(\sqrt{\lambda}\, R_1)}{\psi_n(\sqrt{\lambda}\, R_1)} \geqslant \frac{R_1\, b(R_1)\, b(R_2)}{\lambda^{3/2} + \sqrt{\lambda^3 + R_1^2\, b(R_1)\, b^2(R_2)}} = C_n R_2, \tag{C.28}$$

with

$$C_n = \frac{R_1(c_1 - 1/R_2^2)c_2\sqrt{\lambda}}{1/R_2 + \sqrt{1/R_2^2 + R_1^2(c_1 - 1/R_2^2)c_2^2}} \longrightarrow \sqrt{c_1\lambda} \qquad \text{as } R_2 \to \infty. \tag{C.29}$$

In other words, when λ is fixed by (4.13), the left-hand side of (C.28) can be made arbitrarily large. On the other hand, for a fixed λ, the term $\frac{J_0'(\sqrt{\lambda}\, R_1)}{J_0(\sqrt{\lambda}\, R_1)}$ in the inequality (C.25) is independent of α_1 or R_2. We conclude that the inequality (C.25) is fulfilled for α_1 (and R_2) large enough. ■

LEMMA C.8. *If λ is fixed by (4.13), then there exists α_1 large enough such that for any $n_2 > n_1 \geqslant 2$, one has*

$$\frac{\psi_{n_1}'(\sqrt{\lambda}\, r)}{\psi_{n_1}(\sqrt{\lambda}\, r)} \geqslant \frac{\psi_{n_2}'(\sqrt{\lambda}\, r)}{\psi_{n_2}(\sqrt{\lambda}\, r)} \qquad \forall\, 0 < r < R_2. \tag{C.30}$$

PROOF. Integration of (C.4) from r to R_2 yields

$$\begin{aligned} &-r\left(\psi_{n_1}(\sqrt{\lambda}\, r)\psi_{n_2}'(\sqrt{\lambda}\, r) - \psi_{n_2}(\sqrt{\lambda}\, r)\psi_{n_1}'(\sqrt{\lambda}\, r)\right) \\ &= \int_r^{R_2} \frac{(\pi n_2/\phi_1)^2 - (\pi n_1/\phi_1)^2}{r'}\, \psi_{n_1}(\sqrt{\lambda}\, r')\psi_{n_2}(\sqrt{\lambda}\, r')dr', \end{aligned} \tag{C.31}$$

where the upper limit at $r = R_2$ vanished due to the boundary condition $\psi_n(\sqrt{\lambda}\, R_2) = 0$. Since $\psi_n(\sqrt{\lambda}\, r) \geqslant 0$ over $r \in (0, R_2)$ according to (C.24), the integral is positive that implies (C.30). ■

Note that the sign of inequality is opposite here as compared to the inequality (C.3).

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