

## THE REACTIVE BETA MODEL

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### Abstract

We present a reactive beta model that accounts for the leverage effect and beta elasticity. For this purpose, we derive a correlation metric for the leverage effect to identify the relation between the market beta and volatility changes. An empirical test based on the most popular market-neutral strategies is run from 2000 to 2015 with exhaustive data sets, including 600 U.S. stocks and 600 European stocks. Our findings confirm the ability of the reactive beta model to remove an important part of the bias from the beta estimation and from most popular market-neutral strategies. To examine the robustness of the reactive beta measurement, we conduct Monte Carlo simulations over seven market scenarios against five alternative methods. The results confirm that the reactive model significantly reduces the bias overall when financial markets are stressed.

*JEL Classification:* C5, G01, G11, G12, G32

### I. Introduction

Finding an appropriate measure of market betas is of paramount importance for many financial applications, including market-neutral hedge fund managers who target a near-zero beta. Contrary to common belief, perfect beta-neutral strategies are difficult to achieve in practice, as the mortgage crisis in 2008 exemplified, when most market-neutral funds remained correlated with stock markets and experienced considerable unexpected losses. This exposure to the stock index (Banz 1981; Fama and French 1992, 1993; Carhart 1997; Ang et al. 2006) is even stronger during down market conditions (Moreira and Muir 2017; Agarwal and Naik 2004; Bussière, Hoerova, and Klaus 2015). In such a period of market stress, hedge funds may even add no value (Asness, Krail, and Liew 2001).

In this article, we derive a stock market beta measure that we implement to test the quality of hedging for four popular strategies in the hedge funds industry. The first and most important strategy captures the low-beta anomaly (Black 1972; Black, Jensen, and Scholes 1972; Haugen and Heins 1975; Haugen and Baker 1991; Ang et al. 2006; Baker, Bradley, and Taliaferro 2013; Frazzini and Pedersen 2014; Hong and Sraer 2016) that defies conventional wisdom on the risk and reward trade-off predicted by the capital

asset pricing model (CAPM) (Sharpe 1964). According to this anomaly, high-beta stocks underperform low-beta stocks. Similarly, stocks with high idiosyncratic volatility earn lower returns than stocks with low idiosyncratic volatility (Malkiel and Xu 1997; Goyal and Santa-Clara 2003; Ang et al. 2006, 2009). The related strategy consists of shorting high-beta stocks and buying low-beta stocks. The second important strategy captures the size effect (Banz 1981; Reinganum 1981; Fama and French 1992), in which stocks of small firms tend to earn higher returns, on average, than stocks of larger firms. The related strategy consists of buying stocks with small market capitalization and shorting those with high market capitalization. The third strategy captures the momentum effect (Jegadeesh and Titman 1993; Carhart 1997; Grinblatt and Moskowitz 2004; Fama and French 2012), where past winners tend to continue to show high performance. This strategy consists of buying the past year's winning stocks and shorting the past year's losing stocks. The fourth strategy captures the short-term reversal effect (Jegadeesh 1990), where past winners in the last month tend to show low performance. This strategy consists of buying the past month's losing stocks and shorting the past month's winning stocks, which would be highly profitable if there were no transaction cost and no market impact. Testing the quality of the hedge of the strategies is equivalent to assessing the quality of the beta measurements, which is difficult to realize directly as the true beta is not known.

The implementation of all these strategies requires a reliable estimation of the betas to maintain the hedge. Ordinary least squares (OLS) estimation remains the most frequently employed method, even though it is impaired in the presence of outliers, especially from small companies (Fama and French 2008), illiquid companies (Amihud 2002; Acharya and Pedersen 2005; Ang, Shtaubert, and Tetlock 2013), and business cycles (Ferson and Harvey 1999). In these circumstances, the OLS beta estimator might be inconsistent. To overcome these limitations, our approach consists of renormalizing the returns to make them closer to Gaussian and thus to make the OLS estimator more consistent. In addition, many papers report that betas are time varying (Blume 1971; Fabozzi and Francis 1978; Jagannathan and Wang 1996; Fama and French 1997; Bollerslev, Engle, and Wooldridge 1988; Lettau and Ludvigson 2001; Lewellen and Nagel 2006; Ang and Chen 2007; Engle 2016). This can lead to measurement errors that could create serious bias in the cross-sectional asset pricing test (Shanken 1992; Chan and Lakonishok 1992; Meng, Hu, and Bai 2011; Bali, Engle, and Tang 2017). In fact, firms' stock betas do change over time for several reasons. The firm's assets tend to vary over time via acquiring or replacing new businesses, which makes them more diversified. The betas also change for firms that change in dimension to be safer or riskier. For instance, financial leverage may increase when firms become larger, as they can issue more debt. Moreover, firms with higher leverage are exposed to a more unstable beta (Galai and Masulis 1976; DeJong and Collins 1985). One way to account for the time dependence of betas is to consider regime changes when the return history used in the beta estimation is long enough. Surprisingly, only one paper (Chen, Zhang, and Wu 2005) suggests a solution to capture the time dependence and discusses regime changes for the beta using a multiple structural change methodology. The study shows that the risk related to beta regime changes is rewarded by higher returns. Another approach is to examine the correlation dynamics. Francis (1979) finds that "the correlation with the

market is the primary cause of changing betas . . . the standard deviations of individual assets are fairly stable” (p. 989). This finding calls for special attention to the correlation dynamics addressed in our article but are apparently insufficiently investigated in other works.

Despite the extensive literature on this issue, little attention has been paid to the link between the leverage effect<sup>1</sup> and the beta. The leverage effect is defined as the negative correlation between the securities’ returns and their volatility changes. This correlation induces residual correlations between the stock overperformances and beta changes. In fact, earlier studies have heavily focused on the role of the leverage effect on volatility (Black 1976; Christie 1982; Campbell and Hentchel 1992; Bekaert and Wu 2000; Bouchaud, Matacz, and Potters 2001; Valeyre et al. 2013). Surprisingly, despite its theoretical and empirical underpinnings, the leverage effect has not been considered so far in beta modeling, while it is a measure of risk. We aim to close this gap.

Our article starts by investigating the role of the leverage effect in the correlation measure by extending the reactive volatility model (Valeyre et al. 2013), which efficiently tracks the implied volatility by capturing both the retarded effect induced by the specific risk and the panic effect, which occurs whenever the systematic risk becomes the dominant factor. This allows us to set up a reactive beta model incorporating three independent components, all of which contribute to a reduction in the hedging bias. First, we take into account the leverage effect on beta, where the beta of underperforming stocks tends to increase. Second, we consider a leverage effect on correlation, in which a stock index decline induces an increase in correlations. Third, we model the relation between the relative volatility (defined as the ratio of the stock’s volatility to the index’s volatility) and the beta. When the relative volatility increases, the beta increases as well. All three independent components contribute to a reduction in the biases in the naive regression estimation of the beta and therefore considerably improve hedging strategies.

The main contribution of this article is the formulation of a reactive beta model. The economic intuition behind the reactive beta model is the derivation of a suitable beta measure allowing market beta estimation with reduced bias and a smaller standard deviation. The model is coined “reactive” because the beta measurement is adjusted as soon as prices move. An empirical test is performed based on an exhaustive data set that includes the 600 largest American stocks and the 600 largest European stocks from 2000 to 2015, which includes several business cycles. This test validates the superiority of the reactive beta model over conventional methods.

We further examine the robustness of the reactive beta measurement using Monte Carlo simulation against five alternative methods (OLS, minimum absolute deviation (MAD), trimean quantile regression [TRM], dynamic conditional correlation

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<sup>1</sup> Note that we are not dealing with the restricted definition of the “leveraged beta” that comes from the degree of leverage in the firm’s capital structure. Notice that the market beta may be nonlinearly related to the market return, which could lead to spurious inference in beta measurement (DeBondt and Thaler 1987), whereas the leverage effect could be a major explanation of such nonlinearity. For example, Garlappi and Yan (2011) relate leverage to default probability, Daniel, Jagannathan, and Kim (2012) relate the financial leverage to the operating leverage, Choi (2013) relates leverage to economic conditions, Mitchell and Pulvino (2001) relate leverage and volatility managed portfolios, and Liu, Stambaugh, and Yuan (2018) relate leverage to the beta-idiosyncratic volatility relation. In this context, the time-variation effect in conditional beta adds on this bias (Boguth et al. 2011).

[DCC], and asymmetric absolute deviation [ADCC]) over seven scenarios that reflect various market conditions from calm (Gaussian universe) to stressed (non-Gaussian universe). The results confirm that the reactive beta presents a lower bias when stressed market conditions are included.

## II. The Reactive Beta Model

In this section, we present the reactive beta model with three independent components. First, we take into account the specific leverage effect on the beta. Second, we consider the systematic leverage effect on the correlation. Third, we model the relation between the relative volatility and the beta via nonlinear beta elasticity.

### *The Leverage Effect on Beta*

We first account for relations among returns, volatilities, and the beta, which are characterized by the so-called leverage effect. This component takes into account the phenomenon where a beta increases as soon as a stock underperforms the index. Such a phenomenon can be fairly well described by the leverage effect captured in the reactive volatility model. We call the “specific leverage effect” the negative relation between specific returns and the risk (here, the beta), where the specific return is the nonsystematic part of the returns (a stock’s overperformance). The specific leverage effect on the beta follows the same dynamics as the specific leverage effect introduced in the reactive volatility model.

*The Reactive Volatility Model.* This section aims to capture the dependence of betas on stock overperformance (when a stock is overperforming, its beta tends to decrease). For this purpose, we rely on the methodology of the reactive volatility model (Valeyre et al. 2013) to derive a stable measure of the beta by using the renormalization factor that depends on the stock’s overperformance. The model describes the systematic and specific leverage effects. Systematic leverage, which is due to the panic effect, and specific leverage, which is due to a retarded effect, have very different relaxation times and intensities. These two effects are investigated by Bouchaud, Matacz, and Potters (2001), who introduce the measurement of the returns’ volatility correlation function at different time scales  $\tau$ . They define this measurement as  $L(\tau) = E(r^2(t + \tau)r(t)) / E^2(r^2(t))$ , where  $r(t)$  is the daily return at day  $t$ , and they show that it exhibits an exponential decay curve depending on  $\tau$  with two parameters: the relaxation time and the initial amplitude, which describes the intensity of the leverage. The intensity measured is nine times higher for the stock index than for the single stocks, and the relaxation time is six times smaller for the stock index. The higher intensity and the shorter relaxation times applied to the stock index are explained by the panic effect that occurs as soon as all single stocks decrease at the same time. The low intensity and the longer relaxation time applied to single stocks are explained by the retarded effect: on short time scales, the standard deviation of differences in price is the criteria used by traders to assess the risk, whereas on longer time scales, the standard deviation of returns is used. The retarded effect works as if traders need time to take into account a change in price in the analysis of

the risk. The reactive volatility model reproduces very well the measurement of  $L(\tau)$  for the stock index and for the single stocks.

We start by recalling the construction of the reactive volatility model, which explicitly accounts for the leverage effect on volatility. Let  $I(t)$  be a stock index at day  $t$ . It is well known that arithmetic returns,  $r_I(t) = \delta I(t)/I(t-1)$ , are heteroskedastic, partly because of price–volatility correlations. Throughout the text,  $\delta$  refers to the difference between successive values, for example,  $\delta I(t) = I(t) - I(t-1)$ . The reactive volatility model aims to construct an appropriate “level” of the stock index,  $L(t)$ , to replace the original returns  $\delta I(t)/I(t-1)$  with less heteroskedastic returns  $\delta I(t)/L(t-1)$ .

For this purpose, we first introduce two “levels” of the stock index as exponential moving averages (EMAs) with two time scales: a slow level,  $L_s(t)$ , and a fast level,  $L_f(t)$ . In addition, we denote by  $L_{is}(t)$  the EMA (with the slow time scale) of the price  $S_i(t)$  of stock  $i$  at time  $t$ . These EMAs can be computed using standard linear relations:

$$L_s(t) = (1 - \lambda_s)L_s(t-1) + \lambda_s I(t), \quad (1)$$

$$L_f(t) = (1 - \lambda_f)L_f(t-1) + \lambda_f I(t), \quad (2)$$

$$L_{is}(t) = (1 - \lambda_s)L_{is}(t-1) + \lambda_s S_i(t), \quad (3)$$

where  $\lambda_s$  and  $\lambda_f$  are the weighting parameters of the EMAs that we set to  $\lambda_f = 0.0241$  and  $\lambda_s = 0.1484$ , relying on the estimates by Bouchaud, Matacz, and Potters (2001). The slow parameter corresponds to the relaxation time of the retarded effect for specific risk, whereas the fast parameter corresponds to the relaxation time of the panic effect for systematic risk. These two relaxation times are found to be universal, as they are stable over years and do not change among different mature stock markets. The appropriate levels,  $L(t)$  and  $L_i(t)$ , accounting for the leverage effect on the volatility to correctly normalize the difference in price, are introduced for the stock index and individual stocks, respectively:<sup>2</sup>

$$L(t) = I(t) \left( 1 + \frac{L_s(t) - I(t)}{I(t)} \right) \left( 1 + \ell \frac{L_f(t) - I(t)}{L_f(t)} \right), \quad (4)$$

$$L_i(t) = S_i(t) \underbrace{\left( 1 + \frac{L_{is}(t) - S_i(t)}{S_i(t)} \right)}_{\text{specific risk}} \underbrace{\left( 1 + \ell_i \frac{L_f(t) - I(t)}{L_f(t)} \right)}_{\text{systematic risk}}, \quad (5)$$

with the parameters  $\ell$  and  $\ell_i$  quantifying the leverage. The parameter  $\ell$  is introduced by Valeyre et al. (2013) to reproduce the exponential fit of the returns’ volatility correlation

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<sup>2</sup>In practice, a filtering function is introduced to attenuate the contribution from eventual outliers (extreme events or wrong data). The filter is applied to  $z = \frac{L_s(t) - I(t)}{I(t)}$  and  $z = \frac{L_{is}(t) - S_i(t)}{S_i(t)}$  in equations (4) and (5) and is defined as  $F_\phi(z) = \tanh(\phi z)/\phi$  with  $\phi = 3.3$  (in the limit  $\phi = 0$ , there is no filter:  $F_0(z) = z$ ).

function  $L(\tau)$  at different time scales  $\tau$ . The initial parameters of the exponential fit are estimated on seven major stock indexes so that  $\ell$  is deduced to be approximately 8. If  $\ell = \ell_i$ , the correlation between the stock index and the individual stock  $i$  is not affected by the leverage effect. In turn, if  $\ell > \ell_i$ , the correlation increases when the stock index decreases. Although  $\ell_i$  can generally be specific to the considered  $i$ th stock, we ignore its possible dependence on  $i$  and set  $\ell_i = \ell'$ . Using the levels  $L(t)$  and  $L_i(t)$ , we introduce the normalized returns:

$$\tilde{r}_I = \tilde{r}_I(t) = \frac{\delta I(t)}{L(t-1)}, \quad \tilde{r}_i = \tilde{r}_i(t) = \frac{\delta S_i(t)}{L_i(t-1)} \quad (6)$$

and compute the renormalized variances  $\tilde{\sigma}_I^2$  and  $\tilde{\sigma}_i^2$  through the EMAs as:

$$\tilde{\sigma}_I^2(t) = (1 - \lambda_\sigma) \tilde{\sigma}_I^2(t-1) + \lambda_\sigma \tilde{r}_I^2(t), \quad (7)$$

$$\tilde{\sigma}_i^2(t) = (1 - \lambda_\sigma) \tilde{\sigma}_i^2(t-1) + \lambda_\sigma \tilde{r}_i^2(t), \quad (8)$$

where  $\lambda_\sigma$  is a weighting parameter that has to be chosen as a compromise between the accuracy of the estimated renormalized volatility and the reactivity of that estimation. Indeed, the renormalized returns are constructed to be homoskedastic only at short times because the renormalization based on the leverage effect with short relaxation times ( $\lambda_s$ ,  $\lambda_f$ ) cannot account for long periods of changing volatility related to economic cycles. Because economic uncertainty does not change significantly two months (40 trading days), we set  $\lambda_\sigma$  to  $1/40 = 0.025$ . This sample length leads to a statistical uncertainty of approximately  $\sqrt{1/40} \approx 16\%$ . Finally, these renormalized variances can be converted into the reactive volatility  $\sigma_I(t)$  of the stock index quantifying the systematic risk governed by the panic effect, and the reactive volatility  $\sigma_i(t)$  of each individual stock quantifying the specific risk governed by the leverage effect:

$$\sigma_I(t) = \tilde{\sigma}_I(t) \frac{L(t)}{I(t)}, \quad (9)$$

$$\sigma_i(t) = \tilde{\sigma}_i(t) \frac{L_i(t)}{S_i(t)}. \quad (10)$$

This reactive volatility captures a large part of the heteroskedasticity; that is, a large part of the volatility variation is completely explained by the leverage effect. That is the main result of Valeyre et al. (2013): for instance, if the stock index loses 1%,  $\frac{L(t)}{I(t)}$  increases by  $\ell \times 1\% = 8\%$ , and the stock index volatility increases by 8%. This effect is enough to capture a large part of the VIX (VIX Index is a measure of the one-month implied volatility of the U.S. stock market) variation, with  $R^2 = 0.46$ . In turn, if the stock underperforms the stock index by 1%,  $\frac{L_i(t)}{S_i(t)}$  increases by 1%, and the single stock volatility increases by 1%.

*The Specific Leverage Effect in the Reactive Beta Model.* The volatility estimation procedure naturally affects the estimation of the beta. Many financial

instruments rely on the estimated beta,  $\beta_i$ , which corresponds to the slope of a linear regression of the stocks' arithmetic returns  $r_i$  on the index arithmetic return  $r_I$ :

$$r_i = \beta_i r_I + \epsilon_i, \quad \text{with } r_i = \frac{\delta S_i(t)}{S_i(t-1)}, \quad r_I = \frac{\delta I(t)}{I(t-1)}, \quad (11)$$

where  $\epsilon_i$  is the residual random component specific to stock  $i$ . We consider another beta estimate,  $\tilde{\beta}_i$ , based on the reactive volatility model, in which the renormalized stock returns  $\tilde{r}_i$  are regressed on the renormalized stock index returns  $\tilde{r}_I$ :

$$\tilde{r}_i = \tilde{\beta}_i \tilde{r}_I + \tilde{\epsilon}_i, \quad \text{with } \tilde{r}_i = \frac{\delta S_i(t)}{L_i(t-1)}, \quad \tilde{r}_I = \frac{\delta I(t)}{L(t-1)}. \quad (12)$$

We then obtain a reactive beta measure:

$$\beta_i(t) = \tilde{\beta}_i(t) \frac{\sigma_i(t) \tilde{\sigma}_I(t)}{\sigma_I(t) \tilde{\sigma}_i(t)} = \tilde{\beta}_i \frac{L_{is}(t)I(t)}{L_s(t)S_i(t)}, \quad (13)$$

which includes two improvements:

- $\tilde{\beta}_i$ , which becomes less sensitive to price changes by accounting for the specific leverage effect;
- $\sigma_i \tilde{\sigma}_I / (\sigma_I \tilde{\sigma}_i)$ , which changes instantaneously with price changes.

When taking into account the short-term leverage effect in correlations, the reactive term is reduced to  $\frac{L_{is}(t)I(t)}{L_s(t)S_i(t)}$ . This term has a significant impact, as the beta of underperforming stocks should increase.

### *The Systematic Leverage Effect on Correlation*

*The Empirical Estimation of  $\ell^l$  for Single Stocks.* We use the term “systematic leverage effect” to denote the negative relation between systematic returns and the risk (here, the correlation), where the systematic returns are the nonspecific part of the returns (stock index performance). The systematic leverage effect on the correlation follows the same dynamics as the systematic leverage effect introduced in the reactive volatility model (the phenomenon's duration is approximately seven days for  $\lambda_f = 0.1484$ ). All correlations are affected in the same way by the systematic leverage effect, and single stocks and their stock indexes should also shift in the same direction. This explains why the stock's beta does not change with respect to the index. The implication is that betas are not very sensitive to the systematic leverage effect, in contrast to the specific leverage effect. We consider the impact of the short-term systematic leverage effect on correlation. Assuming that the correlation between each stock and the stock index is the same for all stocks, we can define the implied correlation as:<sup>3</sup>

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<sup>3</sup> See <http://www.cboe.com/micro/impliedcorrelation/impliedcorrelationindicator.pdf>

$$\rho(t) = \frac{\sigma_I^2(t) - \sum_i w_i^2 \sigma_i^2(t)}{\sum_{i \neq j} w_i w_j \sigma_i(t) \sigma_j(t)}, \quad (14)$$

where  $w_i$  represents the weight of stock  $i$  in the index. Denoting

$$e_I(t) = \frac{\widehat{L}_s(t)}{I(t)} - 1, \quad e_i(t) = \frac{\widehat{L}_{is}(t)}{S_i(t)} - 1, \quad (15)$$

we use equations (9) and (10) to obtain:

$$\rho = \frac{\tilde{\sigma}_I^2 (1 + e_I)^2 \left(1 + \ell \frac{L_f - I}{L_f}\right)^2 - \left(1 + \ell' \frac{L_f - I}{L_f}\right)^2 \sum_i w_i^2 (1 + e_i)^2 \sigma_i^2}{\left(1 + \ell' \frac{L_f - I}{L_f}\right)^2 \sum_{i \neq j} w_i w_j \tilde{\sigma}_i \tilde{\sigma}_j (1 + e_i) (1 + e_j)}. \quad (16)$$

If the weights  $w_i$  are small, we can ignore the second term; in addition, if  $e_i$  are small, then

$$\sum_{i \neq j} w_i w_j \tilde{\sigma}_i \tilde{\sigma}_j (1 + e_i) (1 + e_j) \approx (1 + e_I)^2 \tilde{\sigma}_0^2,$$

where  $\tilde{\sigma}_0^2$  is an average of  $\tilde{\sigma}_i^2$ . Keeping only the leading terms of the expansion in terms of the small parameter  $(L_f - I)/L_f$ , we thus obtain:

$$\rho \approx \frac{\tilde{\sigma}_I^2}{\tilde{\sigma}_0^2} \left(1 + 2(\ell - \ell') \frac{L_f - I}{L_f}\right). \quad (17)$$

This relation shows the dynamics of the implied correlation  $\rho$  induced by the leverage effect (accounted for through the factor  $(L_f - I)/L_f$ ). We assume that the same dynamics are applicable to correlations between individual stocks, that is,

$$\rho_{i,j} = \tilde{\rho}_{i,j} \left(1 + 2(\ell - \ell') \frac{L_f - I}{L_f}\right), \quad (18)$$

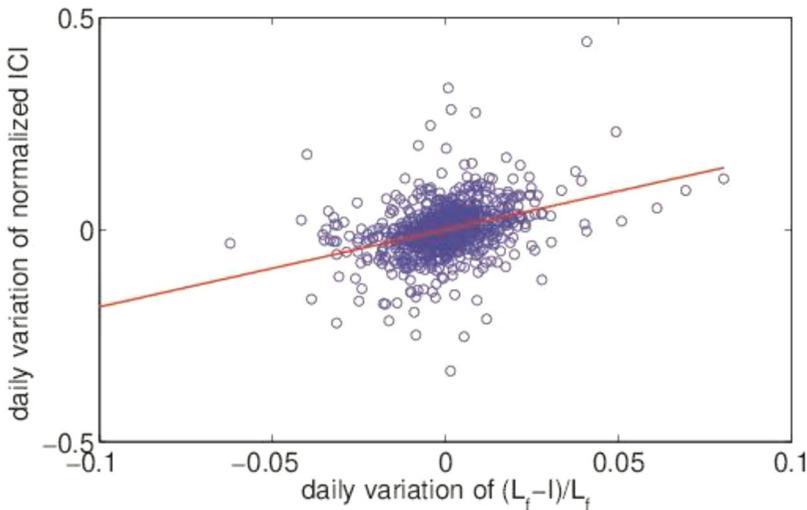
where  $\tilde{\rho}_{i,j}$  are the parameters specific to each pair of stocks  $i$  and  $j$ . From this relation, we derive a measure of correlation accounting for the leverage effect between the single stock  $i$  and the stock index:

$$\rho_i = \tilde{\rho}_i \left(1 + (\ell - \ell') \frac{L_f - I}{L_f}\right), \quad (19)$$

where  $\tilde{\rho}_i$  are the parameters specific to each stock  $i$ . Note that there is no factor 2 in front of  $(\ell - \ell')$  in equation (19) because we have a one-factor model here. We use

equation (19) in the reactive beta model (see equations (34) and (36)) to take into account the varying nature of the correlation in the regression. We rescale the measurement by the normalization factor  $(1 + (\ell - \ell')(L_f - I)/L_f)$  and then recover the variation of the correlation through the denormalization factor  $1/(1 + (\ell - \ell')(L_f - I)/L_f)$ . We emphasize that the parameter  $\ell$  in equation (4) that quantifies the systematic leverage for the stock index is slightly different from the parameter  $\ell'$  in equation (5) that quantifies the systematic leverage for single stocks. According to equation (18), when the market decreases, correlations between stocks increase as  $\ell > \ell'$ , and therefore, the stock index volatility increases more than the single stock's volatility:  $\delta(\sigma_i/\sigma_I) < 0$ . Once again, the beta is, in contrast to the correlation, weakly affected by the systematic leverage effect, as all correlations increase in the same way. More precisely, this means that the impact of the increase in correlation in the beta measurement is compensated by a decrease in the relative volatility:  $\delta(\sigma_i/\sigma_I) < 0$ ; that is, the single-stock volatility increase is lower than that of the stock index volatility. For this reason, the reactive beta model in equations (34) and (36) is not very sensitive to the choice of  $\ell'$ . Nevertheless, we explain in this section how  $\ell'$  is calibrated using the implied volatility index. We measure the level of the systematic leverage effect  $\ell'$  for a single stock by regressing equation (17) with data from the market-implied correlation S&P 500 index. Figure I illustrates the slope of this regression. By regressing  $\frac{L_f - I}{L_f}$  against  $\frac{\rho}{\rho_0}$ , where  $\tilde{\rho}_0$  is the average of  $\rho$ , we deduce that empirically, we can set:

$$\ell - \ell' = 0.91 \pm 0.08, \tag{20}$$



**Figure I. Daily Variations of the Chicago Board Options Exchange S&P 500 Implied Correlation Indices (ICI) Since Their Inception, Divided by Their Mean, versus Daily Variations of the Leverage Factor  $(L_f - I)/L_f$ .** A linear regression (solid line) yields the coefficient  $1.82 \pm 0.16$  (i.e.,  $2(\ell_i - \ell') = 1.82$ ), with  $R^2 = 0.13$  and  $t$ -statistics of 11.4. Sample period is 2007–2015. [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

with a  $t$ -statistic of 11.4. Because  $\ell - \ell' \ll \ell (= 8)$ , we deduce an important result, namely, that the systematic leverage impact on the correlation is more than eight times smaller than the systematic leverage impact on the volatility. The main consequence is that although it is statistically significant, the leverage effect is not a major component of the correlation.

*The Systematic Leverage Effect Component in the Reactive Model.* As discussed earlier, the correlation increases when the stock index price decreases. This effect could generate a bias in the beta measurement, as stock index prices could fluctuate in a sample used to measure the slope. Our solution is to adjust the beta between renormalized returns through the correction factor  $\mathcal{L}(t)$ , defined as

$$\mathcal{L}(t) = 1 + (\ell - \ell') \left( \frac{L_f(t-1) - I(t-1)}{L_f(t-1)} \right). \quad (21)$$

The correction factor  $\mathcal{L}(t)$  should be used to estimate the slope between the stock index and single-stock returns and should then be used to denormalize the slope to obtain the reactive beta that depends directly on  $\mathcal{L}(t)$ .

#### *The Relation between Relative Volatility and Beta*

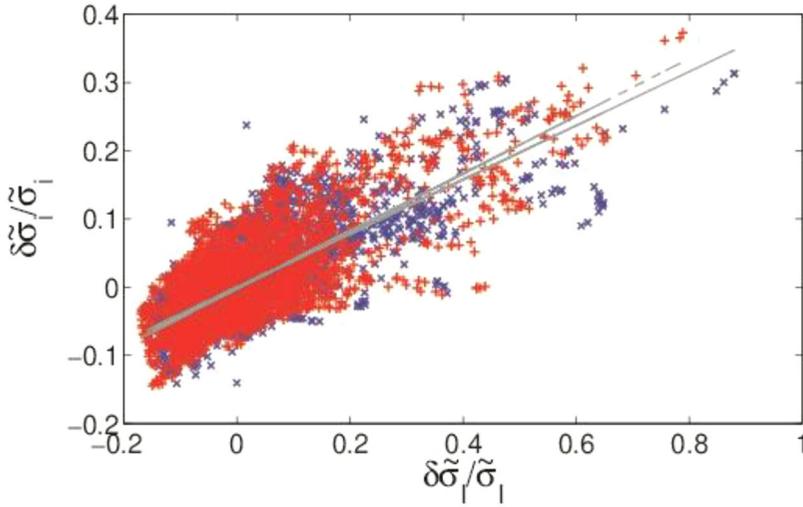
*The Empirical Estimation of Beta Elasticity.* In this part, we identify correlations between the relative volatility and beta changes. We choose the relative volatility, defined as the ratio  $\tilde{\sigma}_i/\tilde{\sigma}_I$ , as an explanatory variable of  $\tilde{\beta}_i$  because  $\tilde{\beta}_i$  is expected to be constant if the ratio  $\tilde{\sigma}_i/\tilde{\sigma}_I$  is constant. However, empirically, the ratio  $\tilde{\sigma}_i/\tilde{\sigma}_I$  can change dramatically between periods of high dispersion (i.e., when stocks are, on average, weakly correlated) and low systematic risk (i.e., when stock indexes are not stressed) and periods of low dispersion and high systematic risk. Figure II illustrates, for both European and U.S. markets, that the dispersion among stocks decreases, on average, when markets become volatile. A linear regression of rescaled daily variations of  $\tilde{\sigma}_i$  yields:

$$\frac{\delta \tilde{\sigma}_i(t)}{\tilde{\sigma}_i(t-1)} \approx 0.4 \frac{\delta \tilde{\sigma}_I(t)}{\tilde{\sigma}_I(t-1)} + \epsilon_i, \quad (22)$$

where  $\epsilon_i$  is the residual (specific) noise. Using the standard rules for infinitesimal increments, we find from this regression the following:

$$\delta \left( \frac{\tilde{\sigma}_i}{\tilde{\sigma}_I} \right) \simeq \frac{\delta \tilde{\sigma}_i}{\tilde{\sigma}_I} - \frac{\tilde{\sigma}_i \delta \tilde{\sigma}_I}{\tilde{\sigma}_I^2} = \frac{\tilde{\sigma}_i}{\tilde{\sigma}_I} \left( \frac{\delta \tilde{\sigma}_i}{\tilde{\sigma}_i} - \frac{\delta \tilde{\sigma}_I}{\tilde{\sigma}_I} \right) \simeq -0.6 \frac{\tilde{\sigma}_i}{\tilde{\sigma}_I} \frac{\delta \tilde{\sigma}_I}{\tilde{\sigma}_I}; \quad (23)$$

that is, the relative volatility  $\tilde{\sigma}_i/\tilde{\sigma}_I$  is relatively stable, but its small variations can still affect the beta estimation. This empirical relation shows that when there is a volatility shock in the market, the stock index volatility increases much faster than the average single stock volatility.



**Figure II. Normalized Daily Variations of  $\tilde{\sigma}_i$ ,  $\delta\tilde{\sigma}_i/\tilde{\sigma}_i = \frac{\tilde{\sigma}_i(t) - \tilde{\sigma}_i(t-1)}{\tilde{\sigma}_i(t-1)}$  versus Normalized Daily Variations of  $\tilde{\sigma}_I$ ,  $\delta\tilde{\sigma}_I/\tilde{\sigma}_I = \frac{\tilde{\sigma}_I(t) - \tilde{\sigma}_I(t-1)}{\tilde{\sigma}_I(t-1)}$  for the European Market (blue crosses) and the U.S. Market (red pluses).**  
 The two gray lines show the linear regressions of the two data sets, with regression coefficients of 0.40 ( $R^2 = 0.60$ ) and 0.42 (with  $R^2 = 0.59$ ) for the European and U.S. markets, respectively. The time frame includes observations from the technology bubble burst, U.S. subprime, and Euro debt crises. Sample period is 1998–2015. [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

Because we want to take into account the impact of the relative volatility change on the beta measurement, we introduce the beta elasticity as the derivative of the beta with respect to the logarithm of the squared relative volatility:

$$f(\tilde{\beta}_i) = \frac{d\tilde{\beta}_i}{d \ln(\tilde{\sigma}_i^2/\tilde{\sigma}_I^2)} = \frac{d\tilde{\beta}_i}{d(\tilde{\sigma}_i/\tilde{\sigma}_I)} \frac{\tilde{\sigma}_i}{2\tilde{\sigma}_I}. \quad (24)$$

We expect that  $f(\tilde{\beta}_i)$  is positive and increasing with  $\tilde{\beta}_i$ . Indeed, we expect that a stock with a low beta should have a stable beta (less sensitive to its relative volatility increase), as the increase in this case is most likely due to a specific risk increase. In such a case, the sensitivity of the beta to the relative volatility is weak. In the opposite case of a high beta, a stock that is highly sensitive to the stock index will face a beta decline as soon as its relative volatility decreases. Consequently, when there is a volatility shock in the market,  $\delta\left(\frac{\tilde{\sigma}_i}{\tilde{\sigma}_I}\right)$  is negative, and therefore, the beta of stocks with a high beta and a high  $f$  is reduced. In turn, the stocks with a low beta are less affected because  $f$  is smaller and  $\delta(\tilde{\sigma}_i/\tilde{\sigma}_I)$  is expected to be less negative.

When the correlation of the stock with the stock index is constant, we can use a linear model:  $f(\tilde{\beta}_i) = \tilde{\beta}_i/2$ . In fact, using the relation  $\tilde{\beta}_i = \tilde{\rho}_i \frac{\tilde{\sigma}_i}{\tilde{\sigma}_I}$  and the assumption that  $\tilde{\rho}_i$  is constant (i.e., it does not depend on  $\frac{\tilde{\sigma}_i}{\tilde{\sigma}_I}$ ), we obtain from equation (24)

$$f = \tilde{\rho}_i \frac{\tilde{\sigma}_i}{2\tilde{\sigma}_I} = \tilde{\beta}_i/2. \text{ In general, however, the correlation can depend on the relative}$$

volatility, and thus, the function  $f$  may be more complicated. To estimate  $f$ , we need the renormalized beta and the relative volatility. For a better estimation, we aim to reduce the heteroskedasticity even further by using an exponential moving regression of the returns  $\tilde{r}_i$  and  $\tilde{r}_I$  that are renormalized by the estimated normalized index volatility  $\tilde{\sigma}_I$ . We denote these renormalized returns as:

$$\hat{r}_i(t) = \frac{\tilde{r}_i(t)}{\tilde{\sigma}_I(t-1)}, \quad \hat{r}_I(t) = \frac{\tilde{r}_I(t)}{\tilde{\sigma}_I(t-1)}. \quad (25)$$

Computing the EMAs,

$$\hat{\phi}_i(t) = (1 - \lambda_\beta)\hat{\phi}_i(t-1) + \lambda_\beta\hat{r}_i(t)\hat{r}_I(t), \quad (26)$$

$$\hat{\sigma}_I^2(t) = (1 - \lambda_\beta)\hat{\sigma}_I^2(t-1) + \lambda_\beta[\hat{r}_I(t)]^2, \quad (27)$$

with  $\lambda_\beta = 1/90$ , we estimate the beta as:

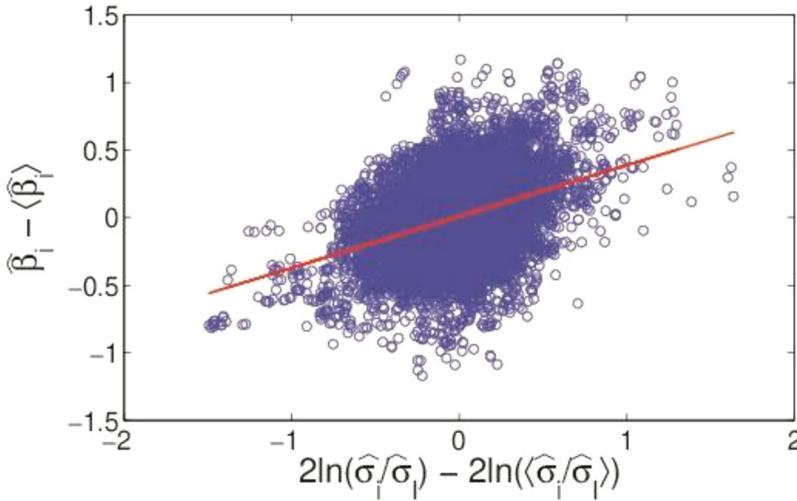
$$\hat{\beta}_i(t) = \frac{\hat{\phi}_i(t)}{\hat{\sigma}_I^2(t)}. \quad (28)$$

Here,  $\hat{\phi}_i$  is an estimation of the covariance between stock index returns and single-stock returns that includes two normalizations: the levels  $L_i$  and  $L$  from the reactive volatility model, and  $\tilde{\sigma}_I$  to further reduce heteroskedasticity. We write  $\hat{\beta}_i$  instead of  $\tilde{\beta}_i$  to stress this particular way of estimating the beta. Similarly, the hat symbol in equation (27) is used to distinguish  $\hat{\sigma}_I(t)$ , computed with renormalized index returns, from  $\tilde{\sigma}_I(t)$ . In principle, the above estimate  $\hat{\beta}$  could be directly regressed to the ratio of earlier estimates of  $\tilde{\sigma}_i$  and  $\tilde{\sigma}_I$  from equation (7). However, to use the normalization by  $\tilde{\sigma}_I$  consistently, we consider the ratio of these volatilities obtained in the renormalized form, that is,  $\hat{\sigma}_i(t)/\hat{\sigma}_I(t)$ , where  $\hat{\sigma}_I(t)$  is given in equation (27), and

$$\hat{\sigma}_i^2(t) = (1 - \lambda_\beta)\hat{\sigma}_i^2(t-1) + \lambda_\beta[\hat{r}_i(t)]^2. \quad (29)$$

Figure III illustrates the sensitivity of the beta to relative volatilities by plotting  $\hat{\beta}_i(t)$  from equation (28) versus  $\ln(\hat{\sigma}_i(t)/\hat{\sigma}_I(t))$  for all stocks  $i$  and times  $t$  from 2000 to 2015, although we display only the time frame 2014–2015 for clarity of illustration. On both axes, we subtract the mean values  $\langle \hat{\beta}_i \rangle$  and  $\ln(\langle \hat{\sigma}_i/\hat{\sigma}_I \rangle)$  averaged over all times in the whole sample. This plot enables us to measure the average of the  $f(\tilde{\beta}_i)$  in equation (24), which is close to  $0.76/2 = 0.38$ .

To obtain the dependence of  $f$  on the beta, we estimate the slope between  $\hat{\beta}_i(t) - \langle \hat{\beta}_i \rangle$  from equation (28) and  $2 \ln(\hat{\sigma}_i(t)/\hat{\sigma}_I(t)) - 2 \ln(\langle \hat{\sigma}_i/\hat{\sigma}_I \rangle)$  locally around each value of  $\hat{\beta}_i$ . For this purpose, we sort all collected values of  $\hat{\beta}_i$  and group them into successive subsets, each with 10,000 points. In each subset, we estimate the slope



**Figure III.** Relation between the Beta  $\hat{\beta}_i$  and the Doubled Logarithm of the Relative Volatility  $\ln(\hat{\sigma}_i/\hat{\sigma}_I)$ , from Which the Mean Values  $\langle \hat{\beta}_i \rangle$  and  $\ln(\langle \hat{\sigma}_i/\hat{\sigma}_I \rangle)$  Were Subtracted (the mean is obtained by averaging over time for each  $i$ ). A linear regression is shown by the solid line:  $\hat{\beta}_i - \langle \hat{\beta}_i \rangle = 0.76[\ln(\hat{\sigma}_i/\hat{\sigma}_I) - \ln(\langle \hat{\sigma}_i/\hat{\sigma}_I \rangle)]$ , with  $R^2 = 0.14$ . For better visualization, only 10,000 randomly selected points are shown (by circles) among 271,958 points from the European data set. Sample period is 2014–2015. [Color figure can be viewed at wileyonlinelibrary.com.]

between  $\hat{\beta}_i(t) - \langle \hat{\beta}_i \rangle$  from equation (28) and  $2 \ln(\hat{\sigma}_i(t)/\hat{\sigma}_I(t)) - 2 \ln(\langle \hat{\sigma}_i/\hat{\sigma}_I \rangle)$  by a standard linear regression over 10,000 points. This regression yields the value of  $f$  of the subset that corresponds to some average value of  $\hat{\beta}_i$ . Repeating this procedure over all subsets, we obtain the dependence of  $f$  on  $\hat{\beta}_i$ , which is plotted in Figure IV. We show that  $f$  increases with the beta. For both European and U.S. markets, we propose the following approximation of the function  $f$ . with three regimes:

$$f(\tilde{\beta}_i) = \begin{cases} 0, & \tilde{\beta}_i < 0.5, \\ 0.6(\tilde{\beta}_i - 0.5), & 0.5 < \tilde{\beta}_i < 1.6, \\ 0.6\tilde{\beta}_i > 1.6. \end{cases} \quad (30)$$

In the first regime, for low-beta stocks (mostly quality and value stocks), the beta elasticity is zero, which is equivalent to the constant beta case. For the intermediate regime, the elasticity increases linearly with  $\tilde{\beta}_i$  and is close to the constant correlation case with  $f(\tilde{\beta}_i) = \tilde{\beta}_i/2$ . In the third regime for high-beta stocks (mostly speculative and growth stocks), the elasticity is constant. The shape of the beta elasticity is similar for the European market and the U.S. market.

*The Nonlinear Beta Elasticity Component in the Reactive Model.* According to equation (30), the sensitivity of the normalized beta to changes in the relative volatility is nonlinear. This elasticity could generate bias in the beta estimation if the relative volatility changes in a sample used to measure the slope. Our solution is to adjust the beta between normalized returns through the correction factor  $\mathcal{F}(t)$ , defined as:

$$\mathcal{F}(t) = 1 + \frac{2f(\tilde{\beta}_i(t))}{\tilde{\beta}_i(t)} \Delta\left(\frac{\tilde{\sigma}_i}{\tilde{\sigma}_I}\right). \quad (31)$$

The function  $f$  is approximated by equation (30),  $\ell - \ell'$  is given by equation (20), and

$$\Delta\left(\frac{\tilde{\sigma}_i}{\tilde{\sigma}_I}\right) = \frac{\tilde{\sigma}_i(t-1)/\tilde{\sigma}_I(t-1) - \sqrt{\kappa_i(t-1)}}{\sqrt{\kappa_i(t-1)}} \quad (32)$$

with

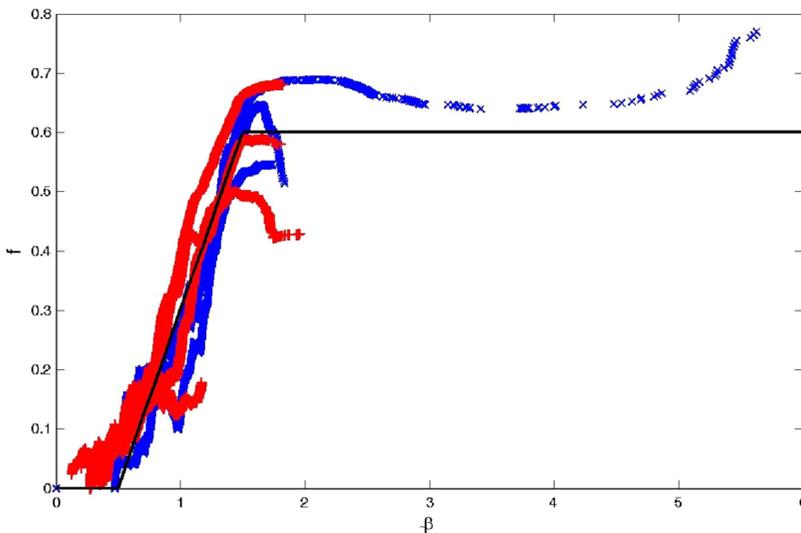
$$\kappa_i(t) = (1 - \lambda_\beta)\kappa_i(t-1) + \lambda_\beta \left(\frac{\tilde{\sigma}_i(t)}{\tilde{\sigma}_I(t)}\right)^2 \quad (33)$$

being the EMA of the squared relative volatility  $(\tilde{\sigma}_i/\tilde{\sigma}_I)^2$ . The  $\Delta(\tilde{\sigma}_i/\tilde{\sigma}_I)$  quantifies deviations of the relative volatility from its average over the sample that will be used to estimate the beta.

The correction factor  $\mathcal{F}(t)$  should be used to estimate the slope between the stock index and single-stock returns and should then be used to denormalize the slope to obtain the reactive beta that depends directly on  $\mathcal{F}(t)$ .

#### Summary of the Reactive Beta Model

In this section, we recapitulate the reactive beta model that combines the three independent components described in the previous sections: the specific leverage effect



**Figure IV.** The Function from  $f$  Equation (24) versus the Beta for the European Market (blue crosses) and the U.S. Market (red pluses). This function is estimated locally for four periods. The solid black line shows the approximation (30). Sample period is 2000–2015. [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

on the beta, the systematic leverage effect on correlation, and the relation between the relative volatility and the beta. Starting with the time series  $I(t)$  and  $S_i(t)$  for the stock index and individual stocks, we compute the levels  $L_f(t)$ ,  $L(t)$ , and  $L_i(t)$  from equations (2), (4), and (5); the normalized stock index and individual stock returns  $\tilde{r}_I(t)$  and  $\tilde{r}_i(t)$  from equation (6); the normalized stock index volatility  $\tilde{\sigma}_I(t)$  from equation (7); the renormalized stock index and individual stock returns  $\hat{r}_I(t)$  and  $\hat{r}_i(t)$  from equation (25); the associated volatilities  $\hat{\sigma}_I(t)$  and  $\hat{\sigma}_i(t)$  from equations (27) and (29); and the renormalized beta  $\hat{\beta}_i(t)$  from equation (28). From these quantities, we reevaluate the covariance between  $\hat{r}_i$  and  $\hat{r}_I$  by accounting for the leverage effects and excluding the other effects. In fact, we compute  $\hat{\Phi}_i(t)$  as an EMA of the normalized covariance of the normalized daily returns:

$$\hat{\Phi}_i(t) = (1 - \lambda_\beta)\hat{\Phi}_i(t - 1) + \lambda_\beta \frac{\hat{r}_i(t)\hat{r}_I(t)}{\mathcal{L}(t)\mathcal{F}(t)}, \quad (34)$$

where  $\mathcal{L}(t)$  and  $\mathcal{F}(t)$  are two correction factors defined in equations (21) and (31) that are used to withdraw bias from the systematic leverage and the beta elasticity. The parameter  $\lambda_\beta$  describes the look-back used to estimate the slope and is set to 1/90, as 90 days of look-back appears to be a good compromise. In fact, for a longer look-back, variations in beta, correlation, and volatilities are expected to happen because of changes in market stress and business cycles and are not taken into account properly by our reactive renormalization. In turn, for a shorter look-back, the statistical noise of the slope would be too high.

Finally, the stable estimate of the normalized beta is

$$\tilde{\beta}_i(t) = \frac{\hat{\Phi}_i(t)}{\hat{\sigma}_I^2(t)}, \quad (35)$$

with  $\hat{\sigma}_I^2(t)$  defined in equation (27) from which the estimated reactive beta of stock  $i$  is deduced as

$$\beta_i(t) = \tilde{\beta}_i(t) \left( \frac{L_i(t)I(t)}{S_i(t)L(t)} \right) \mathcal{L}(t)\mathcal{F}(t). \quad (36)$$

The estimation of the normalized stable beta  $\tilde{\beta}_i(t)$  is close to the slope estimated by an OLS<sup>4</sup> but with exponentially decaying weights to accentuate recent returns and with normalized returns to withdraw different biases. Then, the normalized stable beta  $\tilde{\beta}_i(t)$  is “denormalized” by the factor that combines the three main components: the specific leverage effect on beta  $(L_i/S_i)(I/L)$ , the systematic leverage effect  $\mathcal{L}(t)$ , and nonlinear beta elasticity  $\mathcal{F}(t)$ . The final beta estimation  $\beta_i(t)$  is a reactive dynamic

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<sup>4</sup>We assume that the average of daily returns is zero. That assumption makes sense as at a daily time scale, the average of returns can be completely neglected compared to the standard deviation.

conditional estimation that is adjusted as soon as prices move through the instantaneous variations of the three correction factors.

Every term affects the hedging of a certain strategy:

- The term with  $\mathcal{L}(t)$  does not have a significant impact on the beta, as it is compensated in  $L_i/L$ , which models the short-term systematic leverage effect on the correlation in equations (34) and (36) (introduced in Section II), whereas the levels  $L_i$  and  $L$  are introduced in the reactive volatility model. However, the correlation could be affected by +10% if the market decreases by 10%.
- The term with  $L_i I / (L S_i)$  that models the specific leverage effect on volatilities (introduced in Section II) could affect the beta by 10% if the stocks underperform by 10%. This term affects the hedging of the short-term reversal strategy.
- The term with  $\mathcal{F}(t)$  that models the nonlinear beta elasticity, which is the sensitivity of the beta to the relative volatility (introduced in Section III), could affect the beta by 10% if the relative volatility increases by 10%. This term affects the hedging of the low-volatility strategy.

The reduced version of the reactive beta model, when only the leverage effect is introduced without beta elasticity and stochastic normalized volatilities, defines an interesting class of stochastic processes that appears to be mean reverting with a standard deviation linked to  $\tilde{\sigma}_i \sqrt{1/\lambda_s}$  and a relaxation time linked to  $1/\lambda_s$ .

The reactive beta model is based on the fit of several well-identified effects. Implied parameters work universally for all stock markets ( $\ell - \ell'$  is the only parameter that is fitted only on the U.S. market, as the implied correlations for other countries are not traded). Here, we summarize the different parameters used in the reactive beta model:

- $\lambda_f = 0.1484$ , which describes the relaxation time of 7 days for the panic effect;
- $\lambda_s = 0.0241$ , which describes the relaxation time of 40 days for the retarded effect;
- $l = 8$ , which describes the leverage intensity of the panic effect;
- $\ell - \ell' = 0.91$ , based on implied correlations on the U.S. stock market;
- the different thresholds in the function  $f(\tilde{\beta}_i)$  from equation (30) that describes the nonlinear beta elasticity.

### III. Empirical Findings

#### *Data Description*

We use only daily returns. For the empirical calibration of  $\ell - \ell'$ , we choose the Chicago Board Options Exchange (CBOE) S&P 500 Implied Correlation Index (ICI), which is the first widely disseminated market-based estimate of implied average correlation of the stocks that comprise the S&P 500 Index. This index begins in July 2009, with historical data going back to 2007. We take the front-month correlation index data from 2007 and

roll it to the next contract until the previous one expires. We also use the daily S&P 500 stock index. For the empirical calibration of the other parameters of the reactive beta model, we use the daily S&P 500 stock index and the 600 largest U.S. stocks from January 1, 2000 to May 31, 2015. For the European market, we consider the EuroStoxx50 index and the 600 largest European stocks over the same period. The same data are used for both the calibration of parameters and empirical tests.

For consistency, we keep the parameters of the reactive volatility model that describe the intensity of the panic effect ( $I$ ), the relaxation time of the panic effect ( $\lambda_f$ ), and the relaxation time of the retarded effect ( $\lambda_s$ ) identical to those calibrated before 2000 by Bouchaud, Matacz, and Potters (2001), as they are considered universal.

### *Empirical Results*

In this section, we show that exposure to common risk factors can sometimes lead to a high exposure of market-neutral funds to the stock market index if the betas are not correctly assessed. Indeed, although market-neutral funds should be orthogonal to traditional asset classes, this is not always the case during extreme moves (Fung and Hsieh 1997). For instance, Patton (2009) tests the zero correlation against the nonzero correlation and finds that approximately 25% of the market-neutral funds exhibit significant non-neutrality, concluding that “many market neutral hedge funds are in fact not market neutral, but overall they are, at least, more market neutral than other categories of hedge funds” (p. 2498). The reactive beta model can help hedge funds be more market neutral than others. To demonstrate this, we empirically test the efficiency of hedging using the most popular market neutral strategies (low volatility, short-term reversal, momentum, and size):

- Low-volatility (beta) strategy: buying the stocks with the highest 30% beta and shorting those with the lowest 30% beta (estimated by OLS);
- Short-term reversal strategy: shorting the stocks with the highest 15% one-month returns and buying those with the lowest 15% one-month returns;
- Momentum strategy: buying the stocks with the highest 15% two-year returns and shorting those with the lowest 15% two-year returns;
- Size strategy: buying the stocks with the highest 30% capitalization and shorting those with the lowest 30% capitalization.

The construction of the four most popular strategies that target beta neutrality is explained in Appendix B. The different portfolios are dynamic. The efficiency of the hedge depends on the accuracy of the beta estimation. For each strategy, we compare two methods to estimate the beta that uses only past information to avoid look-ahead bias: OLS (which corresponds to a specific case of our model with  $L_i = S_i$ ,  $L = I$ ,  $\ell = \ell' = 0$ , and  $f = 0$ , with the same exponential weighting scheme) and our reactive method. We analyze two statistics:

- Statistic 1: The CorSTD, which is defined as the standard deviation of the 90-day correlation of the strategy with the stock index returns, describes the

TABLE 1. Efficiency of the hedge.

Statistics	OLS		Reactive	
	Bias	CorSTD	Bias	CorSTD
United States				
Low volatility	-25.54%	21.73%	-16.79%	21.43%
Short-term reversal	13.09%	18.96%	-6.06%	18.50%
Momentum	-6.27%	18.28%	-2.95%	16.54%
Size	-7.56%	17.00%	-1.84%	17.26%
Europe				
Low volatility	-22.39%	19.97%	-14.68%	20.94%
Short-term reversal	13.05%	17.51%	0.64%	14.52%
Momentum	-4.42%	18.03%	-1.55%	17.23%
Size	3.12%	17.15%	3.79%	15.63%

Note: Bias is defined as the correlation over the whole sample between the stock index and each of the ordinary least squares (OLS) and reactive strategies for the U.S. and Europe markets. CorSTD is defined as the standard deviation of the 90-day correlation over the whole sample between the stock index and each of the OLS and reactive strategies for the U.S. and Europe markets. The residual bias for the low-volatility strategy in the reactive method can be explained by the selection bias, as demonstrated in Appendix A. Sample period is 2000–2015.

lack of robustness of the hedge and, consequently, the inefficiency of the beta measurement. The more robust the strategy is, the lower the CorSTD statistics are. If the strategy was well hedged, the correlation fluctuates around 0, within the theoretical 10% standard deviation, and CorSTD is 10% (a CorSTD of 10% is obtained with two independent Gaussian variables for 90-day correlations).

- **Statistic 2:** The bias, which is defined as the correlation of the strategy with the stock index returns on the whole period, describes the bias in the hedge of the strategy and, therefore, the bias of the beta measurement.

These statistics present a proxy for assessing the quality of the beta measurement, which is difficult to realize directly, as true betas are not known.

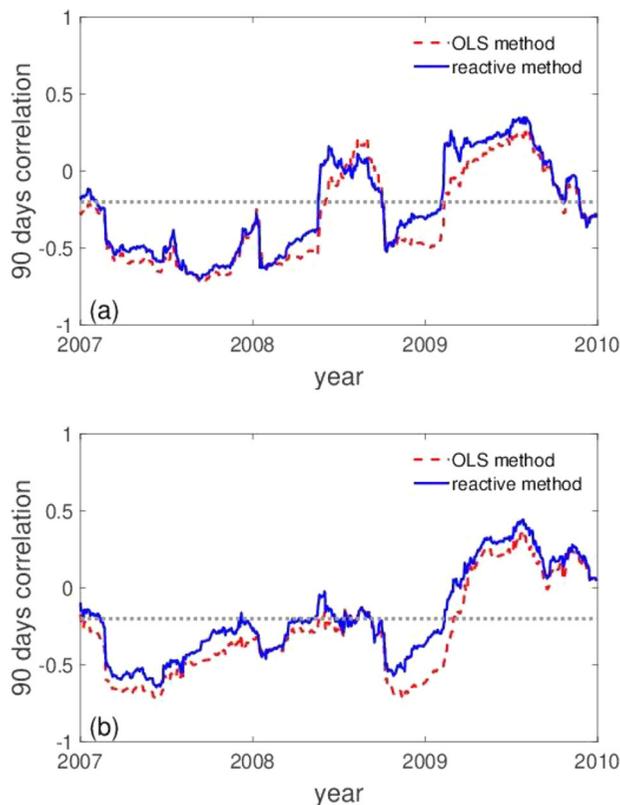
Table 1 summarizes the statistics of the four strategies for the U.S. and Europe markets. We see the highest bias for the low-volatility strategy when hedged with the standard approach ( $-25.5\%$  for United States and  $-22.4\%$  for Europe). The CorSTD is approximately 20%, that is, twice as high as expected if the volatility is stable, which means that the efficiency of the hedge is time varying. This could represent an important risk for the funds of funds managers, where hidden risk could accumulate and arise especially when the market is stressed. Indeed, the bias seems to have been higher by approximately  $-60\%$  for both the United States and Europe when the market was stressed in 2008. The use of the reactive beta model reduces the bias in the low-volatility factor, and the residual bias comes from the selection bias (see Appendix A). When using OLS, the possible loss in 2008 would have been  $-9.6\%$  ( $= -60\% \times 40\% \times 8\%/20\%$ ) for a 40% stock decline with a fund invested entirely in a low-volatility anomaly with a bias of  $-60\%$  and a target annualized volatility of 8% for the fund and for the index.

We also see a significant bias for the short-term reversal strategy when hedged with the standard approach (approximately 13.1% in the United States and Europe). The CorSTD is approximately 18%. The efficiency of the hedge depends on the recent past

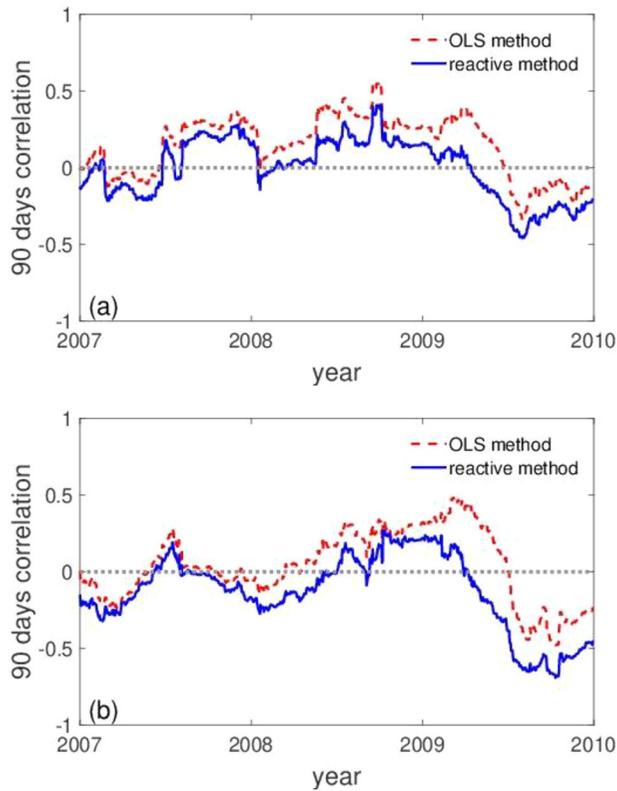
performance of the strategy. As soon as the strategy starts to lose, the efficiency declines and risk rises, as in 2009. Again, we see that the reactive beta model reduces the bias in the short-term reversal factor. The biases and CorSTD are lower for the momentum strategy (in the United States, with a CorSTD of 18.3%) and are of the same magnitude for the size strategy ( $-7.6\%$  in the United States with a CorSTD of 17.0%). The reactive beta model further reduces the bias and the CorSTD. This is also valid for the European market.

We conclude that the reactive beta model reduces the bias of the low-volatility factor when it is stressed by the market. The remaining residual is most likely explained by the selection bias (see Appendix A for a formal proof). The improvement is more significant for the momentum factor and the size factor in the United States only.

We also illustrate these findings by presenting the correlation between the stock index and the low-volatility strategy (Figure V) and the short-term reversal strategy (Figure VI), which are the strategies with the highest bias. We focus on the period surrounding the financial crisis (2007–2010). We can see that the beta computed by OLS



**Figure V. Ninety-Day Correlation of the Low-Volatility Factor with the Stock Index in the European Market (a) and in the U.S. market (b).** Solid and dashed lines present the proposed reactive beta model and the ordinary least squares (OLS) method, respectively. The dotted horizontal line shows the selection bias of  $-19.10\%$ , as shown in Appendix A. A time frame surrounding the financial crisis is chosen. Sample period is 2007–2010. [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]



**Figure VI. Ninety-Day Correlation of the Short-Term Reversal Factor with the Stock Index in the European Market (a) and the U.S. Market (b).** Solid and dashed lines present the proposed reactive beta model and the ordinary least squares (OLS) method, respectively. A time frame surrounding the financial crisis is chosen. Sample period is 2007–2010. [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

was positively and highly exposed to the stock index in 2008. In turn, the exposure was reduced within the reactive model. The improvement becomes even more impressive in extreme cases when the strategies are stressed by the market. We see that in some extreme cases (a stress period with extreme strategies), the common approach could generate high biases ( $-50\%$  for the short-term reversal strategies in 2008–2009 and  $-71\%$  for the beta strategy in 2008). In each case, our methodology allows us to significantly reduce the bias.

#### IV. Robustness Checks

This section presents a robustness check analysis by comparing the quality of several methods for beta measurements against the reactive beta model. We build the comparative analysis based on two important articles. Chan and Lakonishok (1992) enable the assessment of the robustness statistics of some alternative methods to the classical OLS method when assuming implicitly that betas are static and returns are

homoskedastic. This section extends their work by including alternative dynamics beta estimators to be consistent with our reactive model and with the work by Engle (2016), which demonstrates that the betas are significantly time varying using dynamic conditional betas. The models and the methods are presented in detail in Appendix C.

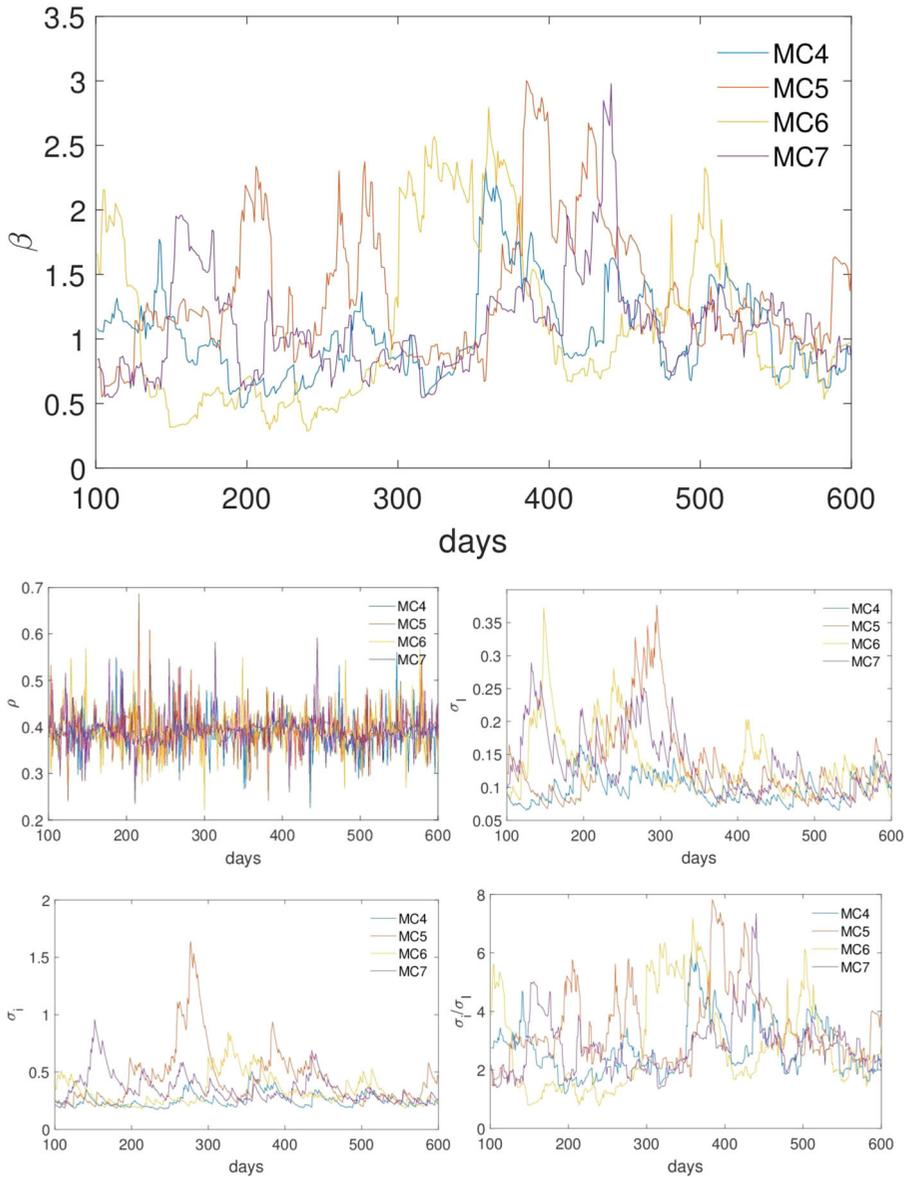
### *Monte Carlo Simulations*

In financial research, simulated data are often used to estimate the error of measurements. For instance, Chan and Lakonishok (1992) build their main results on numerical simulation while applying real data for simple comparison between betas estimated with OLS and quantile regression.

The comparative analysis is based on a two-step procedure. The first step simulates returns using different models that capture some market patterns, and the second step estimates the beta from simulated returns by using our reactive method and alternative methods. We tested the same estimators as used by Chan and Lakonishok (1992), including OLS, MAD, and TRM. We also added two variations of the DCC, which has become a mainstream model to measure the conditional beta when the beta is stochastic (Bollerslev, Engle, and Wooldridge 1988; Bollerslev 1990; Engle 2002; Cappiello, Engle, and Sheppard 2006). We analyze the error of measurements, which we define as the difference between the measured beta and the true beta of the simulated data.

*The First Step: Simulation.* The first step simulates 30,000 paths of  $T=1,000$  consecutive returns for both the stock index ( $r_I$ ) and the single stock ( $r_i$ ). It also allows us to generate 1,000 conditional “true” expected betas per path (Figure VII). To that end, following Chan and Lakonishok (1992), normally distributed residuals and Student- $t$  distributed residuals are considered to take into account the robustness of different methods to outliers.

In our setting, we implement seven Monte Carlo simulations for the returns  $r_i$  and  $r_I$ . In the simulations, we target the realistic case of an unconditional single stock annualized volatility of 40%, an unconditional stock index volatility of 15%, and an unconditional beta of 1. We also target the realistic case of a correlation between the index and the stock of 0.4, as the relative precision of the beta measurement is inversely proportional to the square root of the number of returns when the correlation is close to zero. First, we consider the naive version of the market model, based on equation (11), which we call “the basic market model.” For a constant beta, as in the paper by Chan and Lakonishok (1992), the simulated data are based on the hypothesis of a null intercept, and the beta is set equal to 1 to characterize the ideal case with a Gaussian (MC1) or a Student- $t$  distribution (MC2) for residuals. In the most simple reactive version of the market model, which we call “the reduced reactive market model” (MC3 and MC4), normalized returns  $\tilde{r}_i$  and  $\tilde{r}_I$  are first generated randomly through equation (12) with a normalized beta set to 1. Then, based on the levels  $L_s$ ,  $L_{is}$ , which are, respectively, the slow moving averages of the stock index and the stock prices defined in equation (1), we generate  $\delta I$  and  $\delta S$  defined in equation (6) and then  $r_i$  and  $r_I$ . Finally, we update  $L_s$  and  $L_{is}$ . That model is sufficient to capture the leverage effect on beta with increasing beta as soon as a single stock underperforms the stock index. Even if the normalized beta is set to unity, the denormalized beta in equation (13) becomes time dependent (Figure VII). As



**Figure VII. Simulated Paths for Models MC4–MC7.** The true conditional beta (top), true conditional correlation (middle left), true conditional stock index volatility (middle right), true conditional single stock volatility (bottom left), and true conditional relative volatility (bottom right) are plotted. Paths are limited to 500 days, which are independent from model to model, and capture the same order of magnitude of variation in volatilities, beta, and correlation. MC stands for Monte Carlo simulations. [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

previously, MC3 and MC4 differ by the distribution of residuals, Gaussian (MC3) versus Student- $t$  (MC4).

The full reactive market model (MC5) includes all the components described in Section II, that is, the leverage effect and the nonlinear beta elasticity. For the full

version, we generate stochastic  $\tilde{\sigma}_i$  and  $\tilde{\sigma}_I$ , which generate  $\tilde{r}_i$  and  $\tilde{r}_I$  from equation (12), using the normalized beta fixed to  $\mathcal{F}(t)\mathcal{L}(t)$  (see definitions in equations (31) and (21)). This allows the generation of returns that capture the leverage effect pattern and the empirical nonlinear beta elasticity (Figures III and IV).

We also use another way to generate random returns that captures a time-varying beta through the implementation of the DCC model (Engle 2002, 2016) that generalizes the GARCH(1,1) (The GARCH (1,1) model is a measure of historical volatility introduced by Bollerslev (1986)) process to two dimensions. This is a mainstream model that has two variations: symmetric and asymmetric, the latter capturing the leverage effect. The symmetric and asymmetric versions of the DCC model are denoted as MC6 and MC7, respectively.

To summarize, the seven Monte Carlo simulations are the following:

- MC1: The basic market model in equation (12), where residuals ( $\epsilon_i$ ) are normally distributed, and the constant beta is set to 1.
- MC2: The basic market model in equation (12), where residuals ( $\epsilon_i$ ) follow a Student- $t$  distribution (with three degrees of freedom), and the constant beta is set to 1.
- MC3: The reduced reactive market model in equation (12), where residuals ( $\tilde{\epsilon}_i$ ) are normally distributed with constant volatilities ( $\tilde{\sigma}_i, \tilde{\sigma}_I$ ) and constant renormalized beta ( $\tilde{\beta}$ ) is set to 1, but the denormalized beta is now time dependent (Figure VII). The conditional beta ( $\beta$ ) is a mean-reversion process with a relaxation time  $1/\lambda_s = 40$  days. MC3 includes only the leverage effect and ignores the nonlinear beta elasticity.
- MC4: The reduced reactive market model in equation (12), where residuals ( $\tilde{\epsilon}_i$ ) follow a Student- $t$  distribution (with three degrees of freedom) with constant relative volatility and a constant renormalized beta set to 1, as in MC3.
- MC5: The full reactive market model in equation (12), where residuals ( $\tilde{\epsilon}_i$ ) follow a Student- $t$  distribution (with three degrees of freedom) whose standard deviation ( $s_i$ ) is stochastic and where the normalized stock index return ( $\tilde{r}_I$ ) is a Gaussian whose standard deviation ( $s_I$ ) is also stochastic. We suppose that  $\log(s_I)$  and  $\log(s_i) - \log(s_I)$  follow two independent Ornstein–Uhlenbeck processes (with a relaxation time of 100 days and a volatility of volatility of 0.04). In this way, the stock index annualized volatility could jump up to 40%. The normalized beta set to 1 in MC4 is now set to  $\mathcal{F}(t)\mathcal{L}(t)$  to take into account the nonlinear beta elasticity (see definitions in equations (31) and (21)). Both the leverage effect and stochastic normalized volatilities make the volatilities and the beta defined in equation (36) time dependent (Figure VII).
- MC6: The symmetric DCC model in two dimensions, which generates volatilities of volatilities and a correlation of similar amplitude as MC5 (Figure VII).
- MC7: The asymmetric dynamic conditional correlation (ADCC) model in two dimensions, which generates volatilities of volatilities and a correlation of similar amplitude as MC5 (Figure VII).

In Figure VII, we plot a Monte Carlo path generated for a true beta for MC 3 to 7 (MC1 and MC2 are excluded, as they generate a true beta of 1). We also plot the

conditional correlation and volatilities that are highly volatile and thus make the estimation of the conditional beta complicated.

*The Second Step: Measurements.* The second step is devoted to the analysis of the error measurement of the beta estimations, defined as the difference between the measured beta and the true beta of the simulated data. In our setting, we test five alternative beta estimations that should replicate the true beta as closely as possible. Note that in all five configurations, we use an exponentially weighted scheme to give more weight to recent observations, to be in line with the reactive market model ( $1/\lambda_\beta = 90$ ). Consequently, in a path of  $T = 1,000$  generated returns, only the 90 last returns truly matter (note that Chan and Lakonishok, 1992, is based on the statistics from 35 returns with an equal-weight scheme). The first alternative method is the OLS of the returns, which is also implemented in the empirical test based on real data. Note that the OLS<sup>5</sup> would give the same measurement as our reactive method if the parameters were set differently ( $\lambda_s = 1$ ,  $\lambda_f = 1$ ,  $l = l' = 0$ ,  $f = 0$ ). The square errors in the OLS are weighted by  $(1 - \lambda_\beta)^{T-t}$ . The second method estimates the beta by using the MAD, which is supposed to be less sensitive to outliers because absolute errors are minimized instead of square errors. The absolute errors are weighted by  $(1 - \lambda_\beta)^{T-t}$ . The third alternative is the beta computed from the TRM, which is reputed to be more robust to outliers according to Chan and Lakonishok (1992). The absolute errors are also weighted by  $(1 - \lambda_\beta)^{T-t}$ . The fourth and fifth methods are the conditional beta computed from the DCC model. The DCC method is calibrated using the same exponential  $(1 - \lambda_\beta)^{T-t}$  weights introduced in the log-likelihood function to extract the optimal unconditional volatilities and correlations, and other parameters such as the relaxation time and volatilities of volatilities and volatilities of correlations are set to the values used for the Monte Carlo simulation.

We summarize the reactive method and the five alternative methods implemented to estimate the beta as follows:

- $\beta_{\text{OLS}}$ : beta estimated by the OLS method;
- $\beta_{\text{MAD}}$ : beta estimated by the MAD method;
- $\beta_{\text{TRM}}$ : beta estimated by the TRM;
- $\beta_{\text{DCC}}$ :  $T$ th conditional beta estimated by the DCC model;
- $\beta_{\text{ADCC}}$ :  $T$ th conditional beta estimated by the ADCC model;
- $\beta_{\text{Reactive}}$ : beta estimated by the reactive method in equation (36).

*The Statistics.* We analyze for every path, the error of measurement, defined as the difference between the measured beta based on different methods applied to  $T$  returns and the true beta value at time  $T$ .

To assess the quality of different methods, we use three statistics following Chan and Lakonishok (1992). The first statistic is the bias, which gives the average error of measurement. Obtaining the bias is more informative than simply obtaining an

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<sup>5</sup>We assume that the average of daily returns is zero. This assumption makes sense because at a daily time scale, the average of returns can be completely neglected compared to the standard deviation.

estimated average estimation of the beta because in our case, the true beta is not always 1 but fluctuates around 1 for time-varying models from MC3 to MC7. Because we focus on capturing the leverage effect in the beta measurement, we also define winner (loser) stocks, which are stocks that have outperformed (underperformed) the stock index during the last month. Because of the leverage effect, the OLS method is expected to underestimate the beta for loser stocks and overestimate the beta for winner stocks. It would be interesting to see how robust the improvement of the reactive beta estimation is. We therefore measure the average error among the loser stocks and among the winner stocks. The loser and winner biases are related to the bias in hedging of the short-term reversal strategy measured on real data, and they can confirm the robustness of the empirical measurements. We also define the low- (high-) beta stocks, which are the stocks whose conditional true beta is lower (higher) than 1. We measure the average error among low- and high-beta stocks that are related to the bias in hedging of the low-beta strategy measured from real data and can confirm the robustness of the beta measurement when adding the component describing the nonlinear beta elasticity.

The second statistic is the absolute deviation (ABSD) of a measurement. It reflects the average absolute errors such that the positive and negative sign errors cannot be mutually compensated. It is a measurement of the robustness.

The third statistic, which is equivalent to ABSD, is the inverse of the variance of the errors of measurement ( $\frac{V_{OLS}}{V_m}$ ) to characterize the relative robustness of the alternative beta estimation. The alternative beta method (with subscript  $m$ ) that brings the highest improvement is the one with the highest ratio.

The three statistics are summarized as follows:

- Statistic 1: the bias, the winner bias and the loser bias, the low-beta bias, and the high-beta bias;
- Statistic 2: the ABSD of a measurement;
- Statistic 3: the relative variance statistics  $\frac{V_{OLS}}{V_m}$ .

### *Empirical Tests*

We summarize the statistics in Table 2. We see that all methods are unbiased on average in most Monte Carlo simulations. However, this is misleading, as biases from one group of stocks can be significant and can offset others.

*Winner and Loser Bias.* The estimated  $\beta_{DCC}$  and  $\beta_{ADD}$  appear to be biased as soon as fat tails are included (MC2).

The reactive beta is the only one unbiased for winner and loser stocks when the leverage effect is introduced in Monte Carlo simulations (MC3, MC4, MC5). The biases for winner stocks and loser stocks are significant for all methods except for the reactive beta. The biases are amplified when a fat tail of residual distribution is introduced (MC4). Winner/loser biases can reach 14%. This is in line with the empirical test implemented on real data, where we see that the reactive method reduces the bias of hedging of the short-term reversal strategy (Table 1).

When all components that deviate from the Gaussian market model are mixed in MC5 (fat tails, nonlinear beta elasticity, stochastic volatilities, leverage effect), we see a

TABLE 2. Monte Carlo Robustness Tests.

Method	Bias	Winner Bias	Loser Bias	Low Bias	High Bias	ABSD	Vols/Vm
Panel A. MC1 Gaussian basic market model							
$\beta_{OLS}$	-0.00	-0.00	-0.00			0.16	1.00
$\beta_{Reactive}$	0.00	-0.05*	0.05*			0.18	0.79
$\beta_{DCC}$	0.04*	0.05*	0.03*			0.23	0.51
$\beta_{ADCC}$	0.09*	0.01	0.17*			0.25	0.44
$\beta_{MAD}$	-0.00	0.00	-0.01			0.20	0.65
$\beta_{TRM}$	-0.00	0.00	-0.01			0.20	0.68
Panel B. MC2 <i>t</i> -Student Basic Market Model							
$\beta_{OLS}$	-0.00	0.01	-0.01			0.28	1.00
$\beta_{Reactive}$	0.01	-0.06*	0.08*			0.31	0.82
$\beta_{DCC}$	0.13*	0.14*	0.12*			0.39	0.67
$\beta_{ADCC}$	0.25*	0.15*	0.35*			0.46	0.57
$\beta_{MAD}$	-0.00	-0.00	-0.00			0.22	2.18
$\beta_{TRM}$	-0.00	-0.00	-0.00			0.22	2.24
Panel C. MC3 Gaussian Reduced Reactive Market Model							
$\beta_{OLS}$	-0.00	0.07*	-0.07*	0.07*	-0.07*	0.19	1.00
$\beta_{Reactive}$	-0.00	0.02*	-0.02*	0.02*	-0.02*	0.17	1.27
$\beta_{DCC}$	0.04*	0.10*	-0.02	0.11*	-0.02	0.24	0.62
$\beta_{ADCC}$	0.09*	0.06*	0.12*	0.07*	0.11*	0.24	0.66
$\beta_{MAD}$	-0.01	0.06*	-0.08*	0.06*	-0.08*	0.22	0.73
$\beta_{TRM}$	-0.01	0.06*	-0.08*	0.06*	-0.08*	0.22	0.75
Panel D. MC4 <i>t</i> -Student Reduced Reactive Market Model							
$\beta_{OLS}$	0.01	0.13*	-0.11*	0.12*	-0.10*	0.35	1.00
$\beta_{Reactive}$	-0.01	0.02	-0.04*	0.03	-0.05*	0.31	1.30
$\beta_{DCC}$	0.12*	0.22*	0.02	0.27*	-0.01	0.47	0.84
$\beta_{ADCC}$	0.26*	0.24*	0.28*	0.30*	0.21*	0.52	0.83
$\beta_{MAD}$	-0.03*	0.09*	-0.14*	0.10*	-0.14*	0.27	2.68
$\beta_{TRM}$	-0.03*	0.09*	-0.14*	0.10*	-0.14*	0.27	2.76
Panel E. MC5 <i>t</i> -Student Full Reactive Market Model							
$\beta_{OLS}$	-0.01	0.13*	-0.14*	0.14*	-0.22*	0.50	1.00
$\beta_{Reactive}$	-0.04*	-0.00	-0.07*	0.05*	-0.17*	0.41	1.42
$\beta_{DCC}$	-0.01	0.10*	-0.12*	0.20*	-0.32*	0.52	1.31
$\beta_{ADCC}$	0.10*	0.10*	0.11*	0.29*	-0.17*	0.54	1.32
$\beta_{MAD}$	-0.09*	0.04*	-0.22*	0.09*	-0.37*	0.38	2.43
$\beta_{TRM}$	-0.09*	0.04*	-0.22*	0.09*	-0.36*	0.37	2.46
Panel F. MC6 Gaussian Symmetric DCC Model							
$\beta_{OLS}$	-0.11*	-0.10*	-0.11*	0.06*	-0.27*	0.32	1.00
$\beta_{Reactive}$	-0.07*	-0.11*	-0.02	0.09*	-0.23*	0.33	0.93
$\beta_{DCC}$	-0.01	-0.00	-0.02*	-0.01	-0.01	0.16	4.09
$\beta_{ADCC}$	0.02*	-0.08*	0.12*	0.05*	-0.01	0.22	2.06
$\beta_{MAD}$	-0.14*	-0.13*	-0.15*	0.04*	-0.32*	0.34	0.89
$\beta_{TRM}$	-0.14*	-0.13*	-0.15*	0.04*	-0.32*	0.34	0.90

(Continued)

TABLE 2. Continued.

Method	Bias	Winner Bias	Loser Bias	Low Bias	High Bias	ABSD	Vols/Vm
Panel G. MC7 Gaussian Asymmetric DCC Model							
$\beta_{OLS}$	-0.09*	0.03	-0.24*	0.09*	-0.25*	0.30	1.00
$\beta_{Reactive}$	-0.07*	0.02	-0.17*	0.10*	-0.21*	0.27	1.21
$\beta_{DCC}$	-0.04*	0.04*	-0.15*	-0.00	-0.08*	0.21	2.08
$\beta_{ADCC}$	-0.01	-0.01	-0.01	-0.00	-0.01	0.15	3.74
$\beta_{MAD}$	-0.13*	-0.02	-0.28*	0.06*	-0.29*	0.32	0.92
$\beta_{TRM}$	-0.13*	-0.01	-0.28*	0.06*	-0.29*	0.32	0.92

Note: Statistics are provided for seven Monte Carlo simulations (MC1–MC7) and six methods—ordinary least squares (OLS), reactive model, dynamic conditional correlation (DCC), asymmetric dynamic conditional correlation (ADCC), minimum absolute deviation (MAD), and trimean quantile regression (TRM)—to estimate the beta. We estimated statistics such as bias, which is the average error of beta measurements. Winner/loser biases are the biases among winner/loser stocks. Low/high biases are the biases among low/high beta stocks. ABSD is the average of the error in absolute value. Vols/Vm is the variance of the error in the OLS case divided by the variance of the error.

\*Bias greater than three standard deviations.

kind of cocktail effect, as bias is generated for most methods on average and not only in some groups of stocks. The reactive method provides the best results and is the only method that has no bias.  $\beta_{MAD}$  and  $\beta_{TRM}$ , which were supposed to be robust, appear to perform very poorly, with high bias (average, loser or winner) as soon as the stochastic volatility is added, which is confirmed with MC6 and MC7.

We also see that the reactive model looks to be incompatible with the DCC and ADCC models. Indeed, MC5 generates high bias for  $\beta_{DCC}$  and  $\beta_{ADD}$  in the winner and loser stocks even if the leverage effect and the dynamic beta are implemented in the ADCC. In the same way, MC6 generates bias for the reactive method, which is amplified when leverage effect is generated through MC7. We wonder which model is the most realistic. Both ADCC and the reactive model capture the volatility clustering and leverage effect patterns, but their dynamics are very different. For example, in the reactive model, volatility increases as soon as the price decreases, and it decreases as soon as the price increases. In contrast, the volatility in ADCC increases only if the return is more negative than the unconditional standard deviation ( $\gamma(\sigma_i^2[\xi_i^-(t)]^2 - \tilde{\sigma}_i^2) > 0$ ; see equations (67) and (69). The reactive beta model has three components tailored to three well-identified effects (specific leverage through the retarded effect, systematic leverage through the panic effect, and nonlinear beta elasticity) whose main parameters appear to be stable and universal for all markets. Bouchaud, Matacz, and Potters (2001) measure most of the parameters for seven main stock indexes. The relaxation time is approximately one week for the panic effect ( $\lambda_s = 0.1484$ ), the relaxation time is 40 days for the retarded effect ( $\lambda_s = 0.0241$ ), and the leverage parameter for the panic effect is  $l = 8$ . The systematic leverage parameter on correlation  $\ell - \ell' = 0.91$  is the only one measured through the implied correlation only from the U.S. market. The parameters of the beta elasticity are measured for both the European and the U.S. markets. Whereas the reactive model is tailored to the market, the DCC and ADCC models are more generic

constrained models whose parameters are obtained when optimizing the log-likelihood of returns, which in reality are a result of the complex mixture of the three effects, without focusing on each element. DCC and ADCC do not take into account that, in fact, a large part of the heteroskedasticity comes in reality from the complex leverage effect and not from simple autoregressive conditional heteroskedasticity. The parameters we use to describe the relaxation times, the volatilities of volatility, the volatility of correlation, and the asymmetries of the DCC and ADCC models are based on the work by Sheppard (2017) and Carhart (1997). Relaxation times of 10 days and 13 days are estimated for the U.S. market and are different from those used in the reactive volatility model. In Carhart's Table 5, we see that the relaxation time would be 2.5 days for Belgian stocks (the decay factor is  $\beta = 0.6184$  for the univariate GARCH), 4 days for French stocks (the decay factor is  $\beta = 0.7497$ ), and 14 days for Spanish stocks (the decay factor is  $\beta = 0.9360$ ). It is not surprising to see this variation if simple autoregressive conditional heteroskedasticity cannot capture the complexity of the leverage effect. We think it would be better to apply autoregressive conditional heteroskedasticity to model the residual part of the heteroskedasticity of returns once the part due to the leverage effect is withdrawn through normalized returns from the reactive model. The relaxation time in this case is expected to be a couple of months.

*High- and Low-Beta Bias.* The reactive beta is the only one that reduces the bias for low- and high-beta stocks when stochastic volatility is introduced and when the empirical nonlinear beta elasticity is implemented (MC 5). This is in line with the empirical test applied to real data, where we see that the reactive method reduces the bias of hedging of the low-volatility strategy (Table 1).

*ABSD and  $V_{OLS}/V_m$ .*  $\beta_{OLS}$ , which is the theoretical optimal estimation for Monte Carlo simulated returns with the Gaussian market model (MC1), gives similar statistics to that of the reactive beta for the MC3. In this case (MC3), the reactive method outperforms the other methods. The ABSD of 0.17 is explained by irreducible statistical noise that is intrinsic to any regression based on approximately 90 points with a weak correlation.

When a fat tail is incorporated into the residual (MC4), the ABSD of the reactive beta increases and becomes intermediate between the ABSD of  $\beta_{OLS}$ ,  $\beta_{MAD}$  and  $\beta_{TRM}$ .  $\beta_{MAD}$  and  $\beta_{TRM}$  are more robust in the presence of fat tails. The reactive beta is expected to be as sensitive as the OLS would be due to the outliers. The reactive method could be still improved if a TRM regression were implemented instead of the classical OLS to measure the normalized beta between normalized returns. When stochastic volatility and correlation are introduced (MC5, MC6, and MC7), the reactive beta becomes as robust as  $\beta_{MAD}$  and  $\beta_{TRM}$  based on ABSD.

## V. Open Problems in Other Fields

The estimated beta is used in a wide range of financial applications, including security valuation, asset pricing, portfolio management, and risk management. This also extends to corporate finance in many applications, such as financing decisions to quantify risk associated with debt, equity and assets, and firm valuation when discounting cash flows

using the weighted average cost of capital. The most likely reason is that the beta describes systematic risk that could not be diversified and that should be remunerated. However, as explained, the OLS estimator of the beta is subject to measurement errors, which include the presence of outliers, time dependence, the leverage effect, and the departure from normality.

### *Asset Pricing*

Bali, Engle, and Tang (2017) apply the DCC model by Engle (2016) to assess the cross-sectional variation in expected stock returns. They estimate the conditional beta for the S&P 500 using daily data from 1963 to 2009. They test whether the betas have predictive power for the cross-section of individual stock returns over the next one to five days. They show that there is no link between the unconditional beta and the cross-section of expected returns. Most remarkably, they also show that the time-varying conditional beta is priced in the cross-section of daily returns. At the portfolio level, they indicate that a long–short trading strategy of buying the highest conditional beta stocks and selling the lowest conditional beta stocks yields average returns of 8% per year. Thus, conditional CAPM is empirically valid, whereas unconditional CAPM is not empirically valid. Moreover, Bali, Engle, and Tang show that improvements in beta measurement from unconditional to conditional betas would not have significant pricing impacts on major anomalies (size, book, momentum, etc.). Thus, we can see that DCC greatly changes the pricing of the low-volatility anomaly that disappears and improves the empirical validation of the CAPM but does not change the pricing of other major anomalies. We expect that the reactive method can bring further improvements. Indeed, as revealed by our robustness tests in Section IV, the leverage effect and the nonlinear beta elasticity are likely to generate bias in the DCC estimation. Because our reactive method is designed to correct for these biases, its use can help reveal pricing effects of the dynamic beta on major anomalies. This point is an interesting perspective for future research.

### *Corporate Finance*

To determine a fair discount rate for valuing cash flows, the firm's manager must select the appropriate beta of the project given that the discount rate remains constant over time, though the project may exhibit significant variation over time and the leverage effect because of the debt-to-equity ratio. As such, Ang and Liu (2004) discuss how to discount cash flows with time-varying expected returns in a traditional setup. For instance, although the traditional dividend discount model assumes that the expected return along with the expected rate of cash-flow growth are set as constant, they are time varying and correlated. In practice, in the first step, the manager computes the expected future cash flows from financial forecasts. In the second step, the manager uses a constant discount rate, usually relying on the CAPM for the discounting factor. In contrast, Ang and Liu derive a valuation formula that incorporates the correlation among stochastic cash flows, betas, and risk premia. They show that the greater the magnitude of the difference between the true discount rate and the constant discount rate, the greater the project's misvaluation. They even show that when computing perpetuity values from the discounting model, the potential mispricing can become worse. Ang and Liu conclude

that accounting for time-varying expected returns can lead to different prices from using a constant discount rate from the traditional unconditional CAPM. The impact of the leverage effect and of the nonlinear elasticity of the beta on potential mispricing deserves to be investigated. Indeed, our results seem to indicate that the mispricing might be higher for low- and high-beta stocks over a long period. This could be an interesting topic for future work.

### *Alternative Asset Classes*

Notice that in this article, the reactive beta model is tailored for stocks. However, it could help reduce bias in a context involving assets other than stocks, such as hedge funds or mutual funds. Indeed, the simple market-neutral strategies (short-term reversal, momentum, size) can be extended to simple directional strategies (contrarian, trend following) to model the behaviors of funds managers. Some identified bias in beta measurement described in Section II captured by the market-neutral strategies are also most likely to occur in the directional strategies.

An application of the reactive beta model on hedge funds raises concerns about a better estimation of nonlinearity features that would stem from option-like strategies or higher moments as documented by the literature. Fung and Hsieh (2001) warn that hedge funds employ dynamic trading strategies that have option-like returns even if the manager does not trade in derivatives markets. This means that asset pricing models of investment styles are not designed to capture nonlinear returns that commonly characterize the hedge fund industry. Agarwal and Naik (2004) observe that hedge funds report large losses during crisis episodes, which suggests that they may be bearing significant left-tail risk, particularly during large market downturns. They find that nonlinear option-like payoffs from a wide range of equity-oriented hedge funds resemble a strategy of writing a put option on the equity index. Recall that hedge funds generally employ long–short dynamic strategies to capture nonstandard risk premia, in contrast to mutual funds that employ an overall long position on buy-and-hold strategies to capture standard risk premia such as equity/bond risk premia. Agarwal, Arisoy, and Naik (2017) build on an augmented version of the Fung and Hsieh (2004) seven-factor model to find that hedge funds with greater leverage, longer time in existence, and larger assets under management have more negative uncertainty betas. This echoes the findings of Bali, Brown, and Tang (2017) for stocks that provide evidence of significant nonlinearity in uncertainty premium. Agarwal, Green, and Ren (2018) measure risk-adjusted hedge fund performance using a range of single- and multifactor models to find that, surprisingly, hedge fund flows are better explained by the CAPM alpha than by more sophisticated models. This echoes the findings of Berk and van Binsbergen (2016), who first use capital flows of mutual funds for asset pricing models to finally conclude that the CAPM better explains risk than no model at all.

An application on mutual funds also raises concerns about estimation errors in the individual beta estimate because beta is exposed to estimation errors for individual stocks (see, e.g., Chordia, Goyal, and Shanken 2015). But because mutual funds are themselves diversified portfolios, it should alleviate the estimation error in the beta estimate. This is important because it addresses the controversy in the literature as to

whether some expected return variations associated with factor loadings (betas) are due to economic risk or to mispricing effects linked to this measurement error. At the same time, using portfolios could hide precious information that exists at the individual stock level as documented by the literature (Black, Jensen, and Scholes 1972; Fama and MacBeth 1973). Such an investigation on alternative asset classes is left for future research.

## VI. Conclusion

We propose a reactive beta model with three components that account for the specific leverage effect (when a stock underperforms, its beta increases), the systematic leverage effect (when a stock index declines, correlations increase), and beta elasticity (when relative volatility increases, the beta increases). The three components are fitted and incorporated through elaborate statistical measurements. An empirical test is run from 2000 to 2015 with exhaustive data sets including both American and European securities. We compute the bias for hedging the most popular market-neutral strategies (low volatility, short-term reversal, momentum, and capitalization) using the standard approach of beta measurement and the reactive beta model. Our main findings emphasize the ability of the reactive beta model to significantly reduce the biases of these strategies, particularly during stress periods. We further extend the research design to include robustness checks based on simulated data to compare the reactive method with five alternative methods (OLS, MAD, TRM, DCC, and ADCC) over seven Monte Carlo scenarios reflecting different market conditions from calm (Gaussian residuals, no leverage effect, constant beta) to stress (non-Gaussian residuals, leverage effect, nonlinear beta elasticity, stochastic volatility, nonconstant volatility of volatility, and volatility of correlation). We find that the overall results confirm that the reactive beta presents the lower bias when stressed market conditions are included. Furthermore, the reactive model can be useful in other empirical applications such as asset pricing and corporate finance and alternative asset classes such as hedge funds and mutual funds. This provides a good starting point for future research.

## Appendix A: Selection Bias

Here, we provide evidence that the bias in the beta of the low-volatility factor comes from selection bias: selection of the bottom beta stocks yields the stocks whose beta is underestimated.

The measured beta  $\beta_{im}$  of stock  $i$  is obtained by a standard linear regression of the  $i$ th stock returns,  $r_i$ , to the stock index returns,  $r_I$ ,

$$r_i = \beta_{im}r_I + \epsilon_i, \quad (\text{A1})$$

where  $\epsilon_i$  is the residual return. We suppose that the measured beta of stock  $i$ ,  $\beta_{im}$ , is affected by noise,

$$\beta_{im} = \beta_{iT} + \eta_i, \quad (\text{A2})$$

where  $\beta_{iT}$  is the true beta (which is unknown), and  $\eta_i$  is the error of the measurement inherent to the linear regression. The standard deviation of  $\eta_i$ ,  $\sigma_{\eta_i}$ , depends on the average correlation between the single stock  $i$  and the stock index and on the number  $n$  of independent points used for the regression (which we set at  $n = \frac{1}{\lambda_\beta} = 90$ ):

$$\sigma_{\eta_i} = \frac{\sigma_{\epsilon_i}}{\sigma_I} \frac{1}{\sqrt{n}}, \quad (\text{A3})$$

where  $\sigma_{\epsilon_i}$  is the standard deviation of the residual returns  $\epsilon_i$ . Averaging the above relation over all stocks, we obtain

$$\sigma_\eta = \frac{\langle \sigma_{\epsilon_i} \rangle}{\sigma_I} \sqrt{\lambda_\beta}, \quad (\text{A4})$$

where  $\langle \sigma_{\epsilon_i} \rangle$  denotes the average. According to equation (A1), the standard deviation of the stock returns,  $\sigma_i$ , is

$$\sigma_i = \sqrt{\beta_{im}^2 \sigma_I^2 + \sigma_{\epsilon_i}^2} \approx \sigma_{\epsilon_i}, \quad (\text{A5})$$

because  $(\beta_{im} \sigma_I / \sigma_i)^2 \ll 1$  (stocks are much more volatile than the index). We thus obtain

$$\sigma_\eta \approx \frac{\langle \sigma_i \rangle}{\sigma_I} \sqrt{\lambda_\beta}. \quad (\text{A6})$$

The low-volatility factor is 50% long of the 30% top  $\beta_{im}$  stocks and 50% short of the 30% bottom  $\beta_{im}$  stocks (here, we consider only one sector for simplicity). We adjust the most volatile leg to target a beta-neutral factor if we suppose that  $\eta_i$  are null. In reality, when taking into account the difference between the measured and the true beta, we obtain the beta of the low-volatility factor as:

$$\beta_{\text{lowfactor}} = -50\% \langle \beta_{iT} | i \in \text{Bottom} \rangle + 50\% \frac{\langle \beta_{im} | i \in \text{Bottom} \rangle}{\langle \beta_{im} | i \in \text{Top} \rangle} \langle \beta_{iT} | i \in \text{Top} \rangle \quad (\text{A7})$$

This is essentially the beta-neutral condition that we impose when constructing the factor (see Appendix B). Here,  $\langle \beta_{im} | i \in \text{Bottom} \rangle$  is the average of the measured beta over stocks  $i$  in the 30% bottom in the measured beta values  $\beta_{im}$  (similar for other averages).

Defining  $\Delta\beta_B$  and  $\Delta\beta_T$  as

$$\langle \beta_{iT} | i \in \text{Bottom} \rangle = \langle \beta_{im} | i \in \text{Bottom} \rangle + \Delta\beta_B, \quad (\text{A8})$$

$$\langle \beta_{iT} | i \in \text{Top} \rangle = \langle \beta_{im} | i \in \text{Top} \rangle + \Delta\beta_T, \quad (\text{A9})$$

we rewrite equation (A7) as

$$\begin{aligned}\beta_{\text{lowfactor}} &= -50\%(\langle\beta_{im}|i \in \text{Bottom}\rangle + \Delta\beta_B) \\ &\quad + 50\% \frac{\langle\beta_{im}|i \in \text{Bottom}\rangle}{\langle\beta_{im}|i \in \text{Top}\rangle} (\langle\beta_{im}|i \in \text{Top}\rangle + \Delta\beta_T) \\ &= -50\%\Delta\beta_B + 50\% \frac{\langle\beta_{im}|i \in \text{Bottom}\rangle}{\langle\beta_{im}|i \in \text{Top}\rangle} \Delta\beta_T.\end{aligned}\tag{A10}$$

Given that  $\langle\beta_{im}|i \in \text{Bottom}\rangle \ll \langle\beta_{im}|i \in \text{Top}\rangle$  (as the  $\beta_{im}$  in the top quantile are higher than the  $\beta_{im}$  in the bottom quantile), we obtain the following approximation

$$\beta_{\text{lowfactor}} \approx -50\%\Delta\beta_B.\tag{A11}$$

If one knew the true  $\beta_{iT}$  values and used them to construct the low-volatility factor, the excess  $\Delta\beta_B$  would be zero. However, the true values are unknown, and therefore the measured beta  $\beta_{im}$  is used, which creates selection bias and nonzero  $\Delta\beta_B$ , as shown below.

To estimate  $\Delta\beta_B$ , we consider the true beta  $\beta_{iT}$  and the measurement error  $\eta_i$  as independent random variables and replace the average over stocks by the following conditional expectation

$$\Delta\beta_B = \langle\beta_{iT} - \beta_{im}|i \in \text{Bottom}\rangle \approx \mathbb{E}\{\beta_{iT} - \beta_{im}|i \in \text{Bottom}\} = B.\tag{A12}$$

We then have

$$\begin{aligned}-B &= \mathbb{E}\{\eta_i|i \in \text{Bottom}\} = \int_{-\infty}^{\infty} \eta \mathbb{P}\{\eta_i \in (\eta, \eta + d\eta)|i \in \text{Bottom}\} \\ &= \int_{-\infty}^{\infty} \eta \frac{\mathbb{P}\{\eta_i \in (\eta, \eta + d\eta), i \in \text{Bottom}\}}{\mathbb{P}\{i \in \text{Bottom}\}},\end{aligned}\tag{A13}$$

where we write explicitly the conditional probability. The denominator is precisely the threshold determining the bottom quantile,  $\mathbb{P}\{i \in \text{Bottom}\} = p$ , which we set to 30%. We thus obtain

$$-B = \frac{1}{p} \int_{-\infty}^{\infty} \eta \mathbb{P}\{\eta_i \in (\eta, \eta + d\eta), \beta_{im} - \beta_0 < Q\},\tag{A14}$$

where the event  $i \in \text{Bottom}$  is equivalently written as  $\beta_{im} < \beta_0 + Q$ , where  $Q$  is the value of the measured beta that corresponds to the quantile  $p$ , and  $\beta_0$  is the mean of  $\beta_{im}$ . Using equation (A15) and the assumption that  $\beta_{iT}$  and  $\eta_i$  are independent, we obtain

$$\begin{aligned}
-B &= \frac{1}{p} \int_{-\infty}^{\infty} \eta \mathbb{P}\{\eta_i \in (\eta, \eta + d\eta), \beta_{iT} - \beta_0 < Q - \eta\} \\
&= \frac{1}{p} \int_{-\infty}^{\infty} \eta \mathbb{P}\{\eta_i \in (\eta, \eta + d\eta)\} \mathbb{P}\{\beta_{iT} - \beta_0 < Q - \eta\}. \quad (\text{A15})
\end{aligned}$$

To obtain some quantitative estimates, we make a strong assumption that both  $\beta_{iT}$  and  $\eta_i$  are Gaussian variables, with means  $\beta_0$  and 0 and standard deviations  $\sigma_\beta$  and  $\sigma_\eta$ , respectively. We then obtain

$$-B = \frac{1}{p} \int_{-\infty}^{\infty} d\eta \eta \frac{\exp\left(-\eta^2 / (2\sigma_\eta^2)\right)}{\sqrt{2\pi}\sigma_\eta} \Phi((Q - \eta)/\sigma_\beta), \quad (\text{A16})$$

where

$$\Phi(x) = \int_{-\infty}^x dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \quad (\text{A17})$$

is the cumulative Gaussian distribution. Changing the integration variable, we obtain

$$-B = \frac{\sqrt{2}\sigma_\eta}{p\sqrt{\pi}} \int_{-\infty}^{\infty} dx x \exp(-x^2) \Phi\left(\left(Q - x\sqrt{2}\sigma_\eta\right)/\sigma_\beta\right). \quad (\text{A18})$$

Integrating by parts and omitting technical computations, we obtain

$$B = \frac{\sqrt{2}\sigma_\eta}{p\sqrt{\pi}} \frac{\sigma_\eta}{2\sigma_\beta\sqrt{1+b^2}} \exp\left(-\frac{a^2}{1+b^2}\right), \quad (\text{A19})$$

where  $a = Q/(\sqrt{2}\sigma_\beta)$  and  $b = \sigma_\eta/\sigma_\beta$ . Setting

$$Q = \sigma_\beta\sqrt{2}q, \quad q = \text{erf}^{-1}(2p - 1), \quad (\text{A20})$$

we obtain

$$B = \frac{\sigma_\eta}{p\sqrt{2\pi}} \frac{1}{\sqrt{1 + (\sigma_\beta/\sigma_\eta)^2}} \exp\left(-\frac{q^2}{1 + (\sigma_\eta/\sigma_\beta)^2}\right), \quad (\text{A21})$$

from which

$$\beta_{\text{lowfactor}} \approx -50\% \frac{\sigma_\eta}{p\sqrt{2\pi}} \frac{1}{\sqrt{1 + (\sigma_\beta/\sigma_\eta)^2}} \exp\left(-\frac{q^2}{1 + (\sigma_\eta/\sigma_\beta)^2}\right). \quad (\text{A22})$$

From the data for the United States, we estimate the standard deviation of the measured beta ( $\sigma_\beta = 0.43$ ), the volatility of the stock index ( $\sigma_I = 19.77\%$ ), the volatility of the low-volatility factor ( $3.46\%$ ), and  $\langle\sigma_i\rangle/\sigma_I = 1.53$ . Setting  $\lambda_\beta = 1/90$ , we obtain from equation (A6),  $\sigma_\eta = 1.53\sqrt{1/90} = 0.1613$ . For  $p = 0.3$  (bottom 30%), we obtain  $q = -0.3708$  and, thus,  $\beta_{\text{lowfactor}} \approx 0.0334$  from equation (A22). Finally, we conclude that  $\rho_{\text{lowfactor}} = 3.34\% \frac{19.77\%}{3.46\%} = 19.1\%$ .

## Appendix B: Construction of the Beta-Neutral Factors

We implement the four most popular strategies as four beta-neutral factors that are constructed as follows. First, we split all stocks into six clusters of sectors of similar sizes to minimize sectorial correlations. For each trading day, the stocks of the chosen cluster are sorted according to the indicator (e.g., the capitalization) available the day before (we use the publication date and not the valuation date). The related indicator-based factor is formed by buying the first  $pN$  stocks in the sorted list and shorting the last  $pN$  stocks, where  $N$  is the number of stocks in the considered cluster and  $p$  is a chosen quantile level. As described in Section II, we use  $p = 0.15$  for short-term reversal and long-term momentum factors, and  $p = 0.30$  for the capitalization and low-volatility factors. The other stocks (with intermediate indicator values) are not included (weighted by 0). To reduce the specific risk, the weights of the selected stocks are set inversely proportional to the stock's volatility  $\sigma_i$ , whereas the weights of the remaining stocks are 0. Moreover, the inverse stock volatility is limited to reduce the impact of extreme specific risk. For each trading day, we recompute the weight  $w_i$  as follows

$$w_i = \begin{cases} +\mu_+ \min\{1, \sigma_{\text{mean}}/\sigma_i\}, & \text{if } i \text{ belongs to the first } pN \text{ stocks in the sorted list,} \\ -\mu_- \min\{1, \sigma_{\text{mean}}/\sigma_i\}, & \text{if } i \text{ belongs to the last } pN \text{ stocks in the sorted list,} \\ 0, & \text{otherwise,} \end{cases} \quad (\text{B1})$$

where  $\sigma_{\text{mean}} = \frac{1}{N}(\sigma_1 + \dots + \sigma_N)$  is the mean estimated volatility over the cluster of sectors. In this manner, the weights of low-volatility stocks are reduced to avoid strongly unbalanced portfolios concentrated in such stocks. The two common multipliers,  $\mu_\pm$ , are used to ensure the beta market-neutral condition:

$$\sum_{i=1}^N \beta_i w_i = 0, \quad (\text{B2})$$

where  $\beta_i$  is the sensitivity of stock  $i$  to the market obtained either by OLS or by our reactive method. In every case, the method to estimate beta uses rolling daily returns and

only past information to avoid the look-ahead bias. If the aggregated sensitivity of the long part of the portfolio to the market is higher than that of the short part of the portfolio, its weight is reduced by the common multiplier  $\mu_+ < \frac{1}{2pN}$ , which is obtained from equation (B2) by setting  $\mu_- = \frac{1}{2pN}$  (which implies that the sum of absolute weights  $|w_i|$  does not exceed 1). In the opposite situation (when the short part of the portfolio has a higher aggregated beta), we set  $\mu_+ = \frac{1}{2pN}$  and determine the reducing multiplier  $\mu_- < \frac{1}{2pN}$  from equation (B2). The resulting factor is obtained by aggregating the weights constructed for each supersector. We emphasize that the factors are constructed on a daily basis; that is, the weights are reevaluated daily based on updated indicators. However, most indicators do not change frequently, so the transaction costs related to changing the factors are not significant.

### Appendix C: Description of Alternative Methods

#### *Unconditional Beta*

*The Theory.* Chan and Lakonishok (1992) produce an empirical analysis that describes various robust methods for estimating constant beta, as they provide an alternative to OLS. Their method is built on the work of Koenker and Bassett (1978), which provides robust alternatives to the sample mean using a more complex linear combination of order statistics to face non-Gaussian errors, which are the source of outliers. Instead of minimizing the sum of squared residuals, they consider an estimator that is based on minimizing the criterion, including a penalty function  $\varrho$  on the residuals  $\epsilon$ :

$$\sum_{t=1}^T \varrho_{\theta}(\epsilon_t) \quad (\text{C1})$$

for  $\varrho_{\theta}(\epsilon_t) = \theta|\epsilon_t|$  if  $\epsilon_t \geq 0$ , or  $(1 - \theta)|\epsilon_t|$  if  $\epsilon_t < 0$ , where  $0 < \theta < 1$ .

Chan and Lakonishok (1992) minimize the sum of absolute deviations of the residuals  $\epsilon_{it}$  from the market model instead of the sum of squared deviations. The resulting minimum absolute deviation (MAD) estimator of the regression parameters corresponds to the special case of  $\theta = 1/2$ , where half of the observations lie above the line, and the other half lie below. More generally, large or small values of the weight  $\theta$  attach a penalty to observations with large positive or negative residuals. Varying  $\theta$  between 0 and 1 yields a set of regression quantile estimates  $\widehat{\beta}(\theta)$  that is analogous to the quantiles of any sample of data. However, Chan and Lakonishok recognize that MAD does not prove to be a clearly superior method, and they suggest that it may be improved via linear combinations of sample quantiles such as trimmed means.

For that reason, Chan and Lakonishok (1992) test different combinations of regressions quantiles serving as the basis for the robust estimators. They discuss the general case of the trimmed regression quantile (TRQ) given as a weighted average of the regression quantile statistics:

$$\hat{\beta}_\alpha = (1 - 2\alpha)^{-1} \int_\alpha^{1-\alpha} \hat{\beta}(\theta) d\theta \tag{C2}$$

where  $0 < \alpha < 1/2$  and  $0 < \theta < 1$ .

More specifically, Chan and Lakonishok (1992) suggest a more straightforward and equivalent method that considers estimators that are finite linear combinations of regression quantiles and are computationally simpler:

$$\beta_\omega = \sum_{i=1}^N \omega_i \hat{\beta}(\theta_i), \tag{C3}$$

where weights  $0 < \omega_i < 1$ ,  $i = 1, \dots, N$  and  $\sum_{i=1}^N \omega_i = 1$ . The specific case of the weighted average is given by Tukey’s trimean (TRM) estimator:

$$\hat{\beta}_{\text{TRM}} = 0.25\hat{\beta}(1/4) + 0.5\hat{\beta}(1/2) + 0.25\hat{\beta}(3/4) \tag{C4}$$

*The Application.* Chan and Lakonishok’s (1992) analysis is based mainly on simulated return data, although they add some tests with actual return data. The main advantages of a simulation are that the true values of the underlying parameters are known and that the extent of departures from normality can be controlled. Chan and Lakonishok begin with a baseline simulation with 25,000 replications using data generated from a normal distribution. They also consider the case where the residual term is drawn from a Student-distribution with three degrees of freedom to explain the observed leptokurtosis in daily return data. We follow the same methodology to assess the quality of the OLS, MAD, and TRM beta estimators using Gaussian and *t*-Student residuals in the seven types of Monte Carlo simulations (MC1, . . . , MC7).

To replicate the exponential weight scheme of the reactive model ( $\lambda_\beta = 1/90$ ), equation (C1) is replaced by

$$\sum_{t=1}^T (1 - \lambda_\beta)^{T-t} Q_\theta(\epsilon_t). \tag{C5}$$

### Conditional Beta

*The Theory.* The first application of time-varying beta was proposed in Bollerslev, Engle, and Wooldridge (1988), where the beta was computed as the ratio of the conditional covariance to the conditional variance. Engle (2002) generalizes Bollerslev’s (1990) constant correlation model by making the conditional correlation matrix time dependent with the dynamic conditional correlation (DCC) model, which constrains the time-varying conditional correlation matrix to be positive definite and the number of parameters to grow linearly by a two-step procedure. The first step requires the GARCH variances to be estimated univariately. Their parameter estimates remain constant for the next step. The second stage is estimated conditioned on the parameters estimated in the first stage.

Hereafter, we extend the modeling of the DCC beta to include an asymmetric term in the conditional variance equation. For asymmetry in the conditional variance, we select the GJR-GARCH(1,1) specification by Glosten, Jagannathan, and Runkle (1993), which assumes a specific parametric form with leverage effect in the conditional variance (DCC-GJR beta). The basic idea is that negative shocks at period  $t-1$  have a stronger impact on the conditional variance at period  $t$  than positive shocks. Note that even though the conditional distribution is Gaussian, the corresponding unconditional distribution can still present excess kurtosis.

We select the ADCC model by Cappiello, Engle, and Sheppard (2006) to incorporate asymmetry in correlation.<sup>6</sup> The case mixing asymmetry located in the variance equation (GJR-GARCH) and in the correlation equation (ADCC) is examined (ADCC-GJR GARCH). In our article, the symmetric GARCH DCC is simply called DCC, and the asymmetric ADCC-GJR is simply called ADCC.

Let us consider  $r_i$  and  $r_I$  as the returns of a single stock and the stock index, respectively. We assume that their respective conditional variances follow a (GJR-) GARCH(1,1) specification. The stock return  $r_i$  is defined by its conditional volatility,  $\sigma_i$ , and a zero-mean white noise  $\xi_i(t)$ :

$$r_i(t) = \sigma_i(t-1)\xi_i(t). \quad (C6)$$

The conditional variation specification of the stock return is the following:

$$\sigma_i^2(t) = (1 - a - b - \gamma/2)\tilde{\sigma}_i^2 + a\sigma_i^2(t-1)[\xi_i(t)]^2 + b\sigma_i^2(t-1) + \gamma\sigma_i^2[\xi_i^-(t)]^2, \quad (C7)$$

where  $\tilde{\sigma}_i$  is unconditional volatility, and  $a$ ,  $b$ , and  $\gamma$  are parameters reflecting, respectively, the ARCH, GARCH, and asymmetry effects. When  $\gamma = 0$ , the specification collapses to a GARCH model; otherwise, it stands for the GJR-GARCH model, where the asymmetric term is defined as  $\xi_i^-(t) = \xi_i(t)$  if  $\xi_i(t) > 0$ , or  $\xi_i^-(t) = 0$  otherwise.

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<sup>6</sup>There is a rich literature documenting asymmetry in correlation overall during bear markets. To cite a few examples, Ang and Bekaert (2000) find evidence that the presence of a high-volatility and high-correlation regime tends to coincide with a bear market. Longin and Solnik (2001) find that the correlation among large negative returns is much larger than the correlation among large positive returns. Forbes and Rigobon (2002) warn that the correlation can increase only because the volatility increases even if the beta remains constant. To that end, there has been a controversy in the literature on the statistical significance of such an asymmetry. For this purpose, Ang and Chen (2002) develop a summary statistic that quantifies the degree of asymmetry in correlations across downside and upside markets relative to a particular model. They find that stocks from either small firms, value firms, or low-past-returns firms tend to exhibit greater asymmetric correlations. Hong, Tu, and Zhou (2006) extend the Ang and Chen analysis to a model-free approach so that if symmetry is rejected, the data cannot be modeled by any symmetrical distributions. They find that the betas can be asymmetric even if there is no asymmetry in the correlation. They also find strong evidence of asymmetries for both size and momentum portfolios, but no evidence for book-to-market portfolios. Jiang, Wu, and Zhou (2018) extend Hong, Tu, and Zhou's correlation-based test approach to find that asymmetry is much more pervasive than previously thought. Indeed, they address asymmetry beyond the second moment, as the correlation coefficient is a measure of linear dependence, captured by the market beta, between individual stock returns and the market portfolio return. In contrast to Hong, Tu, and Zhou, they find evidence of asymmetry in some portfolios sorted by the book-to-market ratio.

The stock index return  $r_I$  is defined by its conditional volatility,  $\sigma_I$ , and a zero-mean white noise  $\xi_I(t)$  that is correlated to  $\xi_i(t)$ :

$$r_I(t) = \sigma_I(t-1)\xi_I(t). \quad (C8)$$

The conditional variance specification of the stock index return is the following:

$$\sigma_I^2(t) = (1 - a - b - \gamma/2)\tilde{\sigma}_I^2 + a\sigma_I^2(t-1)[\xi_I(t)]^2 + b\sigma_I^2(t-1) + \gamma\sigma_I^2[\xi_I^-(t)]^2. \quad (C9)$$

We define the normalized conditional variance diagonal terms as follows:

$$q_{ii}(t) = (1 - a_\rho - b_\rho - \gamma_\rho/2) + a_\rho\xi_i(t-1)\xi_i(t-1) + b_\rho q_{ii}(t-1) + \gamma_\rho\xi_i^-(t-1)\xi_i^-(t-1) \quad (C10)$$

$$q_{II}(t) = \left(1 - a_\rho - b_\rho - \frac{\gamma_\rho}{2}\right) + a_\rho\xi_I(t-1)\xi_I(t-1) + b_\rho q_{II}(t-1) + \gamma_\rho\xi_I^-(t-1)\xi_I^-(t-1). \quad (C11)$$

The normalized conditional covariance term  $q_{iI}(t)$  is given by:

$$q_{iI}(t) = \left(1 - a_\rho - b_\rho - \frac{\gamma_\rho}{4}\right)\tilde{\rho} + a_\rho\xi_i(t-1)\xi_I(t-1) + b_\rho q_{iI}(t-1) + \gamma_\rho\xi_i^-(t-1)\xi_I^-(t-1). \quad (C12)$$

When  $\gamma_\rho = 0$ , the specification collapses to a DCC model; otherwise, it stands for the ADCC model, where the asymmetric term is defined as  $\xi_i^-(t) = \xi_i(t)$  if  $\xi_i(t) > 0$ , or  $\xi_i^-(t) = 0$  otherwise.

The conditional correlation between  $\xi_I(t+1)$  and  $\xi_i(t+1)$  is then updated by:

$$\rho_{iI}(t) = q_{iI}(t)/\sqrt{q_{II}(t)q_{ii}(t)}. \quad (C13)$$

The beta DCC and beta ADCC estimation are defined in the same way:

$$\beta_{DCC}(t) = \frac{\rho_{iI}(t)\sigma_i(t)}{\sigma_I(t)}. \quad (C14)$$

The log-likelihood function is optimized to calibrate the parameters  $\tilde{\rho}$ ,  $\tilde{\sigma}_I$ , and  $\tilde{\sigma}_i$  for estimation:

$$L_{DCC} = \frac{1}{2} \sum_t^T (L_V(t) + L_C(t)) \quad (C15)$$

$$L_V(t) = -2 \log(2\pi) - \xi_I(t)^2 - \xi_i(t)^2 - 2 \log(\sigma_I(t)) - 2 \log(\sigma_i(t)) \quad (C16)$$

$$L_C(t) = -\log(\det(R(t))) - U'(t)R(t)^{-1}U(t) - U'(t)U(t), \quad (C17)$$

with  $\det$  as the determinant of a matrix, and

$$R(t) = \begin{bmatrix} 1 & \rho_{il}(t) \\ \rho_{il}(t) & 1 \end{bmatrix}, \quad U(t) = \begin{bmatrix} \xi_i(t) \\ \xi_l(t) \end{bmatrix}. \quad (C18)$$

*The Application.* For Monte Carlo simulation purposes:

- $\xi_i(t)$  is either generated randomly in MC6 and MC7 according to a standard Gaussian or measured through returns  $r_i(t)$  and  $\sigma_i(t-1)$  for beta DCC estimation.
- $\gamma = 0$  for MC6 and beta DCC estimation but  $\gamma > 0$  for MC7 and beta ADCC, which captures the asymmetry term of the GJR-GARCH.
- $\xi_l(t)$  is either generated randomly in MC6 and MC7 according to a standard Gaussian random variable that is correlated with the random variable  $\xi_i(t)$  (the correlation between  $\xi_i(t)$  and  $\xi_l(t)$  is  $\rho_{il}(t-1)$ ) or is measured through returns  $r_l(t)$  and  $\sigma_l(t-1)$  for beta DCC estimation.
- $\gamma_\rho = 0$  for MC6 and beta DCC but  $\gamma_\rho > 0$  for MC7 and beta ADCC, which captures the asymmetry term of the ADCC.

The fixed parameters that are supposed to be known when testing the beta DCC are set to U.S. market estimates from Sheppard (2017):

- fixed parameters for the univariate symmetric GARCH(1,1) process (MC6, i.e., DCC):  $b = 0.89$ ,  $b$  is the decay coefficient, and  $1/(1-b)$  is related to the number of days the process needs to mean revert;  $a = 0.099$  describes the level of the volatility of the volatility.
- fixed parameters for the univariate asymmetric GJR-GARCH(1,1,1) process (MC7, i.e., ADCC):  $b = 0.901$ ,  $b$  is the decay coefficient, and  $1/(1-b)$  is related to the number of days the process needs to mean revert;  $a = 0.0$ ,  $a + \gamma/2$  describe the level of the volatility of the volatility;  $\gamma = 0.171$ ,  $\gamma$  describes the asymmetry.

The fixed parameters that are supposed to be known when testing the DCC and ADCC betas are set to U.S. market estimates from Capiello, Engle, and Sheppard (2006):

- fixed parameters for the symmetric cross-term process (MC6, i.e., DCC):  $b_\rho = 0.9261$ ,  $b_\rho$  is the decay coefficient and is linked to the relaxation time;  $a_\rho = 0.0079$  describes the level of the volatility.
- fixed parameters for the asymmetric cross-term process (MC7, i.e., ADCC):  $b_\rho = 0.9512$ ,  $b_\rho$  is the decay coefficient and is linked to the relaxation time;  $a_\rho = 0.0020$ ,  $a_\rho + \gamma_\rho/4$  describes the level of the volatility of the correlation;  $\gamma_\rho = 0.0040$ ,  $\gamma_\rho$  describes the asymmetry.

The fixed parameters that are not known when testing the DCC beta and are estimated through the optimization of log-likelihood are set by MC simulation to:

- $\tilde{\rho} = 0.15/0.4$ , unconditional correlation;
- $\tilde{\sigma}_I = 0.15/\sqrt{255}$ ,  $\tilde{\sigma}_i = 0.4/\sqrt{255}$  unconditional stock index volatility;
- $\tilde{\sigma}_i = 0.4/\sqrt{255}$  unconditional single stock volatility.

To replicate the exponential weight scheme in the reactive model ( $\lambda_\beta = 1/90$ ), equation (C15) is replaced by

$$L_{DCC} = \frac{1}{2} \sum_t^T (1 - \lambda_\beta)^{T-t} (L_V(t) + L_C(t)). \quad (\text{C19})$$

## References

- Acharya, V., and L. H. Pedersen, 2005, Asset pricing with liquidity risk, *Journal of Financial Economics* 77, 375–410.
- Agarwal, V., Y. E. Arisoy, and N. Y. Naik, 2017, Volatility of aggregate volatility and the cross-section of hedge fund returns, *Journal of Financial Economics* 125, 491–510.
- Agarwal, V., T. C. Green, and H. Ren, 2018, Alpha or beta in the eye of the beholder: What drives hedge fund flows? *Journal of Financial Economics* 127, 417–34.
- Agarwal, V., and N. Y. Naik, 2004, Risks and portfolio decisions involving hedge funds, *Review of Financial Studies* 17, 63–98.
- Amihud, Y., 2002, Illiquidity and stock returns: Cross-section and time-series effects, *Journal of Financial Markets* 5, 31–56.
- Ang, A., and G. Bekaert, 2000, International asset allocation with time-varying correlations, *Review of Financial Studies* 15, 1137–87.
- Ang, A., and J. Chen, 2002, Asymmetric correlations of equity portfolios, *Journal of Financial Economics* 63, 443–94.
- Ang, A., and J. Chen, 2007, CAPM over the long run: 1926–2001, *Journal of Empirical Finance* 14, 1–40.
- Ang, A., R. Hodrick, Y. Xing, and X. Zhang, 2006, The cross-section of volatility and expected returns, *Journal of Finance* 61, 259–99.
- Ang, A., R. Hodrick, Y. Xing, and X. Zhang, 2009, High idiosyncratic volatility and low returns: International and further U.S. evidence, *Journal of Financial Economics* 91, 1–23.
- Ang, A., and J. Liu, 2004, How to discount cashflows with time-varying expected returns, *Journal of Finance* 59, 2745–83.
- Ang, A., A. Shtauber, and P. C. Tetlock, 2013, Asset pricing in the dark: The cross-section of OTC stocks, *Review of Financial Studies* 26, 2985–3028.
- Asness, C., R. Krail, and J. Liew, 2001, Do hedge funds hedge? *Journal of Portfolio Management* 28, 6–19.
- Baker, M., B. Bradley, and R. Taliaferro, 2013, The low beta anomaly: A decomposition into micro and macro effects, Working Paper, *Harvard Business School*.
- Bali, T. G., S. J., Brown, and Y. Tang, 2017, Is economic uncertainty priced in the cross-section of stock returns? *Journal of Financial Economics* 126, 471–89.
- Bali, T. G., R. F. Engle, and Y. Tang, 2017, Dynamic conditional beta is alive and well in the cross section of daily stock returns, *Management Science* 63, 3760–79.
- Banz, R. W., 1981, The relationship between return and market value of common stocks, *Journal of Financial Economics* 9, 3–18.
- Bekaert, G., and G. Wu, 2000, Asymmetric volatility and risk in equity markets, *Review of Financial Studies* 13, 1–42.
- Berk, J. B., and J. H. van Binsbergen, 2016, Assessing asset pricing models using revealed preference, *Journal of Financial Economics* 119, 1–23.
- Black, F., 1972, Capital market equilibrium with restricted borrowing, *Journal of Business* 4, 444–55.
- Black, F., 1976, Studies in stock price volatility changes, *American Statistical Association, Proceedings of the Business and Economic Statistics Section*, 177–181.

- Black, F., M. Jensen, and M. Scholes, 1972, The capital asset pricing model: Some empirical tests, in M. Jensen, ed.: *Studies in the Theory of Capital Markets* (Praeger, New York), 79–121.
- Blume, M. E., 1971, On the assessment of risk, *Journal of Finance* 26, 1–10.
- Boguth, O., M. Carlson, A. Fisher, and M. Simutin, 2011, Conditional risk and performance evaluation: Volatility timing, overconditioning, and new estimates of momentum alphas, *Journal of Financial Economics* 102, 363–89.
- Bollerslev, T., 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics*, 31, 307–327.
- Bollerslev, T., 1990, Modelling the coherence in short-run nominal exchange rates: A multivariate generalized ARCH model, *Review of Economics and Statistics* 72, 498–505.
- Bollerslev, T., R. Engle, and J. Wooldridge, 1988, A capital asset pricing model with time-varying covariances, *Journal of Political Economy* 96, 116–31.
- Bouchaud, J.-P., A. Matacz, and M. Potters, 2001, Leverage effect in financial markets: The retarded volatility model, *Physical Review Letters* 87, 1–4.
- Bussière, M., M. Hoerova, and B. Klaus, 2015, Commonality in hedge fund returns: Driving factors and implications, *Journal of Banking & Finance* 54, 266–80.
- Campbell, J. Y., and L. Hentchel, 1992, No news is good news: An asymmetric model of changing volatility in stock returns, *Journal of Financial Economics* 31, 281–318.
- Cappiello, L., R. Engle, and K. Sheppard, 2006, Asymmetric dynamics in the correlations of global equity and bond returns, *Journal of Financial Econometrics* 4, 537–72.
- Carhart, M., 1997, On persistence in mutual fund performance, *Journal of Finance* 52, 57–82.
- Chan, L., and J. Lakonishok, 1992, Robust measurement of beta risk, *Journal of Financial and Quantitative Analysis* 27, 265–82.
- Chen, A-S., T.-W. Zhang, and H. Wu, 2005, The beta regime change risk premium, Working Paper, National Chung Cheng University.
- Choi, J., 2013, What drives the value premium? The role of asset risk and leverage, *Review of Financial Studies* 26, 2845–75.
- Chordia, T., A. Goyal, and J. Shanken, 2015, Cross-sectional asset pricing with individual stocks: Betas versus characteristics, Working Paper, Emory University.
- Christie, A., 1982, The stochastic behavior of common stock variances—Value, leverage, and interest rate effects, *Journal of Financial Economic Theory* 10, 407–32.
- Daniel, K., R. Jagannathan, and S. Kim, 2012, Tail risk in momentum strategy returns, NBER Working Paper No. 18169.
- DeBondt, W. M., and R. H. Thaler, 1987, Further evidence on investor overreaction and stock market seasonality, *Journal of Finance* 42, 557–81.
- DeJong, D., and D. W. Collins, 1985, Explanations for the instability of equity beta: Risk-free rate changes and leverage effects, *Journal of Financial and Quantitative Analysis* 20, 73–94.
- Fabozzi, J. F., and C. J. Francis, 1978, Beta random coefficient, *Journal of Financial and Quantitative Analysis* 13, 101–16.
- Engle, R. F., 2002, Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models, *Journal of Business and Economic Statistics* 20, 339–50.
- Engle, R. F., 2016, Dynamic conditional beta, *Journal of Financial Econometrics* 14, 643–67.
- Fama, E. F., and K. R. French, 1992, The cross-section of expected returns, *Journal of Finance* 47, 427–65.
- Fama, E. F., and K. R. French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.
- Fama, E. F., and K. R. French, 1997, Industry costs of equity, *Journal of Financial Economics* 43, 153–93.
- Fama, E. F., and K. R. French, 2008, Dissecting anomalies, *Journal of Finance* 63, 1653–78.
- Fama, E. F., and K. R. French, 2012, Size, value, and momentum in international stock returns, *Journal of Financial Economics* 105, 457–72.
- Fama, E. F., and J. D. MacBeth, 1973, Risk, return and equilibrium: empirical tests, *Journal of Political Economy* 81, 607–36.
- Ferson, W. E., and C. R. Harvey, 1999, Conditioning variables and the cross-section of stock returns, *Journal of Finance* 54, 1325–60.
- Forbes, K., and R. Rigobon, 2002, No contagion, only interdependence: Measuring stock market comovements, *Journal of Finance* 57, 2223–61.
- Francis, J. C., 1979, Statistical analysis of risk surrogates for NYSE stocks, *Journal of Financial and Quantitative Analysis* 14, 981–97.
- Frazzini, A., and L. Pedersen, 2014, Betting against beta, *Journal of Financial Economics* 111, 1–25.

- Fung, W., and D. A. Hsieh, 1997, Empirical characteristics of dynamic trading strategies: The case of hedge funds, *Review of Financial Studies* 10, 275–302.
- Fung, W., and D. A. Hsieh, 2001, The risk in hedge fund strategies: Theory and evidence from trend followers, *Review of Financial Studies* 14, 313–41.
- Fung, W., and D. A. Hsieh, 2004, Hedge fund benchmarks: A risk-based approach, *Financial Analyst Journal* 60, 65–81.
- Galai, D., and R. W. Masulis, 1976, The option pricing model and the risk factor of stock, *Journal of Financial Economics* 3, 53–81.
- Garlappi, L., and H. Yan, 2011, Financial distress and the cross-section of equity returns, *Journal of Finance* 66, 789–822.
- Glosten, L. R., R. Jagannathan, and D. E. Runkle, 1993, On the relation between the expected value and the volatility of nominal excess return on stocks, *Journal of Finance* 48, 1779–1801.
- Goyal, A., and P. Santa-Clara, 2003, Idiosyncratic risk matters! *Journal of Finance* 58, 975–1007.
- Grinblatt, M., and T. Moskowitz, 2004, Predicting stock price movements from past returns: The role of consistency and tax-loss selling, *Journal of Financial Economics* 71, 541–79.
- Haugen, R. A., and N. L. Baker, 1991, The efficient market inefficiency of capitalization-weighted stock portfolios, *Journal of Portfolio Management* 17, 35–40.
- Haugen, R. A., and A. Heins, 1975, Risk and the rate of return on financial assets: Some old wine in new bottles, *Journal of Financial and Quantitative Analysis* 10, 775–84.
- Hong, H., and D. Sraer, 2016, Speculative betas, *Journal of Finance* 71, 2095–2144.
- Hong, Y., J. Tu, and G. Zhou, 2006, Asymmetries in stock returns: Statistical tests and economic evaluation, *Review of Financial Studies* 20, 1547–81.
- Jagannathan, R., and Z. Wang, 1996, The conditional CAPM and the cross-section of expected returns, *Journal of Finance* 51, 3–53.
- Jegadeesh, N., 1990, Evidence of predictable behavior of securities returns, *Journal of Finance* 45, 881–98.
- Jegadeesh, N., and S. Titman, 1993, Returns to buying winners and selling losers: Implications for stock market efficiency, *Journal of Finance* 48, 65–91.
- Jiang, L., K. Wu, and G. Zhou, 2018, Asymmetry in stock comovements: An entropy approach, *Journal of Financial and Quantitative Analysis* 53, 1479–1507.
- Koenker, R., and G. Bassett, 1978, Regression quantiles, *Econometrica* 46, 33–50.
- Lettau, M., and S. Ludvigson, 2001, Resurrecting the (C)CAPM: A cross-sectional test when risk premia are time-varying, *Journal of Political Economy* 109, 1238–87.
- Lewellen, J., and S. Nagel, 2006, The conditional CAPM does not explain asset pricing anomalies, *Journal of Financial Economics* 82, 289–314.
- Liu, J., R. F. Stambaugh, and Y. Yuan, 2018, Absolving beta of volatility’s effects, *Journal of Financial Economics* 128, 1–15.
- Longin, F., and B. Solnik, 2001, Extreme correlation of international equity markets, *Journal of Finance* 56, 649–76.
- Malkiel, B. G., and Y. Xu, 1997, Risk and return revisited, *Journal of Portfolio Management* 23, 9–14.
- Meng, J. G., G. Hu, and J. A. Bai, 2011, A simple method for estimating betas when factors are measured with error, *Journal of Financial Research* 34, 27–60.
- Mitchell, M., and T. Pulvino, 2001, Characteristics of risk and return in risk arbitrage, *Journal of Finance* 56, 2135–75.
- Moreira, A., and T. Muir, 2017, Volatility-managed portfolios, *Journal of Finance* 72, 1611–44.
- Patton, A. J., 2009, Are “market neutral” hedge funds really market neutral? *Review of Financial Studies* 22, 2295–2330.
- Reinganum, R., 1981, Misspecification of capital asset pricing: empirical anomalies based on earnings yields and market values, *Journal of Financial Economics* 9, 19–46.
- Roll, R., 1977, A critique of the asset pricing theory’s tests: Part I: On past and potential testability of the theory, *Journal of Financial Economics* 4, 129–76.
- Shanken, J., 1992, On the estimation of beta-pricing models, *Review of Financial Studies* 5, 1–33.
- Sharpe, W. F., 1964, Capital asset prices: A theory of market equilibrium under risk, *Journal of Finance* 19, 425–42.
- Sheppard, K., 2017, Financial econometrics notes, Chapter 7, University of Oxford, pp. 417–482. [https://www.kevinshppard.com/images/5/5a/Financial\\_Econometrics\\_2018-2019.pdf](https://www.kevinshppard.com/images/5/5a/Financial_Econometrics_2018-2019.pdf).
- Valeyre, S., D. S. Grebenkov, S. Aboura, and Q. Liu, 2013, The reactive volatility model, *Quantitative Finance* 13, 1697–1706.